Output stabilization of switching systems with linear discrete-time modes

Paolo Caravani, Elena De Santis
Department of Electrical Engineering
University of L’Aquila
Monteluco di Roio, 67040 L’Aquila, Italy.
caravani,desantis@ing.univaq.it

Abstract — The discrete component of the hybrid state of a discrete-time linear switching system is assumed uncontrolled and unobserved. Conditions of stabilizability for this class of systems are given in terms of a new definition of control invariance, based on the realization of a discrete observer that permits reconstruction of the discrete state in certain intervals of the time basis. The paper highlights the relationship between the minimum dwell time of the system and its stabilizability. An almost complete characterization of stabilizability is offered in terms of certain subsets of the continuous state space. These sets are amenable to tractable parametric procedures for controller synthesis.

I. INTRODUCTION

The reconstruction of the discrete state of an arbitrarily switching linear system from measurements of the continuous output component has been studied in several works, both in the continuous and discrete time domain ([7], [4], [1], [2]). However, the subsequent logical step - stabilization - is to our knowledge still an open question. In the assumption of full information on the continuous state and no information on the discrete state, we address in this paper the stabilizability (and the stabilization) problem. A classical and widely popular approach to this problem is to handle uncertainty with robust control techniques. Namely, conditions are sought to stabilize (in an appropriate sense) the entire class of continuous state models comprising the switching system. Recent developments in hybrid systems [5] have shown how to exploit information about the discrete state to obtain less conservative stability conditions. However, much of the work in this area is based either on full knowledge of the state or, full knowledge of the discrete state and partial knowledge of the continuous state [8]. We are aware of no works that do the opposite, as we do here. When the continuous state component is known, the uncontrolled and unobserved discrete state can in certain conditions be reconstructed, thus providing useful information for improvement of control performance - chiefly stabilization - over the worst-case or robust approach. In this paper we describe a discrete state observer (Sec. III) that, on the basis of a bounded collection of past observations, returns an estimation of the current discrete state, which converges to the actual value in a finite number of steps. In Sec. IV we state the equivalence between stabilizability from the output of the switching system and existence of an appropriate controlled invariant set. The notion of observer-based stabilizability is introduced and characterized. In particular robust stabilizability implies observer-based stabilizability. Such a characterization paves the way for a computationally feasible controller design, as shown in Sec. V.

NOTATION. $\mathbb{I}, \mathbb{R}$ are the set of natural, real numbers. $L, M, N$ the set of integers from 1 to $L, M, N$. If $\mathcal{A}$ is a set, $\text{card}(\mathcal{A}), \text{int}(\mathcal{A})$ denote cardinality and interior. Inclusion $\mathcal{A} \subset \mathcal{B}$ is non-proper, unless $\mathcal{A} \notin \mathcal{B}$ is specified. $\text{Im}(\mathcal{A}), \text{Ker}(\mathcal{A})$ denote range-space and null-space of a matrix $\mathcal{A}$. For a function $f : \mathbb{I} \to \mathbb{R}^n$, the symbol $f|_{[a,b]}$ denotes the vector of components $f(a), f(a + 1) \ldots f(b)$. Acronym DES stands for discrete event system, LMI for linear matrix inequality.

II. DEFINITIONS

The class of switching systems we consider in this paper is defined in the following:

Definition 1: A discrete-time linear switching system without reset is a tuple

$$S = (\Xi, \Theta, S, E), \quad (1)$$

where:

- $\Xi = \mathbb{N} \times \mathbb{R}^n$ is the hybrid state space, where:
  - $\mathbb{N}$ is the discrete state space and $i \in \mathbb{N}$ a discrete state;
  - $\mathbb{R}^n$ is the continuous state space and $x \in \mathbb{R}^n$ a continuous state;
- $\Theta = M \times \mathbb{R}^m$ is the hybrid input space, where:
  - $\mathbb{M}$ is the discrete disturbance space and $j \in \mathbb{M}$ a discrete disturbance;
  - $\mathbb{R}^m$ is the continuous control input space and $u \in \mathbb{R}^m$ a continuous control;
- $S$ is a map associating to each discrete state $i \in \mathbb{N}$ the linear dynamical control system $x(t + 1) = A_i x(t) + B_i u(t)$, where $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$ are constant matrices, $x(t) \in \mathbb{R}^n$ is the continuous state, $u(t) \in \mathbb{R}^m$ is the continuous control input;
- $E = \mathbb{N} \times \mathbb{M} \times \mathbb{N}$ is a collection of transitions and $e = (i,j,h) \in E$ a transition from discrete state $i$ to discrete state $h$ under disturbance $j$. 


We assume that the discrete disturbance is not available for measurement. A hybrid time basis $\tau$ is an infinite or finite sequence of sets of integers, called intervals, $I_k = \{t \in \mathbb{I} : t_k < t \leq t'_k\}$, with $t'_k > t_k$ and $t'_k = t_{k+1}$. Let $card(\tau) = L + 1$. If $L < \infty$, then $t'_L$ can be finite or infinite. Given a hybrid time basis $\tau$, the integers $t'_k$ are called switching times. Denote by $T$ the set of all hybrid time bases. A temporal evolution of $S$ is defined as follows.

