# Robotics, Geometry and Control - Rigid body motion and geometry

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The material for these slides is largely taken from the *text*

- A Mathematical Introduction to Robotic Manipulation - R. Murray, Z. Li and S. Sastry, CRC Press, 1994.

# Rigid body motion

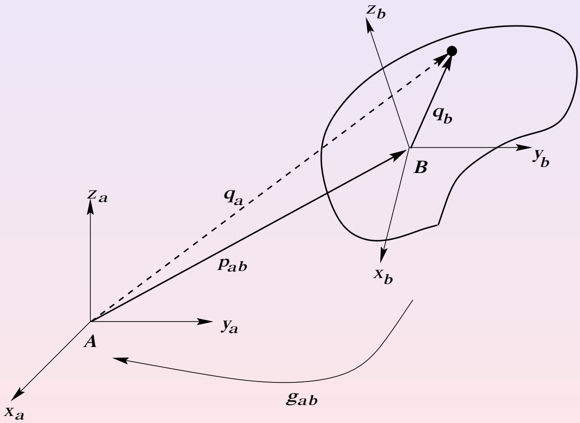


Figure: Rigid body motion

# Rigid body motion

## Definition

Rigid body motion is characterized by two properties

- The distance between any two points remains invariant
- The orientation of the body is preserved. (A right-handed coordinate system remains right-handed)

# $SO(3)$ and $SE(3)$

- Two groups which are of particular interest to us in robotics are $SO(3)$ - the special orthogonal group that represents rotations - and $SE(3)$ - the special Euclidean group that represents rigid body motions.
- Elements of $SO(3)$ are represented as $3 \times 3$ real matrices and satisfy
$$R^T R = I$$
with $det(R) = 1$.
- An element of $SE(3)$ is of the form $(p, R)$ where $p \in R^3$ and $R \in SO(3)$.

# Frames of reference or coordinate frames

- Rigid body motions are usually described using two frames of reference. One is called the *body frame* that remains fixed to the body and the other is the *inertial frame* that remains fixed in inertial space.
- Position of a point $p$ on the rigid body after the body undergoes a rotation of angle $\theta$ about the inertial z-axis. The transformation (or matrix) that maps the initial coordinates of $p$ to the final coordiantes of $p$ (both in the inertial frame) is the rotation matrix

$$R_{z\theta} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Rotations

- Similarly for the $y$ and $x$ axes with angles $\alpha$ and $\beta$, we have

$$R_{y\alpha} = \begin{pmatrix} \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & 1 & 0 \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix} \qquad R_{x\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{pmatrix}$$

# Cross product and rotations

- **The Cross Product** between two vectors $a = (a_1, a_2, a_3)^T$ and $b = (b_1, b_2, b_3)^T$ in $R^3$ is defined as

$$a \times b = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

- The operation could be represented as the multiplication by a matrix as

$$b \xrightarrow{a} a \times b = \hat{a} b \quad \text{where} \quad \hat{a} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

Notice that $\hat{a}$ is a skew-symmetric matrix.

# Angular velocity vector and skew-symmetric representation

- The angular velocity vector of a rigid body can be expressed with respect to a certain basis as

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \quad \equiv \quad \hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

- Often it is expressed as a scalar (magnitude) multiplied to a unit vector (axis of rotation)

# Skew-symmetric matrices and rotations

- Consider a rotation about the z-axis. In the standard basis, the $z$ axis is $(0, 0, 1)$ which in the skew-symmetric matrix form is

$$\hat{\omega} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- The set of skew-symmetric matrices in $R^{3\times 3}$ with the bracket operation defined as

$$[X, Y] = XY - YX$$

  forms a Lie algebra and is denoted as $so(3)$.
- In vector notation, the Lie bracket on $so(3)$ between two elements $\omega_1, \omega_2 \in R^3$ is given by

$$[\omega_1, \omega_2] = \omega_1 \times \omega_2 \qquad OR \quad [\hat{\omega}_1, \hat{\omega}_2] = \hat{\omega_1}\hat{\omega_2} - \hat{\omega_2}\hat{\omega_1}$$

# A differential equation

- Velocity of a point $q$ on a rigid body undergoing pure rotation. Let the axis of rotation be defined by $\omega$ and we assume unit angular velocity. Then
$$\dot{q}(t) = 1\omega \times q(t) = \hat{\omega} q(t)$$
- Time-invariant linear differential equation which may be integrated to give
$$q(t) = e^{\hat{\omega} t} q(0)$$
- Rotation about the axis $\omega$ at unit velocity for $\theta$ units of time gives a net rotation of

$$R(\omega, \theta) = e^{\hat{\omega}\theta} \qquad e(\cdot) : so(3) \to SO(3)$$

# Parametrizing rotational motion and a Lie algebra $so(3)$

- Parametrize a rotation motion in time by a differentiable curve $c(.) : R \supset I \rightarrow SO(3)$. The set

$$g \triangleq \{\frac{dc}{dt}|_{t=0} : c(t) \in SO(3) \quad \text{and} \quad c(0) = I\}$$

is the set of all tangent vectors at the identity of $SO(3)$. It consists of all skew-symmetric matrices of dimension 3 called $so(3)$.

