Observability and Detectability of Linear Switching Systems: A Structural Approach*

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Abstract

We define observability and detectability for linear switching systems as the possibility of reconstructing and respectively of asymptotically reconstructing the hybrid state of the system from the knowledge of the hybrid outputs for a suitable choice of the control input. We derive a necessary and sufficient condition for observability that can be verified computationally. A characterization of control inputs ensuring observability of the switching system is given. We prove that checking detectability of a linear switching system is equivalent to checking asymptotic stability of a suitable switching system with guards extracted from it, thus providing interesting links to Kalman decomposition and the theory of stability of hybrid systems.

keywords: linear switching systems, observability, location observability, detectability, Kalman decomposition, hybrid systems.

1 Introduction

Research in the area of hybrid systems addresses significant application domains with the aim of developing further understanding of the implications of the hybrid model on control algorithms and to evaluate whether using this formalism can be of substantial help in solving complex, real-life, control problems. In many application domains, hybrid controller synthesis problems are addressed by assuming full hybrid state information, although in many realistic situations state measurements are not available. Hence, to make hybrid controller synthesis relevant, the design of hybrid state observers is of fundamental importance. A step towards a procedure for the synthesis of these observers is the analysis of observability and detectability of hybrid systems.

Observability has been extensively studied both in the continuous ([24], [28]) and in the discrete domains (see e.g. [34], [35]). In particular, Sontag in [36] defined a number of observability concepts and analyzed their relations for polynomial systems. More recently, various researchers investigated observability of hybrid systems. The definitions of observability and the criteria to assess this property varied depending on the class of systems under consideration and on the knowledge that is assumed at the output. Vidal et al. [39] considered autonomous switching systems and proposed a definition of observability based on the concept of indistinguishability of continuous initial states and

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discrete state evolutions from the outputs in free evolution. In [13] and [3] observability of switching systems (with control) is investigated, while in [16] observability of internal variables arising from the interconnection of switching systems is addressed. Critical observability for safety critical switching systems was introduced in [12], where a set of "critical" states must be reconstructed immediately since they correspond to hazards that may yield catastrophic events. Incremental observability was introduced in [7] for the class of piecewise affine systems. Incremental observability implies that different initial states always give different outputs independently of the applied input. A characterization of observability and the definition of a hybrid observer for the class of autonomous piecewise affine systems can be found in [10]. In [19] observability of autonomous hybrid systems is analyzed by using abstraction techniques. In [21] the definitions of observability of [38] and the results of [5] on the design of an observer for deterministic hybrid systems were extended to discrete–time stochastic linear autonomous hybrid systems. In [6], the notion of generic final–state determinability proposed by Sontag [36] was extended to hybrid systems and sufficient conditions were given for linear hybrid systems.

While observability of hybrid systems was addressed in these papers, a general notion of detectability has not been introduced as yet. To the best of our knowledge, the only contribution dealing with detectability can be found in [31] where detectability was defined for the class of jump linear systems as equivalent to the existence of a set of linear gains ensuring the convergence to zero of the estimation error in a stochastic setting. In this paper, general notions of observability and detectability are introduced for the class of linear switching systems, though our definitions apply to more general classes of hybrid systems, since they involve only dynamical properties of the executions that are generated by the hybrid system. The observability notion proposed here for linear switching systems and those proposed for other classes of hybrid systems in [7], [38], [39] are shown not to be equivalent in [13]. Further, we derive a computable necessary and sufficient condition for assessing this property. As a by–product, we obtain an explicit characterization of the class of control inputs ensuring observability and show that this class contains "almost all" inputs. We then characterize detectability using a Kalman–like approach. In particular, we show that checking detectability of a linear switching system is equivalent to checking the asymptotic stability of a suitable linear switching system with guards associated with the original system. This result is clearly related to the classical detectability analysis of linear systems [9]. It is important because it allows the use of a wealth of existing results on the stability of switching and hybrid systems (see e.g. [8, 40, 27, 22, 33, 26, 1, 32, 20]).

The paper is organized as follows. In Section 2, we introduce linear switching systems and the notions of observability and detectability. Section 3 is devoted to finding conditions for the reconstruction of the discrete component of the hybrid state. In Section 4 we give a complete characterization of observability and some general results characterizing detectability of linear switching systems. Section 5 shows a Kalman–like decomposition of switching systems that reduces detectability for linear switching systems to asymptotic stability of linear switching systems with guards. Section 7 includes some technical proofs of results established in Section 3. Finally, Section 6 offers concluding remarks.

2 Preliminaries and basic definitions

Aim of this section is to introduce the preliminary definitions and the problem setting of this paper. Section 2.1 introduces the class of systems that we focus on and Section 2.2 formalizes the dynamical properties under study.