**Definition 2:** (Linear switching system execution). An execution of a linear switching system without reset is the tuple $\chi = (\xi, \tau, u, \xi)$ where $\xi \in \Xi$ is the hybrid initial state, $\tau$ is the hybrid time basis, and $u : \mathbb{I} \rightarrow \mathbb{R}^m$ is the control input, and $\xi : \mathbb{I} \times \mathbb{L} \rightarrow \Xi$ the hybrid state evolution. The hybrid state evolution $\xi$ is defined as follows:

$$\xi(t_1, t) = \xi_1,$$

$$\xi(t, k) = (i(k), x(t, k)),$$

$$\xi(t_{k+1}, k + 1) = (i(k + 1), x(t'_k, k)),$$

where $i(k)$ denotes the discrete state during interval $I_k$, $e_k = (i(k), j(k), i(k + 1)) \in E$ and $x(t, k)$ is the (unique) solution at time $t \in I_k$ of the dynamic system $S(i(k))$, with initial time $t_k$, initial condition $x(t_k) = x(t, k)$ and control inputs $u(t_k) \ldots u(t - 1)$.

We assume that the continuous state is known at any $t$. More precisely, the information available on the system evolution is represented by a function $\eta : \mathbb{I} \rightarrow \mathbb{R}^n$, which for any $k = 1, ..., L$ is defined as:

$$\eta(t) = x(t, k), t \in [t_k, t'_k - 1].$$

Notice that the transition at time $t$ from one discrete state to another may not be detected observing $\eta(t)$ hence an observability notion for the discrete state is required.

The definition of observability we propose is based on the existence of at least one input–output experiment such that the discrete state is reconstructed, at least in some intervals of the time basis. We will have to keep track of past output observations through a sliding window of length $\Delta$.

In this section we adapt definitions and results of [5] to the present context.

**Definition 3:** Given an integer $\Delta \geq 1$, a linear switching system $S$ is $\Delta$-observable if there exist two functions $\tilde{u} : \mathbb{I} \rightarrow \mathbb{R}^m$ and $\tilde{f} : \mathbb{R}^{n(\Delta + 1)} \times \mathbb{R}^{m\Delta} \rightarrow 2^\mathbb{N}$ such that

$$\tilde{f}\left(\eta_{[t-\delta,t]}, \tilde{u}_{[t-\delta,t-1]}\right) = \{i(k)\}, \forall t \in [t_k + \Delta, t'_k],$$

in any interval $k$ of the time basis satisfying $t'_k - t_k \geq \Delta$. The system $S$ is called observable if there exists $\Delta$ for which it is $\Delta$-observable.

Notice that, although the definition is void on those executions whose time basis do not contain intervals of length $\geq \Delta$, it is non-void over the set $\mathcal{E}$ of all possible executions. In this sense $\Delta$-observability expresses (as it should) a system-specific property over the set $\mathcal{E}$.

Notice that, by choosing $\tilde{f} : \mathbb{R}^{n(\Delta + 1 + \delta)} \times \mathbb{R}^{m(\Delta + k)} \rightarrow 2^\mathbb{N}$ such that $\tilde{f} = \overline{f}$ for $k \geq 0$, $\Delta$-observability implies $(\Delta + k)$-observability, for $k \geq 0$.

For $\Delta = 1$, the condition (3) trivializes to $\tilde{f}(\eta_{[t-1,t]}, \tilde{u}_{[t-1,t-1]}) = \{i(k)\}, \forall t \in [t_k + 1, t'_k]$. That is, since we assumed $t'_k - t_k \geq 1$ for any $k \in \mathbb{L}$, in whatever execution there exists at least one time-point in each interval of the time basis at which the current discrete state is reconstructed. The same conclusion holds, if we assume a minimum dwell time $\delta \geq 1$ (i.e. any execution is such that $t'_k - t_k \geq \delta$, for any $k \in \mathbb{L}$ and $\Delta \leq \delta$.

In essence, $\Delta$-observability requires the existence of a function of past observations returning one singleton corresponding to the current discrete state, over intervals of length $\Delta$ of the time basis. While this is clearly necessary to reconstruct the current discrete state, it is unfortunately not sufficient due the possibility of stealth modes. Over some time interval of length $\Delta$, the dynamics of mode $j$ may be indistinguishable from the dynamics resulting from a switch, occurred in that interval, between modes $i$ and $h$, and $\tilde{f}$ may return stealth mode $\{j\}$ when $h$ is true. This motivates

**Definition 4:** Function $\tilde{f}$ is an observer for a switching system $S$ observable with $\{\tilde{u}, \tilde{f}\}$ if, denoting $t^*$ the instant at which a switching occurred at $t_k$ is detected, $i$ is the current discrete state of $S$ whenever $\tilde{f} = \{i\}$ and $t^* \leq t < t_{k+1}$.

