Robust Stability

• Robust stability against time-invariant and time-varying uncertainties
• Parameter dependent Lyapunov functions
• Semi-infinite LMI problems
• From nominal to robust performance
Consider linear time-invariant system

\[
\dot{x}(t) = A(\delta)x(t)
\]

where \(A(.)\) is a continuous function of the parameter vector

\[
\delta = (\delta_1, ..., \delta_p)
\]

which is only known to be contained in uncertainty set

\[
\delta \subset \mathbb{R}^p.
\]

Robust stability analysis

Is system asymptotically stable for all possible parameters \(\delta \in \delta\)?

Example: Mass- or load-variation of controlled mechanical system.
Example

Academic example with **rational** parameter-dependence:

\[
\dot{x} = \begin{pmatrix}
-1 & 2\delta_1 & 2 \\
\delta_2 & -2 & 1 \\
3 & -1 & \frac{\delta_3 - 10}{\delta_1 + 1}
\end{pmatrix} x.
\]

The parameters \(\delta_1\), \(\delta_2\), \(\delta_3\) are bounded as

\[
\delta_1 \in [-0.5, 1], \quad \delta_2 \in [-2, 1], \quad \delta_3 \in [-0.5, 2].
\]

Hence \(\delta\) is actually a polytope (box) with eight generators:

\[
\delta = [-0.5, 1] \times [-2, 1] \times [-0.5, 2] = \co \left\{ \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} : \delta_1 \in \{-0.5, 1\}, \quad \delta_2 \in \{-2, 1\}, \quad \delta_3 \in \{-0.5, 2\} \right\}.
\]
Relation to Optimization

Spectral abscissa of square matrix $A$: $\rho_a(A) = \max_{\lambda \in \sigma(A)} \frac{1}{2}(\lambda + \bar{\lambda})$.

**Obvious fact:** $A(\delta)$ is Hurwitz for all $\delta \in \delta$ if and only if $\rho_a(A(\delta)) < 0$ for all $\delta \in \delta$.

**Two main sources for trouble:**
- $\rho_a(A(\delta))$ is far from convex/concave in the variable $\delta$.
- Inequality has to hold at infinitely many points.

Hence typically computational approaches fail:
- Cannot find global maximum of $\rho_a(A(\delta))$ over $\delta$.
- Even if $\delta$ is a polytope, not sufficient to check the generators.
- Even more trouble if $\delta$ is no polytope.
The uncertain system $\dot{x} = A(\delta)x$ with $\delta \in \delta$ is defined to be **quadratically stable** if there exists $X \succ 0$ with

$$A(\delta)^T X + X A(\delta) \prec 0 \quad \text{for all } \delta \in \delta.$$

**Why name?** $V(x) = x^T X x$ is quadratic Lyapunov function.

**Why relevant?** Implies that $A(\delta)$ is Hurwitz for all $\delta \in \delta$.

**How to check?** Easy if $A(\delta)$ is affine in $\delta$ and $\delta = \text{co}\{\delta^1, \ldots, \delta^N\}$ is a polytope with moderate number of generators: Verify whether

$$X \succ 0, \quad A(\delta^k)^T X + X A(\delta^k) \prec 0, \quad k = 1, \ldots, N$$

is feasible.
Example

If $A(\delta)$ is not affine in $\delta$, a parameter transformation often helps!

In example introduce $\delta_4 = \frac{\delta_3 - 10}{\delta_1 + 1} + 12$. Test quadratic stability of

$$\begin{pmatrix}
-1 & 2\delta_1 & 2 \\
\delta_2 & -2 & 1 \\
3 & -1 & \delta_4 - 12
\end{pmatrix}, \ (\delta_1, \delta_2, \delta_4) \in \delta = [-0.5, 1] \times [-2, 1] \times [-9, 8].$$

LMI-Toolbox: System quadratically stable for $$(\delta_1, \delta_2, \delta_4) \in r\delta$$ with largest possible factor $r \approx 0.45$.