# Some examples

- Consider a rotation about the axis $(0, 0, 1)$ in the standard basis parametrized as follows

$$R_z(.) : t \rightarrow SO(3) \qquad R_z(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Let the axis of rotation be $\omega = (1, 0, 0)$, assume unit angular velocity and let the time of rotation be $t$. Then the rotation achieved is

$$e^{\hat{\omega} t} = I + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -t \\ 0 & t & 0 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -t \\ 0 & t & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -t \\ 0 & t & 0 \end{pmatrix} + \ldots$$

# The exponential map

- Given the axis of rotation, the angular velocity and the time of rotation, the exponential map denoted by ”exp” gives the actual rotation. Mathematically, the exponential map is a transformation from $so(3)$ to $SO(3)$ given as

$$\exp(\hat{\omega}) \triangleq I + \hat{\omega} + \hat{\omega}^2/2! + \ldots \in SO(3)$$

- **Remark**: The axis of rotation $\omega$ is often normalized such that $\|\omega\| = 1$ and the angular velocity vector written as $\alpha\omega$ where $\alpha$ is the magnitude of the angular velocity.

# The exponential map

## Lemma

*The exponential map from the Lie algebra $so(3)$ to the group $SO(3)$ is a many-to-one map that is surjective. (A given rotation ($\in SO(3)$)can be obtained in more than one ($\in so(3)$) way).*

**Example:** Consider the elements of $so(3)$

$$\begin{pmatrix} 0 & -\alpha & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -\alpha - 2\pi & 0 \\ 0 & 0 & \alpha + 2\pi \\ 0 & 0 & 0 \end{pmatrix}$$

Both yield the same value for $\exp(\cdot)$

# Axis of rotation and angular velocity

- Consider the rotation of a rigid body where the rotation is parametrized in time by a curve $R(t) \in SO(3)$. Let the position (coordinates) of a point fixed on the rigid body, with respect to an inertial frame (A) and a body fixed frame (B) be $q_a$ and $q_b$ respectively.

$$q_a = R_{ab} q_b \quad v_a = \dot{R}_{ab} q_b \quad \text{since} \quad \dot{q}_b = 0$$

## Lemma

*The matrices $\dot{R}(t)R^{-1}(t)$ and $R^{-1}(t)\dot{R}(t)$ both belong to the Lie algebra $so(3)$.*

# Skew-symmetric representations

- Consider the identity

$$R(t)R^{-1}(t) = I$$

On differentiation we have

$$\dot{R}(t)R^{-1}(t) = -R(t)\dot{R^{-1}}(t) = -(\dot{R}(t)R^{-1}(t))^T$$

Recall $R^T = R^{-1}$.

- Similarly $R^{-1}(t)\dot{R}(t)$ is also skew-symmetric.
- Now

$$v_a = \dot{R}_{ab}q_b = \dot{R}_{ab}R_{ab}^{-1}R_{ab}q_b = \dot{R}_{ab}R_{ab}^{-1}q_a = \hat{\omega}_{ab}^s q_a$$

where $\hat{\omega}_{ab}^s \triangleq \dot{R}_{ab}R_{ab}^{-1} \in so(3)$ is termed the *spatial angular velocity*. Physically, the spatial angular velocity corresponds to the instantaneous angular velocity of the object as seen from the spatial coordinate frame.

# Body and spatial angular velocities

- Similarly, we define the *body angular veocity* as

$$\hat{\omega}^b_{ab} \triangleq R^{-1}_{ab}\dot{R}_{ab} \in so(3)$$

The body angular velocity describes the instantaneous angular velocity in the instantaneous body frame.

- The velocity of the point in the body frame is

$$v_b \triangleq R^{-1}_{ab} v_a = R^{-1}_{ab}\hat{\omega}^s_{ab} q_a = \hat{\omega}^b_{ab} q_b$$

# Euclidean motions and groups

- Suppose $q_a$ and $q_b$ are coordinates of a point $q$ relative to frames $A$ and $B$, respectively.

$$q_a = p_{ab} + R_{ab} q_b$$

  Here $p_{ab}$ represents the position of the origin of the frame $B$ with respect to frame $A$ in frame $A$ coordinates and $R_{ab}$ is the orientation of frame $B$ with respect to frame $A$ .