2.1 Switching systems

In this section we introduce the class of linear switching systems (LSw–systems) and the class of linear switching systems with guards (GLSw–systems), that generalize the class defined in [13], following the general model of hybrid automata (see e.g. [29, 37]).
The inputs of a GLSw–system are a discrete and unknown disturbance $\sigma$ and a continuous control input $u$. The hybrid state $\xi$ is composed of two components: the discrete state $q_i$ belonging to a finite set $Q$ and the continuous state $x$ belonging to the linear space $\mathbb{R}^n$, whose dimension $n_i$ depends on $q_i$. The hybrid output has a discrete and a continuous component as well, the former associated to the discrete states and to the transitions between discrete states, the latter associated to the continuous state. The evolution of the discrete state is governed by a Finite State Machine; a transition $e = (q_i, \sigma, q_h)$ may occur at time $t$ from the discrete state $q_i$ to the discrete state $q_h$, if the discrete disturbance $\sigma$ occurs at time $t$ and the continuous component $x$ of the hybrid state at time $t$ is in a given region $G(e) \subset \mathbb{R}^n$, called guard \cite{30}, depending on $e$. The evolution of the continuous state is described by a set of linear dynamical systems, controlled by the continuous input $u$, and whose matrices depend on the current discrete state $q_i$. Whenever a transition $e$ occurs, the continuous state $x$ is instantly reset to a new value $R(e)x$, where $R(e)$ is a matrix depending on the transition $e$. More formally,

Definition 1 A linear switching system with guards (GLSw–system) $S$ is a tuple

$$
(S, \Theta, \Upsilon, S, E, \gamma, G, R),
$$

where:

- $\Xi = \bigcup_{i \in Q} \{q_i\} \times \mathbb{R}^{n_i}$ is the hybrid state space, where:
  - $Q = \{q_i, i \in J\}$ is the discrete state space, $J = \{1, 2, \ldots , N\}$;
  - $\mathbb{R}^{n_i}$ is the continuous state space associated with the discrete state $q_i \in Q$;
- $\Theta = \Sigma \times \mathbb{R}^m$ is the hybrid input space, where:
  - $\Sigma = \{\sigma_i, i \in J_1\}$ is the discrete disturbance space, $J_1 = \{1, 2, \ldots , N_1\}$;
  - $\mathbb{R}^m$ is the continuous control input space;
- $\Upsilon = P \times \mathbb{R} \times \mathbb{R}^l$ is the hybrid output space, where:
  - $P = \{p_i, i \in J_2\} \cup \{\epsilon\}$ is the discrete output space, $J_2 = \{1, 2, \ldots , N_2\}$, with $\epsilon$ the null event;
  - $\mathbb{R}^l$ is the continuous output space;
- $S$ is a map associating to each discrete state $q_i \in Q$ the linear dynamical control system:

$$
S(q_i) : \begin{cases}
\dot{x}(t) = A_i x(t) + B_i u(t), \\
y(t) = C_i x(t),
\end{cases}
$$

where $x(t) \in \mathbb{R}^{n_i}$ is the continuous state, $u(t) \in \mathbb{R}^m$ is the continuous control input, $y(t) \in \mathbb{R}^l$ is the continuous output at time $t \geq 0$, $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m}$ and $C_i \in \mathbb{R}^{l \times n_i}$;

- $E \subset Q \times \Sigma \times Q$ is a collection of transitions; a transition $e = (q_i, \sigma, q_h) \in E$ with $q_i = q_h$ will be called in–loop transition;

- $\gamma = (\gamma_Q, \gamma_E)$ is the discrete output function where $\gamma_Q : Q \to P$ and $\gamma_E : E \to P$ associate an output symbol to each discrete state and respectively to each transition; a transition $e$ with $\gamma_E(e) = \epsilon$ will be called silent;

- $G$ is the guard map associating to every $e = (q_i, \sigma, q_h) \in E$ the linear subspace $G(e) \subset \mathbb{R}^n$;

- $R$ is the reset function associating to every $e = (q_i, \sigma, q_h) \in E$ the reset matrix $R(e) \in \mathbb{R}^{n_i \times n_i}$; $R(e) \neq I$, $\forall e = (q_i, \sigma, q_h) \in E$. 

3
Given a $GLSw$–system $S$, if $G(e) = \mathbb{R}^n$ for any $e = (q_i, \sigma, q_h) \in E$, then $S$ is called linear switching system ($LSw$–system) and for simplicity, the guard $G$ is omitted in the tuple (1), i.e. $S = (\Xi, \Theta, T, S, E, \gamma, R)$.

A $GLSw$–system $S$ is said to be autonomous if there is no continuous input acting on the plant, i.e. if $\Theta = \Sigma \times \{0\}$.