Quadratically stable for shrunk set $0.45\delta$. Not for $r\delta$ with $r > 0.45$. The critical factor is called **quadratic stability margin**.
Time-Varying Parametric Uncertainties

Now assume that the parameters $\delta(t)$ vary with time, and that they are known to satisfy $\delta(t) \in \delta$ for all $t$. Check stability of

$$\dot{x}(t) = A(\delta(t))x(t), \quad \delta(t) \in \delta.$$ 

The uncertain system with time-varying parametric uncertainties is exponentially stable if there exists $X \succ 0$ with

$$A(\delta)^TX + XA(\delta) \prec 0 \quad \text{for all} \quad \delta \in \delta.$$

Proof will be given for a more general result in full detail.

Therefore quadratic stability does in fact imply robust stability for arbitrary fast time-varying parametric uncertainty. If bounds on velocity of parameters are known, this test is conservative.
Let us hence assume that the parameter-curves $\delta(.)$ are continuously differentiable and are only known to satisfy

$$\delta(t) \in \delta \quad \text{and} \quad \dot{\delta}(t) \in v \quad \text{for all time instances.}$$

Here $\delta \subset \mathbb{R}^p$ and $v \subset \mathbb{R}^p$ are given compact sets (e.g. polytopes).

**Robust stability analysis**
Verify whether the linear time-varying system

$$\dot{x}(t) = A(\delta(t))x(t)$$

is exponentially stable for all parameter-curves $\delta(.)$ that satisfy the above described bounds on value and velocity.

Search for suitable Lyapunov function.
Main Stability Result

Theorem

Suppose $X(\delta)$ is continuously differentiable on $\delta$ and satisfies

$$X(\delta) > 0, \sum_{k=1}^{p} \partial_k X(\delta) v_k + A(\delta)^T X(\delta) + X(\delta) A(\delta) < 0$$

for all $\delta \in \delta$ and $v \in v$. Then there exist constants $K > 0, a > 0$ such that all trajectories of the uncertain time-varying system satisfy

$$\|x(t)\| \leq K e^{-a(t-t_0)} \|x(t_0)\| \text{ for all } t \geq t_0.$$ 

- Covers many tests in literature. Study the proof to derive variants!
- Condition is in general sufficient only!
- Is also necessary in case that $v = \{0\}$: Time-invariant uncertainty.
Proof

Continuity & compactness ... exist \( \alpha, \beta, \gamma > 0 \) such that for all \( \delta \in \delta, v \in v \):

\[
\alpha I \preceq X(\delta) \preceq \beta I, \quad \sum_{k=1}^{p} \partial_k X(\delta) v_k + A(\delta)^T X(\delta) + X(\delta) A(\delta) \preceq -\gamma I.
\]

Suppose that \( \delta(t) \) is any admissible parameter-curve and let \( x(t) \) denote a corresponding state-trajectory of the system. Here is the crucial point:

\[
\frac{d}{dt} x(t)^T X(\delta(t)) x(t) = x(t)^T \left[ \sum_{k=1}^{p} \partial_k X(\delta(t)) \dot{\delta}_k(t) \right] x(t) +
\]

\[
+ x(t)^T \left[ A(\delta(t))^T X(\delta(t)) + X(\delta(t)) A(\delta(t)) \right] x(t).
\]

Since \( \delta(t) \in \delta \) and \( \dot{\delta}(t) \in v \) we can hence conclude

\[
\alpha \| x(t) \|^2 \preceq x(t)^T X(\delta(t)) x(t) \preceq \beta \| x(t) \|^2, \quad \frac{d}{dt} x(t)^T X(\delta(t)) x(t) \leq -\gamma \| x(t) \|^2.
\]
Proof

The proof is now finished as the one for LTI systems given earlier.