### III. Observability

We now characterize observability of linear discrete-time switching systems with measured continuous state in terms of necessary and sufficient conditions. Under these conditions, the reconstruction of the current discrete state is possible with almost all functions $u : \mathbb{I} \rightarrow \mathbb{R}^m$ meaning that, for any real $\varepsilon > 0$ and for any $\pi$ such that the condition is not verified, there exists a “good” function $\tilde{u}$ such that $||\tilde{u}(t) - \pi(t)|| \leq \varepsilon$, $\forall t \in \mathbb{I}$.

Given $i, h \in \mathbb{N} \times \mathbb{N}$, define the following augmented linear system $S_{ih}$:

$$z(t + 1) = A_{ih}z(t) + B_{ih}u(t)$$

$$y_{ih}(t) = C_{ih}z(t)$$

$$A_{ih} = \begin{pmatrix} A_i & 0 \\ 0 & A_h \end{pmatrix}, B_{ih} = \begin{pmatrix} B_i \\ B_h \end{pmatrix}, C_{ih} = (I | -I).$$

Let $\mathcal{V}_{ih} \subset \mathbb{R}^{2n}$ be the maximal controlled invariant subspace for system $S_{ih}$ contained in $\ker(C_{ih})$, i.e. the maximal subspace $F \subset \mathbb{R}^{2n}$ satisfying the following sets inclusion:

$$A_{ih} F \subset F + \text{im}(B_{ih}),$$

$$F \subset \ker(C_{ih}).$$

and let the gain $K_{ih}$ be such that $(A_{ih} + B_{ih} K_{ih}) \mathcal{V}_{ih} \subset \mathcal{V}_{ih}$. Note that, if $\mathcal{V}_{ih}$ is a proper subspace of $\mathbb{R}^{2n}$, there exists a unique matrix $K_{ih}$ satisfying the above requirement and for a given $z(\tilde{t}) \in \mathcal{V}_{ih}$ the functions $u(t) = K_{ih}z(t) + \nu_{ih}(t), t \geq \tilde{t},$ with $\nu_{ih}(t) \in B_{ih}^{-1}(\mathcal{V}_{ih}), \forall t \geq \tilde{t},$ represent all the inputs that maintain the state evolution in $\mathcal{V}_{ih}$, starting from $z(\tilde{t})$. We next state two Lemmas without proofs (available on request).
Lemma 5: Given \((i, h) \in \mathbb{N} \times \mathbb{N}\), if there exists an integer \(k, k \in [0, 2n - 1]\), such that \(A^k_i B_i \neq A^k_h B_h\), then \(B_i^{-1}(V_{ih}) \neq \mathbb{R}^m\).

Lemma 6: Given \((i, h) \in \mathbb{N} \times \mathbb{N}\), if \(z(t) \notin V_{ih}\), then for any input function \(u\) there exists \(k \in \mathbb{N}, 0 \leq k \leq n\) such that \(z(t + k) \notin \text{Ker}(C_{ih})\).

Theorem 7: Let \(S\) be a linear switching system. Then

i) \(S\) is observable if and only if \(\forall (i, h) \in \mathbb{N} \times \mathbb{N}, \exists k \in \mathbb{I}, k \in [0, n]\) such that \(A^k_i B_i \neq A^k_h B_h\);

ii) If \(S\) is observable, it is \((n + 1)\)-observable for almost all input functions. In particular, it is \((n + 1)\)-observable with any \(\tilde{u}\) such that \(\tilde{u}(t) \notin \tilde{U}\) where

\[
\tilde{U} = \bigcup_{(i, h) \in \mathcal{J}(t)} K_{ih} \left( \begin{pmatrix} \eta(t) \\ \eta(t) \end{pmatrix} \right) + B^{-1}_{ih}(V_{ih})
\]

and the discrete state returned by \(\hat{f} \left( \eta_{[t-n-1,t]} \right) = \hat{u}_{[t-n-1,t-1]} \) is associated to the dynamic system whose solution \(x\), with initial time \(t - n - 1\), initial condition \(x(t - n - 1) = \eta(t - n - 1)\) satisfies \(x_{[t-n-1,t]} = \eta_{[t-n-1,t]} \), \(\forall t \in [t_k + \Delta, t_{k+1} - 1] \), \(\forall k \in \mathbb{L}, t'_{k+1} - t_k \geq \Delta + 1\).