The analysis we carry out proves that detectability of a $LSw$–system $S$ is equivalent to asymptotic stability of an autonomous $GLSw$–system associated to $S$.

We now define the semantics of $GLSw$–systems. We assume that the discrete disturbance is not available for measurement, and that the class of admissible continuous inputs is the set $U$ of piecewise continuous functions $u : \mathbb{R} \to \mathbb{R}^m$. Denote by $0$ the identically zero control function. As defined in [29], a hybrid time basis $\tau$ is an infinite or finite sequence of sets $I_j = \{t \in \mathbb{R} : t_j \leq t \leq t'_j\}$, with $t'_j = t_{j+1}$; let be $\text{card}(\tau) = L + 1$. If $L < \infty$, then $t'_L$ can be finite or infinite. A hybrid time basis $\tau$ is said to be finite, if $L < \infty$ and $t'_L < \infty$ and infinite, otherwise. Given a hybrid time basis $\tau$, time instants $t'_j$ are called switching times. Denote by $T$ the set of all hybrid time bases. The switching systems temporal evolution is defined as follows.

**Definition 2** (GLSw–system execution) An execution $\chi$ of a $GLSw$–system $S$ is a collection:

$$((\xi_0, \tau, \sigma, u, \xi, \eta),)$$

with hybrid initial state $\xi_0 \in \Xi$, hybrid time basis $\tau \in T$, discrete disturbance $\sigma : \mathbb{N} \to \Sigma$, control input $u \in U$, hybrid state evolution $\xi : \mathbb{R} \times \mathbb{N} \to \Xi$ and hybrid output evolution $\eta : \mathbb{R} \times \mathbb{N} \to \Upsilon$. The hybrid state evolution $\xi$ is defined as follows:

$$\xi(t_0, 0) = \xi_0,$$

$$\xi(t, j) = (q(j), x(t, j)), \quad t \in I_j, j = 0, 1, ..., L,$$

$$x(t'_j, j) \in G(e_j), \quad j = 0, 1, ..., L - 1,$$

$$\xi(t_{j+1}, j + 1) = (q(j + 1), R(e_j)x(t'_j, j)), \quad j = 0, 1, ..., L - 1,$$

where $q : \mathbb{N} \to Q$ and for any $j = 0, 1, ..., L$, $e_j = (q(j), \sigma(j), q(j + 1)) \in E$ and $x(t, j)$ is the (unique) solution at time $t \in I_j$ of the dynamical system $S(q(j))$, with initial time $t_j$, initial condition $x(t_j, j)$ and control law $u$. The output evolution of $S$ is specified by the function $\eta$, which for any $j = 0, 1, ..., L$ is defined as:

$$\eta(t, j) = \begin{cases} \gamma q(q(j)), \gamma E \gamma_{j-1} \gamma (e_{j-1}), C_t x(t_j, j), & t = t_j, \\ \gamma q(q(j)), \gamma E \gamma (e_{j-1}), C_t x(t_j, j), & t \in (t_j, t'_j), \end{cases}$$

where $\eta(e_{-1}) = \epsilon$ and $C_t$ is the output matrix associated with the current discrete state $q(j) = q_j$.

In hybrid system control theory, some assumptions are often made for avoiding undesired phenomena on the executions, such as blocking or Zeno behaviours. A formal definition of blocking and Zeno hybrid systems can be found in [30]. Roughly speaking, a hybrid system is said to be blocking, if it generates at least one execution with finite hybrid time basis that cannot be "continued" to an execution with infinite hybrid time basis. A hybrid system is said to be Zeno, if it generates at least one execution with infinite transitions, within a finite time interval. The semantics of $GLSw$–systems, as formally specified by Definition 2, do not allow blocking behaviours. On the other hand, an execution of a $GLSw$–system may be Zeno.

We assume here that the system is non Zeno and that multiple simultaneous transitions are not allowed.

**Assumption 3** Given a $GLSw$–system, for any execution $\chi$ of $S$ the hybrid time basis $\tau$ is such that $t'_j - t_j > 0$ and for any $t > 0$, the cardinality of the set $\{t_j : t_j \leq t\}$ is finite.

Moreover since in this paper we study observability and detectability, the last one being an asymptotic property, we suppose that:
Assumption 4 Given a $GLSw$–system, any execution $\chi$ of $S$ is characterized by an infinite hybrid time basis.