Define \( \xi(t) := x(t)^T X(\delta(t)) x(t) \) to infer that

\[
\|x(t)\|^2 \leq \frac{1}{\alpha} \xi(t), \quad \xi(t) \leq \beta \|x(t)\|^2, \quad \dot{\xi}(t) \leq -\frac{\gamma}{\beta} \xi(t).
\]

The latter inequality leads to

\[
\xi(t) \leq \xi(t_0) e^{-\frac{\gamma}{\beta}(t-t_0)} \text{ for all } t \geq t_0.
\]

With the former inequalities we infer

\[
\|x(t)\|^2 \leq \frac{\beta}{\alpha} e^{-\frac{\gamma}{\beta}(t-t_0)} \|x(t_0)\|^2 \text{ for all } t \geq t_0.
\]

Can choose \( K = \sqrt{\beta/\alpha} \) and \( a = \gamma/(2\beta) \).
Extreme Cases

- Parameters are **time-invariant**: \( v = \{0\} \).

  Have to find \( X(\delta) \) satisfying
  \[
  X(\delta) \succeq 0, \quad A(\delta)^T X(\delta) + X(\delta) A(\delta) \prec 0 \quad \text{for all} \quad \delta \in \delta.
  \]

- Parameters vary **arbitrarily fast**:

  Have to find parameter-independent \( X \) satisfying
  \[
  X \succ 0, \quad A(\delta)^T X + X A(\delta) \prec 0 \quad \text{for all} \quad \delta \in \delta.
  \]

  Is identical to **quadratic stability** test!

Can apply subsequently suggested numerical techniques in both cases!
Specializations

Have derived general results based on Lyapunov functions which still depend \textit{quadratically} on the state (which is restrictive) but which allow for non-linear (smooth) dependence on the uncertain parameters.

Is pure algebraic test and does not involve system- or parameter-trajectory.

Not easy to apply:

- Have to find \textbf{function} satisfying partial differential LMI
- Have to make sure that inequality holds for all $\delta \in \delta$, $\nu \in \nu$.

Allows to easily derive specializations which are or can be implemented with LMI solvers. We just consider a couple of examples.
Example: Affine System - Affine Lyapunov Matrix

Suppose $A(\delta)$ depends affinely on parameters:

$$A(\delta) = A_0 + \delta_1 A_1 + \cdots + \delta_p A_p.$$ 

Parameter- and rate-constraints are boxes:

$$\delta = \{ \delta \in \mathbb{R}^p : \delta_k \in [\underline{\delta}_k, \bar{\delta}_k] \}, \quad v = \{ v \in \mathbb{R}^p : v_k \in [\underline{v}_k, \bar{v}_k] \}$$

These are the convex hulls of

$$\delta_g = \{ \delta \in \mathbb{R}^p : \delta_k \in \{\underline{\delta}_k, \bar{\delta}_k\} \}, \quad v_g = \{ v \in \mathbb{R}^p : v_k \in \{\underline{v}_k, \bar{v}_k\} \}$$

Search for affine parameter dependent $X(\delta)$:

$$X(\delta) = X_0 + \delta_1 X_1 + \cdots + \delta_p X_p$$

and hence $\partial_k X(\delta) = X_k$. 
Example: Affine System - Affine Lyapunov Matrix

With \( \delta_0 = 1 \) observe that

\[
\sum_{k=1}^{p} \partial_k X(\delta)v_k + A(\delta)^T X(\delta) + X(\delta)A(\delta) = \\
= \sum_{k=1}^{p} X_k v_k + \sum_{\nu=0}^{p} \sum_{\mu=0}^{p} \delta_\nu \delta_\mu (A_\nu X_\mu + X_\mu A_\nu).
\]

Is affine in \( X_1, \ldots, X_p \) and \( v_1, \ldots, v_p \) but quadratic in \( \delta_1, \ldots, \delta_p \).