Proof: i) Necessity: obvious. Sufficiency: observe first that if \(a_{ih}, a_{ihh}\) are the coefficients of the characteristic polynomials of \(A_i, A_{ih}\) we have

\[
\begin{align*}
(a_{i0} + a_{i1} A_i + \cdots + a_{i,n-1} A_i^{n-1})B_i &= -A_i^n B_i \\
(a_{h0} + a_{h1} A_i + \cdots + a_{h,n-1} A_i^{n-1})B_h &= -A_h^n B_h
\end{align*}
\]

whence

\[
\begin{align*}
-A_i^{n+1} B_i &= a_{i0} A_i B_i + \cdots + a_{i,n-1} A_i^{n} B_i \\
-A_h^{n+1} B_h &= a_{h0} A_i B_i + \cdots + a_{h,n-1} A_h^{n} B_h
\end{align*}
\]

thus, using \(A_i^{n+1} B_i = A_h^{n+1} B_h\) for \(k \in [0, n]\) in (6,7)

\[
Q = (a_{i0} - a_{h0})B_i + \cdots + (a_{i,n-1} A_i^{n-1} B_i = 0
\]

and in (8,9)

\[
\begin{align*}
A_h^{n+1} B_h - A_i^{n+1} B_i &= (a_{h0} - a_{i0})A_i B_i + \cdots \\
+a_{h,n-1} - a_{i,n-1} A_i^{n} B_i &= A_i Q = 0.
\end{align*}
\]

showing \(A_i^k B_i \neq A_h^k B_h\) for \(k \in [0, n]\) implies the same for \(k > n\), in particular, for \(k \in [0, 2n - 1]\).

If \(\forall (i, h) \in \mathbb{N} \times \mathbb{N}, \exists k \in \mathbb{I}, k \in [0, 2n - 1]\) such that \(A_i^k B_i \neq A_h^k B_h\), then \(B_i^{-1}(V_{ih}) \neq \mathbb{R}^m\) (Lemma 5).

Therefore, if \(z(t) = \begin{pmatrix} \eta(t) \\ \eta(t) \end{pmatrix} \in V_{ih}\), for some \((i, h) \in \mathbb{N} \times \mathbb{N}\), then any input \(u(t) \notin U\) is such that \(z(t + 1) \notin V_{ih}, \forall (i, h) \in \mathcal{J}(t)\), and by Lemma 6 \(z(t+n+1) \notin \text{Ker}(C_{ih})\).

ii) see sufficiency of i).

Contrary to observability of linear systems, observability of linear switching systems requires the input function to be appropriately selected (for example, two modes with zero initial state cannot be distinguished taking \(u(t) \equiv 0\)). Let \(U\) be the set of all input functions for which \(S\) is observable. The set \(U\) contains the class of functions taking values in the complement of \(U\) in \(\mathbb{R}^m\).

We now characterize an observer for \(S\), that is we seek a condition such that, whenever \(\hat{f}\) returns a discrete state in \(\mathbb{N}\), then that discrete state is the current discrete state of \(S\).

The next result shows that \(\Delta\)-observability implies the existence of an observer.

Theorem 8: Let \(S\) be \(\Delta\)-observable for all \(u \in U\), with minimum dwell time \(\delta \geq \Delta\). Then there exists a function \(\hat{f}\) which is an observer for almost all inputs \(u \in U\), as per def. 4.

Proof: Since \(\delta \geq \Delta\) and the initial time is a switching time, \(\Delta\)-observability ensures existence of a function \(\hat{f}\) returning the current discrete state in the first interval of the time basis. Proceeding by induction, suppose there exists a function \(\hat{f}\) such that the statement holds in interval \(I_{k-1}\) that is, the discrete state \(i \ (k-1) = i\) has been reconstructed. If a switching is detected at some later time \(t^* \in \left[t'_{k-1} + 1, t'_{k-1} + \Delta\right]\), the unknown switching time \(t_{k-1}\) must satisfy \(t'_{k-1} \in \left[t^* - \Delta, t^* - 1\right]\), where \(t^*\) is known. It is convenient to write \(t'_{k-1} = t^* - \Delta + \mu\), \(\mu \in [0, \Delta - 1]\). Consider the time interval \([t^* - \Delta, t^*]\) and suppose a switch from \(i\) to \(h\) occurred. Then the sequence of the past \(\Delta\) observations up to time \(t^*\) can be expressed as \(\eta|_{[t^* - \Delta, t^*]} = f_{ih,\mu} \eta(t^*) + G_{ih,\mu} u|_{[t^* - \Delta, t^* - 1]}\) with matrices \(G_{ih,\mu}, F_{ih,\mu}\) of appropriate structure (omitted for lack of space).

For an observed \(\eta(t^*)\), the behaviour of the system \(j\) (stealth mode) is identical to the behaviour resulting from a switching from \(i\) to \(h\), i.e. \((F_{ih,\mu} - F_{jj,\mu})\eta(t^*) + (G_{ih,\mu} - G_{jj,\mu}) u|_{[t^* - \Delta, t^* - 1]} = 0\) if and only if \(u|_{[t^* - \Delta, t^* - 1]}\) belongs to the set \(v + \text{Ker}(G_{ih,\mu} - G_{jj,\mu}) \subset \mathbb{R}^m\Delta\) where \(v\) is a vector depending on \(\eta(t^*)\). It can be proved that

\[
\forall i, j, h, j \neq i, h, G_{ih,\mu} - G_{jj,\mu} \neq 0, \forall \mu \in [0, \Delta - 1]
\]

hence \(\text{Ker}(G_{ih,\mu} - G_{jj,\mu})\) has dimension less than \(m\Delta\). Therefore, for almost all inputs \(u \in U\) it is possible to define a function \(\hat{f}\) which returns the value \(\{h\}\) whenever \(i \ (k) = h\).