- Appending a "1" to the coordinates of a point ( to render the group operation as the usual matrix multiplication)

$$\bar{q}_a = \begin{pmatrix} q_a \\ 1 \end{pmatrix} = \begin{pmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_b \\ 1 \end{pmatrix} = \bar{g}_{ab} \bar{q}_b$$

# The Special Euclidean group $SE(3)$

- Rigid body motion can be described in terms of members of $SE(3)$ - the special Euclidean group expressing rigid body orientation/position.
- An element of $SE(3)$ is of the form $(p, R)$ where $p \in R^3$ describes the coordinates of the origin of a *body fixed frame* with respect to an *inertial frame* and $R \in SO(3)$ describes the orientation of the body-fixed frame with respect to the inertial frame.
- To make the group operation a matrix multiplication in the usual sense, $(p, R)(\in SE(3))$ is in the form of a $4 \times 4$ matrix

$$\begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix}$$

**Question** : Show that the representation of $SE(3)$ as above satisfies the conditions of a group under the operation of matrix multiplication

# Rigid body velocities

- Rigid body **velocities** are described by elements of $se(3)$ as

$$\begin{pmatrix} M & v \\ 0 & 0 \end{pmatrix}$$

where $M \in so(3)$ (recall axis of rotation) and $v \in R^3$.

# Twist representations

- The velocity of a point **p** on the link shown in the figure can be given as

$$\dot{p}(t) = \omega \times r(t)$$

where $r(t)$ is the vector from the origin of the body-frame to the point **p**. This can be rewritten as

$$\dot{p}(t) = \omega \times (p(t) - q(t))$$

Note that $p(t)$ and $q(t)$ here are expressed in the inertial frame.

- We rewrite the above equation in homogeneous coordinates. First we define

$$\hat{\xi} \triangleq \begin{pmatrix} \hat{\omega} & v \\ 0 & 0 \end{pmatrix}$$

with $v \triangleq (-\omega \times q)$

# A differential equation and the exponential map

- Then

$$\begin{pmatrix} \dot{p} \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{\omega} & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ 1 \end{pmatrix} \quad \text{or} \quad \dot{\bar{p}} = \hat{\xi}\bar{p}$$

- We once again have a set of first order differential equations whose solution can be expressed as

$$\bar{p}(t) = e^{\hat{\xi}t}\bar{p}(0)$$

## Lemma

*The exponential map from the Lie algebra $se(3)$ to the group $SE(3)$ is a many-to-one map that is surjective.*

## Remark

*Notice once again, like in the case of rotations, more than one element in $se(3)$ may generate the same rigid body motion ($\in SE(3)$)*

# Twist representations

- The matrix $\hat{\xi}$ is called a **twist** and belongs to $se(3)$ - the Lie algebra associated with the Lie group $SE(3)$. The twist **coordinates** are given as

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} \quad v \in R^3, \omega \in R^3$$

- The Lie bracket on $se(3)$ between two elements $\xi_1, \xi_2 \in R^6$ is given by

$$[\hat{\xi}_1, \hat{\xi}_2] = \hat{\xi}_1 \hat{\xi}_2 - \hat{\xi}_2 \hat{\xi}_1$$

or in twist coordinates

$$[\xi_1, \xi_2] = \begin{bmatrix} \omega_1 \times v_2 - \omega_2 \times v_1 \\ \omega_1 \times \omega_2 \end{bmatrix}$$

# Motion in a plane

- If the rigid body motion is restricted to a plane (two-dimensional) then the group reduces to $SE(2)$ and the Lie algebra to $se(2)$. The Lie bracket on $se(2)$ between two elements $\xi_1, \xi_2 \in R^3$ is given by

$$[\hat{\xi}_1, \hat{\xi}_2] = \hat{\xi}_1\hat{\xi}_2 - \hat{\xi}_2\hat{\xi}_1$$

or in twist coordinates

$$[\xi_1, \xi_2] = \begin{bmatrix} \omega_1 \times v_2 - \omega_2 \times v_1 \\ 0 \end{bmatrix}$$