**Relaxation:** Include additional constraint \( A_\nu^T X_\nu + X_\nu^T A_\nu \succeq 0 \).

- Implies that it suffices to guarantee required inequality at generators. Why? Partially convex function on box!

- Extra condition renders test stronger. **Still sufficient for RS!**
Partially Convex Function on Box

Suppose that $S \subset \mathbb{R}^n$. The Hermitian-valued function $F : S \to \mathbb{H}^m$ is said to be partially convex if the one-variable mapping

$$t \to F(x_1, \ldots, x_{l-1}, t, x_{l+1}, \ldots, x_n)$$

defined on $\{t \in \mathbb{R} : (x_1, \ldots, x_{l-1}, t, x_{l+1}, \ldots, x_n) \in S\}$ is convex for all $x \in S$ and $l = 1, \ldots, n$.

Suppose $S = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and that $F : S \to \mathbb{H}^m$ is partially convex. Then $F(x) \prec 0$ for all $x \in S$ if and only if

$$F(x) \prec 0 \text{ for all } x \in \{a_1, b_1\} \times \cdots \times \{a_n, b_n\}.$$

Proof is simple exercise. Various variants suggested in literature.
Example: Affine System - Affine Lyapunov Matrix

Robust exponential stability guaranteed if

There exist $X_0, \ldots, X_p$ with $A_T^ν X_ν + X_ν A_ν \succeq 0$, $ν = 1, \ldots, p$, and

$$\sum_{k=0}^{p} X_k δ_k \succeq 0, \quad \sum_{k=1}^{p} X_k v_k + \sum_{ν=0}^{p} \sum_{μ=0}^{p} δ_ν δ_μ (A_T^ν X_μ + X_μ A_ν) \prec 0$$

for all $δ \in δ_g$ and $ν \in ν_g$ and $δ_0 = 1$.

• This test is implemented in LMI-Toolbox.

For rate-bounded uncertainties often much less conservative than quadratic stability test.

• Understand the arguments in proof to derive your own variants.
General Recipe to Reduce to Finite Dimensions

Restrict the search to a chosen finite-dimensional subspace.

For example choose scalar continuously differentiable basis functions \( b_1(\delta), \ldots, b_N(\delta) \) and search for the coefficient matrices \( X_1, \ldots, X_N \) in the expansion

\[
X(\delta) = \sum_{\nu=1}^{N} X_{\nu} b_{\nu}(\delta) \quad \text{with} \quad \partial_k X(\delta) = \sum_{\nu=1}^{N} X_{\nu} \partial_k b_{\nu}(\delta).
\]

Have to guarantee for all \( \delta \in \delta \) and \( \nu \in \nu \) that

\[
\sum_{\nu=1}^{N} X_{\nu} b_{\nu}(\delta) \succ 0, \quad \sum_{\nu=1}^{N} \left( \sum_{k=1}^{p} X_{\nu} \partial_k b_{\nu}(\delta) v_k + [A(\delta)^T X_{\nu} + X_{\nu} A(\delta)] b_{\nu}(\delta) \right) \prec 0.
\]

Is finite dimensional but still semi-infinite LMI problem!
Remarks

• If systematically extending the set of basis functions one can improve the sufficient stability conditions. Example: Polynomial basis

\[ b(k_1,\ldots,k_p)(\delta) = \delta_1^{k_1} \cdots \delta_p^{k_p}, \quad k_\nu = 0, 1, 2, \ldots, \quad \nu = 1, \ldots, p. \]

If \( \delta \) is star-shaped one can prove: If the partial differential LMI has a continuously differentiable solution then it also has a polynomial solution (of possibly high degree).

Polynomial basis is a generic choice with guaranteed success.

• One can just grid \( \delta \) and \( \nu \) to arrive at finite system of LMI’s.