We will denote \(U_o\) the class of functions for which an observer of \(S\) exists.

The evolution of the observer can be regarded as a DES with three states. Observer stays in state LEARN till the discrete state is reconstructed, at which time it goes to REACH where \(S\) cannot switch for \(\delta - \Delta\) steps. After this interval, a switch may occur and observer goes to WAIT, until a new switch of \(S\) is detected – and LEARN re-starts.

IV. STABILIZABILITY

Assumption 1: \(S\) is \(\Delta\)-observable and for any execution of \(S\), there is a minimum dwell time \(\delta, t'_{k} - t_{k} \geq \delta \geq \Delta, \)
∀k ∈ \mathbb{L}.

A control strategy \( \varphi \) is a function \( \varphi : \mathbb{R}^{n(\Delta+1)} \rightarrow \mathbb{R}^m \). A switching system \( S \) together with a control strategy \( \varphi \) is called controlled switching system and its executions with \( u(t) = \varphi(\eta|_{t-\Delta t}) \), \( t \geq 0 \), with \( \eta(t) = 0 \), if \( t < 0 \), are called controlled executions.

In this section stabilizability will be linked to an appropriate version of the concept of controlled invariance. Usually control invariance is the ability to confine the state into certain regions of the state space. In the present hybrid-state context, subsets of the hybrid space are of the form \( \bigcup_{i \in \mathbb{L}} i \times \Psi_i \), with \( \Psi_i \subset \mathbb{R}^n \). Since all transitions between discrete states are possible, uncontrolled and unknown, we cannot hope to keep the continuous state in any of the subsets \( \Psi_i \) - as would be possible if \( i \) were known. So the invariance notion we seek should be modified in such a fashion that, as long as no switchings occur and the discrete state is known, ordinary invariance is recovered. And, in intervals from the last switch of length greater than the dwell time, when a new switch can occur, we should ensure that once the new discrete state and an associated new subset become known, we are able to reach it and confine the continuous state in it, as we did in the old. The notion is clearly observer-based, as it cannot do away with the learning mechanism of the discrete state (in what subset \( \Psi_i \) the continuous component is to be confined). Therefore, observer-based invariance should prescribe behavior of the system only in intervals of the time basis longer than the observer-based invariance should prescribe behavior of the discrete state. More precisely, from an initial state \( \xi(\Delta, t) \) is reconstructed (with \( u \in U \)) and \( \eta^O(t') = \eta(1) \). The value remains unchanged until, at time \( t'' \), observations are no longer compatible with \( \xi(1) \), meaning a new switch is detected. At some later time \( t'' \in [t', t'' + \Delta] \) the new current discrete state is reconstructed. Therefore \( \eta^O(t') = \eta(2) \) and \( \eta^O(t) = \eta \), \( \forall t \in [t'', t'' - 1] \), and so on. Obviously \( \eta^O(t) = \hat{f}(\eta|_{t-\Delta t}, \hat{u}|_{t-\Delta t-1}) \), \( \forall t \in [t_k + \Delta, t'_k] \). It is clear that if a property of the switching system can be enforced with an observer-based control strategy \( \varphi^O \) it can also be enforced with a control strategy \( \varphi \), although the reverse implication is not necessarily true.

In what follows we shall analyze the cases of observer-based controlled invariance of \( \Psi \) and of observer-based stabilizability of \( S \), with obvious meaning of the terminology. Let us denote by \( R^{-a}_i(\Lambda) \) the set of all states in \( \mathbb{R}^n \) from which it is possible to reach a set \( \Lambda \subset \mathbb{R}^n \) in a steps of time in mode \( i \), i.e.

\[
\forall x \in R^{-a}_i(\Lambda), \exists u : I \rightarrow \mathbb{R}^m : A^n_i x + \left( A^{n-1}_i B_i \ldots A_i B_i B_i \right) u|_{[0,a-1]} \in \Lambda.
\]

Let \( r^{\Lambda} \) be a subset of \( \Lambda \subset \mathbb{R}^n \) whence is possible to remain in \( \Lambda \) for \( \Delta \) steps of time robustly, i.e. with a control law independent of the discrete state

\[
\forall x \in r^{\Lambda}, \exists u : I \rightarrow \mathbb{R}^m : A^n_h x + \left( A^{n-1}_h B_h \ldots A_h B_h B_h \right) u|_{[0,k-1]} \in \Lambda,
\]

Then we can state

\[
\exists k = 1, \ldots, \Delta, \forall h \in \mathbb{N}.
\]

\[\text{Proof:} \] Sufficiency is obvious. Necessity: suppose \( S \) stabilizable. Starting from any \( \xi_0 \in \rho B \) at time 0, the hybrid state \( \xi(\Delta, 0) = (i(1), x(\Delta, 1)) \) is reconstructed at time \( \Delta \leq \delta \), because \( S \) is \( \Delta \)-observable. Suppose \( i(1) = h \).