Here

$$\xi_i = \begin{bmatrix} v_i \\ \omega_i \end{bmatrix} \qquad v_i \in R^2, \omega_i \in R$$

$$\xi \triangleq (v, \omega)$$

# Screw motions

- An important concept in robotics which helps express the motion of revolute and prismatic joints is the notion of a **screw**. As the name implies, it is the type of motion exhibited by a screw. If $h$ is the pitch of the screw, a rotation of $\theta$ radians along an axis $\omega$ results in a linear translation of $h\theta$ along the axis of the screw.
- We ask: What is the twist that achieves a certain screw motion ? From the figure and elementary physics, we can express the net motion of the screw in an inertial frame as

$$q \ \text{(the position vector to } q) + e^{\hat{\omega}}\theta(p - q) \ \text{(the vector from } q \text{ to } s)$$

$$+h\theta\omega \ \text{(the vector from } s \text{ to } f)$$

which can be expressed as

$$\begin{pmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ 1 \end{pmatrix}$$

# Screw motions

- We equate this to a motion achieved by a twist $(v, \omega)$

$$\begin{pmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ 1 \end{pmatrix}$$

# Twist representations - Body and Spatial velocities

## Lemma

*The matrice $\dot{g}(t)g^{-1}(t)$ and $g^{-1}(t)\dot{g}(t)$ both belong to the Lie algebra $se(3)$.*

Consider

$$g = \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \qquad g^{-1} = \begin{pmatrix} R^T & -R^T p \\ 0 & 1 \end{pmatrix}$$

and

$$g^{-1}\dot{g} = \begin{pmatrix} R^T \dot{R} & R^T \dot{p} \\ 0 & 0 \end{pmatrix}$$

$$\dot{g}g^{-1} = \begin{pmatrix} \dot{R}R^T & -\dot{R}R^T p + \dot{p} \\ 0 & 0 \end{pmatrix}$$

# Spatial velocity - rotation and translation

Now

$$\bar{q}_a = \begin{pmatrix} q_a \\ 1 \end{pmatrix} = \begin{pmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_b \\ 1 \end{pmatrix} = \bar{g}_{ab} \bar{q}_b$$

- The *spatial velocity* is

$$v_a = \dot{g}_{ab} q_b = \dot{g}_{ab} g_{ab}^{-1} g_{ab} q_b = \dot{g}_{ab} g_{ab}^{-1} q_a$$

- The matrix $\dot{g}_{ab} g_{ab}^{-1} \triangleq \hat{V}_{ab}^s \in se(3)$ has the form of a twist given by

$$\begin{pmatrix} \dot{R}_{ab} R_{ab}^T & -\dot{R}_{ab} R_{ab}^T p_{ab} + \dot{p}_{ab} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \hat{\omega}_{ab}^s & v_{ab}^s \\ 0 & 0 \end{pmatrix}$$

# Body velocity

- Similar to the case of rotations, we can describe a *body frame velocity* (the angular and translational velocity relative to the instantaneous body frame) as

$$\hat{V}^b_{ab} \triangleq g^{-1}_{ab}\dot{g}_{ab} = \begin{pmatrix} R^T_{ab}\dot{R}_{ab} & R^T_{ab}\dot{p}_{ab} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \hat{\omega}^b_{ab} & v^b_{ab} \\ 0 & 0 \end{pmatrix} \in se(3)$$

# Relating the body and spatial representations

- These two velocities (spatial and body) are related by a similarity transformation as

$$\hat{V}^b_{ab} = g^{-1}_{ab}\hat{V}^s_{ab} g_{ab}$$

- The $6 \times 6$ matrix which transforms twist coordinates ($\in R^6$) in one reference frame to another is called the *adjoint transformation* ($Ad_{g_{ab}} : \mathcal{G} \rightarrow \mathcal{G}$) associated with $g_{ab}$.

$$V^s_{ab} = \begin{pmatrix} v^s_{ab} \\ \omega^s_{ab} \end{pmatrix} = \begin{pmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{pmatrix} \begin{pmatrix} v^b_{ab} \\ \omega^b_{ab} \end{pmatrix}$$

where

$$Ad_{g_{ab}} \triangleq \begin{pmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{pmatrix}$$

# Body velocity

## Lemma

*If $\hat{\xi} \in se(3)$ is a twist with twist coordinates $\xi \in R^6$, then for any $g \in SE(3)$, $g\hat{\xi}g^{-1}$ is a twist with twist coordinates $Ad_g\xi \in R^6$*

## Lemma

*The exponential map from $se(3)$ to $SE(3)$ is a surjective map. Hence every rigid transformation $g$ can be written as the exponential of some twist.*

## Remark

*Notice once again, like in the case of rotations, more than one element in $se(3)$ may generate the same rigid body motion ($\in SE(3)$)*