  **Trouble:** Huge LMI system. No guarantees at points not in grid.
Semi-Infinite LMI-Constraints

Nominal stability could be reduced to an LMI feasibility test: Does there exist a solution $x \in \mathbb{R}^n$ of some LMI

$$F_0 + x_1 F_1 + \cdots + x_n F_n \prec 0.$$ 

We have now seen that testing robust stability can be reduced to the following question: Does there exist some $x \in \mathbb{R}^n$ with

$$F_0(\delta) + x_1 F_1(\delta) + \cdots + x_n F_n(\delta) \prec 0 \text{ for all } \delta \in \delta$$

where $F_0(\delta), \ldots, F_n(\delta)$ are Hermitian-valued functions of $\delta \in \delta$.

Is a generic formulation for the robust counterpart of an LMI feasibility test in which the data matrices are affected by uncertainties.
From Nominal to Robust Performance

Recall how we obtained from nominal stability characterizations the corresponding robust stability tests against time-varying rate-bounded parametric uncertainties.

As a major beauty of the dissipation approach, this generalization works without any technical delicacies for performance as well!

Only for time-invariant uncertainties the frequency-domain characterizations make sense. For time-varying uncertainties (and time-varying systems) we have to rely on the time-domain interpretations.

We provide an illustration for quadratic performance!
Robust Quadratic Performance

Uncertain input-output system described as

$$\dot{x}(t) = A(\delta(t))x(t) + B(\delta(t))w(t)$$
$$z(t) = C(\delta(t))x(t) + D(\delta(t))w(t).$$

with continuously differential parameter-curves $\delta(.)$ that satisfy

$$\delta(t) \in \delta \text{ and } \dot{\delta}(t) \in v \quad (\delta, v \subset \mathbb{R}^p \text{ compact}).$$

Robust quadratic performance: Exponential stability and existence of $\epsilon > 0$ such that for $x(0) = 0$, for all parameter curves and for all trajectories

$$\int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T \begin{pmatrix} Q_p & S_p \\ S_p^T & R_p \end{pmatrix} \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} \, dt \leq -\epsilon \|w\|_2^2.$$

$L_2$-gain, passivity, ...
Characterization of Robust Quadratic Performance

Suppose there exists a continuously differentiable Hermitian-valued $X(\delta)$ such that $X(\delta) \succ 0$ and

$$\begin{pmatrix}
\sum_{k=1}^{p} \partial_k X(\delta) v_k + A(\delta)^T X(\delta) + X(\delta) A(\delta) & X(\delta) B(\delta) \\
B(\delta)^T X(\delta) & 0
\end{pmatrix} +$$

$$+ \begin{pmatrix}
0 & I \\
C(\delta) & D(\delta)
\end{pmatrix}^T P_p \begin{pmatrix}
0 & I \\
C(\delta) & D(\delta)
\end{pmatrix} \prec 0$$

for all $\delta \in \delta$, $v \in v$. Then the uncertain system satisfies the robust quadratic performance specification.

Numerical search for $X(\delta)$: Same as for stability!
Extends to other LMI performance specifications!
Sketch of Proof

**Exponential stability:** Left-upper block is
\[
\sum_{k=1}^{p} \partial_k X(\delta) v_k + A(\delta)^T X(\delta) + X(\delta) A(\delta) + C(\delta)^T R_p C(\delta) \preceq 0.
\]

Can hence just apply our general result on robust exponential stability.

**Performance:** Add to right-lower block \(\epsilon I\) for some small \(\epsilon > 0\) (compactness). Left- and right-multiply inequality with \(\text{col}(x(t), w(t))\) to infer
\[
\frac{d}{dt} x(t)^T X(\delta(t)) x(t) + \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T P_p \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} + \epsilon w(t)^T w(t) \leq 0.
\]

Integrate on \([0, T]\) and use \(x(0) = 0\) to obtain
\[
x(T)^T X(\delta(T)) x(T) + \int_0^T \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T P_p \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \leq -\epsilon \int_0^T w(t)^T w(t) dt.
\]

Take limit \(T \to \infty\) to arrive at required quadratic performance inequality.