Then the controlled execution starting from \( \xi(\Delta, 1) \) has to evolve in a subset \( \Psi_h \) of \( \rho B \), controlled invariant for system \( S(h) \), because a switching from \( h \) to some other discrete state could never occur. If a switching from \( h \) to \( j \) occurs at time \( t'_1 \), then the stabilizing control law has to steer the continuous state to a subset \( \Psi_j \) of \( \varepsilon B_j \), controlled invariant for system \( S(j) \), in \( \Delta \) steps of time. Therefore

\[
\exists u : I \rightarrow \mathbb{R}^m : A^n_k x + \left( A^{n-1}_k B_k \ldots A_k B_k B_k \right) u|_{[0,k-1]} \in \rho B,
\]

\[\text{int}(\rho B), \Psi_{k} \text{ is bounded, hence the result follows.} \]

The property of \( \Delta \)-observability implies the existence of an observer which returns the actual discrete state of \( S \) at most \( \Delta \) steps after the last switch, and before a new switch occurs. Therefore we can define an observer-based strategy \( \varphi^O : (\mathbb{N} \cup \{ \epsilon \}) \times I \rightarrow \mathbb{R}^n \rightarrow \mathbb{R}^m \) whose first argument is the output \( \eta^O \) of the observer

\[
\eta^O : I \rightarrow \mathbb{N} \cup \{ \epsilon \}
\]

with symbol \( \epsilon \) denoting "unknown discrete state". The function \( \eta^O \) gives observation-compatible information about the discrete state. More precisely, from a initial \( \eta^O(0) \) assumed equal to \( \epsilon \), at some time \( 0 < t' \leq \Delta \) the discrete state \( i(1) \) is reconstructed (with \( u \in U \)) and \( \eta^O(t') = i(1) \). The value remains unchanged until, at time \( t'' \), observations are no longer compatible with \( i(1) \), meaning a new switch is detected. At some later time \( t'' \in [t', t'' + \Delta] \) the new current discrete state is reconstructed. Therefore \( \eta^O(t) = i(2) \) and \( \eta^O(t) = \epsilon \), \( \forall t \in [t'', t'' - 1] \), and so on. Obviously \( \eta^O(t) = \hat{f}(\eta|_{t-\Delta t}, \hat{u}|_{t-\Delta t-1}) \), \( \forall t \in [t_k + \Delta, t'_k] \).
Theorem 12: Suppose the set \( \Psi = \bigcup_{i \in \mathbb{N}} \{i\} \times \Psi_i \) is such that \( \forall i \in \mathbb{N} \):

a) \( \Psi_i \subset \Sigma \)

b) \( \forall x \in \Psi_i, \exists u \in \mathbb{R}^m : A_{ix}x + B_{iu} \in \text{int}(\Sigma), \forall h \in \mathbb{N} \)

where \( \Sigma = \text{rob}^\Delta \left( \bigcap_{h \in \mathbb{N}} \text{int}(\Omega_h) \right) \) and \( \Omega_i = R^{-(\delta - \Delta)}(\Psi_i) \).

Proof: Consider the time interval \( t_1 + \Delta \) and suppose the initial continuous state \( x_0 \) is in \( \Sigma \), with initial discrete state unknown. By definition of \( \Sigma \), there exists a control sequence \( u^{\tau}_{|0,\Delta-1} \) (r for robust) such that the continuous state remains in \( \bigcap_{h \in \mathbb{N}} \text{int}(\Omega_h) \), for any \( t = 0, 1, \ldots, \Delta \). Therefore there exists a sufficiently small \( \varepsilon > 0 \) such that the continuous state remains in \( \bigcap_{h \in \mathbb{N}} \text{int}(\Omega_h) \), for any \( t = 0, 1, \ldots, \Delta \), for any control sequence in the set \( u^{\tau}_{|0,\Delta-1} + \varepsilon E^\Delta \), and this sequence belongs to \( \mathcal{U}^\varepsilon \). This implies that it is possible to remain in \( \bigcap_{h \in \mathbb{N}} \text{int}(\Omega_h) \) with some control law \( u \in \mathcal{U} \).

Therefore at time \( t_1 + \Delta \) the state \( i(1) \) is reconstructed. As we are guaranteed that for \( \delta - \Delta \) time points the discrete state won’t change, it is possible to reach \( \Psi_{i(1)} \) at time \( t_1 + \delta \) with unrestricted control. Since condition b) holds, if \( t^*_2 \) is the first time-point (past \( t_1 + \delta \)) at which a new switching can be deduced from function \( \eta \), condition b) (first part) implies that there exists a control law \( u \in \mathcal{U} \) that keeps the continuous state in \( \Psi_{i(1)} \) from \( t_1 + \delta \) to \( t^*_2 - 1 \). At time \( t^*_2 \) the discrete state is again unknown but by condition b) (second part) the continuous state is guaranteed to be in \( \Sigma \), thus regenerating the information conditions extant at time \( 0 \).

The reverse implication comes by definition of observer-based strategy and controlled invariance.

Corollary 13: If there exists a set \( \Psi = \bigcup_{i \in \mathbb{N}} \{i\} \times \Psi_i \) bounded, observer-based controlled invariant with respect to a set \( \Sigma \) having the origin as an interior point then \( \mathcal{S} \) is stabilizable. Conversely, if \( \mathcal{S} \) is observer-based stabilizable then there exists a set \( \Psi = \bigcup_{i \in \mathbb{N}} \{i\} \times \Psi_i \) bounded, such that \( \forall i \in \mathbb{N} \):

a") \( \Psi_i \subset \Sigma \)

b") \( \forall x \in \Psi_i, \exists u \in \mathbb{R}^m : A_{ix}x + B_{iu} \in \text{int}(\Sigma), \forall h \in \mathbb{N} \)

Note that if \( \mathcal{S} \) is observer-based stabilizable, the above corollary states the existence of a set \( \Psi \) which is not observer-based controlled invariant (compare conditions b, Thm 12 and b”, Cor. 13) since in general we cannot guarantee this property with input functions in \( \mathcal{U} \). It is easy to verify that such a set is \( \Delta \)-controlled invariant.

Remark 14: If \( \delta = \Delta \), then conditions a,b in Thm 12 imply \( \Omega_i = \Psi_i \) and therefore the conditions

\( \Psi_i \subset \Sigma = \text{rob}^\Delta \left( \bigcap_{h \in \mathbb{N}} \Psi_h \right) \)

\( \forall x \in \Psi_i, \exists u \in \mathbb{R}^m : A_{ix}x + B_{iu} \in \text{int}(\Sigma), \forall h \in \mathbb{N} \)

imply that \( \Delta \) is a robust controlled invariant set, i.e. whose invariance is obtained with a control law independent of \( i \).

Moreover, in the framework of the above corollary, \( \Sigma \) is bounded with the origin in \( \text{int}(\Sigma) \). Therefore, if \( \delta = \Delta \) observer-based stabilizability implies robust stabilizability.

V. CONTROLLER SYNTHESIS

In this section we study a control synthesis based on invariant ellipsoids. For references see [3], [6].

A. recalls

The Shur complement condition is

\[
\begin{bmatrix}
Q & S \\
S^t & R
\end{bmatrix} > 0 \iff R - S^tQ^{-1}S > 0 \quad \text{and} \quad Q > 0.
\]

Let \( \mathcal{E}(X^{-1}) = \{x : x^tX^{-1}x \leq 1\} \) be an ellipsoid (when an inverse is indicated, the matrix is assumed positive definite). It is known that \( u = Kx \) makes \( \mathcal{E}(X^{-1}) = \{x : x^tX^{-1}x \leq 1\} \) an invariant ellipsoid for \( x^+ = (A + BK)x \) if and only if \( \mathcal{E}(X^{-1}) \subset \mathcal{E}((A + BK)^tX^{-1}(A + BK)) \). If the condition is strengthened to \( \mathcal{E}(X^{-1}) \subset \text{int}(\mathcal{E}((A + BK)^tX^{-1}(A + BK))) \) then there exists a positive definite matrix \( X \in \mathbb{R}^{n \times n} \) and a matrix \( Y \in \mathbb{R}^{m \times n} \) that satisfy the LMI

\[
\begin{bmatrix}
X & AX + BY \\
XA^t + Y^tB^t & X
\end{bmatrix} > 0,
\]

and \( K = YX^{-1} \). Also, if \( (A_h, B_h) \) are elements of a finite collection of matrices of same dimensions as \( (A, B) \) indexed by \( h \), then

\[
x \in \mathcal{E}(X^{-1}) \Rightarrow (A_h + B_hK)x \in \text{int}((A_h + B_h)^tZ^{-1})
\]

for all \( h \) of the collection if and only if

\[
\begin{bmatrix}
Z \\
XA^t_h + Y^tB^t_h
\end{bmatrix} > 0
\]

and \( K = YX^{-1} \) with \( X, Y \) satisfying (10). Condition (11) expresses a particular form of ellipsoidal inclusion that will be useful below, namely

\[
\mathcal{E}(X^{-1}) \subset \mathcal{E}((A_h + B_hYX^{-1})^tZ^{-1}(A_h + B_hYX^{-1})).
\]
B. control strategy

The minimum number of steps to reach \( \mathcal{E}(X^{-1}) \) from \( x_0 \) for system \( x^+ = Ax + Bu \), is \( \tau^* = \min \tau : x_0'X^{-1}x_0 \leq 1 \) where \( \mathcal{E}(X^{-1}) = R^{-\tau}(\mathcal{E}(X^{-1})) \) and the min time control law can be expressed as a set of (time-varying) feedback gains from the state \( x_0 \).

Now we parametrize the sets appearing in Thm 12 in the family of ellipsoids and assume linear state feedback. Let us use notation

\[
P_i^t = [B_i|A_iB_i|\ldots|A_i^{t-1}B_i|] \in \mathbb{R}^{n \times m}, \quad t \in [1, \alpha]
\]

\[
Q_i^t = [P_i^t|0| \ldots |0] \in \mathbb{R}^{n \times \Delta m}, \quad t \in [1, \Delta],
\]

with \( \alpha = \delta - \Delta \) and introduce gain matrices

\[
F = [F_1^t|\ldots|F_{\Delta}^t], \quad G_i = [G_i^t|\ldots|G_i^{t-1}^t], \quad K_i
\]

with \( i \in \mathbb{N} \). Our “decision variables” are \( F, G_i, K_i \).

\( F \): time-varying, open-loop gains to be used as soon as a switch is detected, to remain in \( \bigcap \Omega_h \) for \( \Delta \) steps, robustly. Strategy applies as the observer enters \( \text{LEARN} \) mode.

\( G_i \): time-varying, open-loop gains to drive \( x \) inside \( \Psi_i \) in min time (\( \leq \delta - \Delta \) steps) as soon as \( i \) becomes known. Strategy applies when the observer enters \( \text{REACH} \) mode.

\( K_i \): constant state-feedback gains \( (u_i = K_i x) \) to be used once \( i \) is known, \( x \in \Psi_i \) and no switch is detected. Strategy applies when the observer enters \( \text{WAIT} \) mode.

C. ellipsoidal parametrization

In Thm 12 we replace \( = \subset \) in the definition of \( \Sigma \). While slightly more stringent, this is necessary if the intersection of ellipsoids is to be replaced by an ellipsoid. Assume therefore the following parametrization of the sets appearing in Thm 12

\[
\Psi_i = \{ x : x'X_i^{-1}x \leq 1 \} \quad i \in \mathbb{N}
\]

\[
\Sigma = \{ x : x'Z_i^{-1}x \leq 1 \}.
\]

Express first conditions a) and b) of Thm 12.

\[
\forall i, h \in \mathbb{N}, \quad x'X_i^{-1}x \leq 1 \Rightarrow x'Z_i^{-1}x \leq 1 \quad (12)
\]

\[
x'(A_i + B_iK_i)'X_i^{-1}(A_i + B_iK_i)x < 1 \quad (13)
\]

\[
x'(A_i + B_iK_i)'Z_i^{-1}(A_i + B_iK_i)x < 1. \quad (14)
\]

Next, express the properties \( \Sigma \) should have. Let \( \mathcal{E}(T^{-1}) \) be an ellipsoid in \( \bigcap_{h \in \mathbb{N}} \Omega_h \).

Firstly, it must be possible to stay in the interior of \( \mathcal{E}(T^{-1}) \) robustly for at most \( \Delta \) steps, starting from any point of \( \Sigma \)

\[
x'Z^{-1}x \leq 1 \Rightarrow x'(A_i^t + Q_i^tF)'T^{-1}(A_i^t + Q_i^tF)x < 1, \quad (15)
\]

\( \forall i \in \mathbb{N}, \ t \in [1, \Delta] \).

Secondly, it should be possible to reach \( \Psi_i \) from any point of \( \mathcal{E}(T^{-1}) \) in at most \( \alpha = \delta - \Delta \) steps

\[
x'T^{-1}x \leq 1 \Rightarrow x'(A_i^\alpha + P_i^\alpha G_i)x < 1, \quad (16)
\]

\( \forall i \in \mathbb{N} \).

D. LMI formulation

We now express each of the above conditions (12-16) in LMI form (asterisks are symmetric terms). As each expresses an ellipsoidal inclusion, (10,11) will be used

\[
Z - X_i \geq 0, \quad \forall i \quad (17)
\]

\[
\begin{bmatrix}
X_i & A_iX_i + B_iY_i \\
* & X_i
\end{bmatrix} > 0, \quad \forall i \quad (18)
\]

\[
\begin{bmatrix}
Z & A_iX_i + B_iY_i \\
* & X_i
\end{bmatrix} > 0, \quad \forall i, h \quad (19)
\]

\[
\begin{bmatrix}
T & A_i^tZ + Q_i^tL \\
* & Z
\end{bmatrix} > 0, \quad \forall i, t \in [1, \Delta] \quad (20)
\]

\[
\begin{bmatrix}
X_i & A_i^tT + P_i^tH_i \\
* & T
\end{bmatrix} > 0, \quad \forall i \quad (21)
\]

This is a set of LMI’s in the variables \( X_i, Y_i, T, H_i, Z, L \). If feasible, a solution to the synthesis problem is

\[
F = LZ^{-1}; \quad \text{observer in state LEARN}
\]

\[
G_i = H_iT^{-1}; \quad \text{observer in state REACH}
\]

\[
K_i = Y_iX_i^{-1}; \quad \text{observer in state WAIT}.
\]

REFERENCES