We suppose that the set $T$ of hybrid time bases satisfies Assumptions 3 and 4. Given an execution $\chi$, the pair $(\tau, q)$ is called discrete state evolution associated with $\chi$. Given a $GLSw$–system $S$ and two executions of $S$ with the same discrete state evolution $(\tau, q)$,

$$\chi_1 = ((q_0, x_{01}), \tau, \sigma, \xi_1, \eta_1),$$
$$\chi_2 = ((q_0, x_{02}), \tau, \sigma, \xi_2, \eta_2),$$

where $\tau = \{I_j\}_{j=0,1,...,L}$, $\xi_1(t,j) = (q(j), x_1(t,j))$ and $\xi_2(t,j) = (q(j), x_2(t,j))$, $t \in I_j$, $j = 0, 1, ..., L$, and given any $\alpha, \beta \in \mathbb{R}$, we define the tuple:

$$\chi = \alpha \chi_1 \oplus \beta \chi_2 := ((q_0, \alpha x_{01} + \beta x_{02}), \tau, \sigma, \alpha u_1 + \beta u_2, \xi, \eta),$$

(3)

where:

$$\xi(t,j) = \xi_1(t,j) + \xi_2(t,j) := (q(j), \alpha x_1(t,j) + \beta x_2(t,j)), t \in I_j, j = 0, 1, ..., L,$$

and where $\eta$ can be derived from $\xi$, as done in (2). Since by definition of $G$, for any $j = 0, 1, ..., L-1$ and any $\alpha, \beta \in \mathbb{R}$

$$x_1(t_j',j), x_2(t_j',j) \in G(e_{j+1}) \Rightarrow \alpha x_1(t_j',j) + \beta x_2(t_j',j) \in G(e_{j+1}),$$

(4)

the tuple $\chi$ defined in (3) is an execution of $S$.

Given a $LSw$–system $S$ and any execution $\chi = ((q_0, x_0), \tau, \sigma, \xi, \eta)$ of $S$ define the free execution $\chi_f = ((q_0, x_0), \tau, \sigma, 0, \xi_1, \eta_1)$, where we recall $0$ is the identically zero control function, and the forced execution $\chi_f = ((q_0, 0), \tau, \sigma, u, \xi_f, \eta_f)$, that are characterized by the same discrete state evolution $(\tau, q)$ of $\chi$. It is readily seen that:

$$\chi = \chi_f \oplus \chi_f.$$  

(5)

In the following we refer to $\eta_f$ and $\chi_f$ as the free response and respectively the forced response associated with execution $\chi$.

Remark 5 It is worth to point out that decomposition (5) does not hold for a general $GLSw$–system $S$, since the converse of implication (4) does not hold in general. In fact tuples $\chi_f$ and $\chi_f$ are executions of a $LSw$–system, obtained by setting $G(e) = \mathbb{R}^n$ for any $e = (q_1, \sigma, q_0) \in E$ in the $GLSw$–system $S$, but they are not in general executions of $S$.

2.2 Observability and Detectability

In linear system theory, observability and detectability deal with the exact and (respectively) the asymptotic reconstruction of the state, on the basis of the knowledge of the continuous input and of the continuous output that is accessible from the environment. In this section, we generalize those notions to the class of $GLSw$–systems.

First of all we need to introduce, for $GLSw$–systems, the notion of observed (hybrid) output that represents the (hybrid) signals that are accessible from the environment for reconstructing the hybrid state. Given a $GLSw$–system $S$ and an execution $\chi = (\xi_0, \tau, \sigma, u, \xi, \eta)$ of $S$, define

$$\eta_0 : \mathbb{R} \rightarrow \mathcal{Y},$$

such that:

$$\eta_0(t) = \eta(t_j), t \in [l_j, l_j'), j = 0, 1, ..., L.$$

The restriction of $\eta_0$ to the interval $[l_j, l_j')$ is said to be the observed output at time $t$ of the execution $\chi$ and the symbol $\mathcal{Y}_0$ denotes the set of all observed outputs at some time $t$. According to the
definition of observed output, the transition from one discrete state to another may not be visible from the observed output.

For formally introducing the notions of observability and detectability, we need to equip the hybrid state space with a metric. Given a GLSw–system $S$, define:

$$
\delta : \Xi \times \Xi \to \mathbb{R}^+ \cup \{ \infty \},
$$

such that, for any $(q_i, x_i), (q_h, x_h) \in \Xi$,

$$
\delta((q_i, x_i), (q_h, x_h)) = \begin{cases} 
\infty, & \text{if } q_i \neq q_h, \\
\|x_i - x_h\|_{n_i}, & \text{if } q_i = q_h,
\end{cases}
$$

(6)

where $\|x\|_{n_i}$ is the Euclidean norm of $x$ in $\mathbb{R}^{n_i}$. The pair $(\Xi, \delta)$ is a metric space [25]. Metric (6) has been obtained by combining classical metrics for discrete spaces and for Euclidean spaces.

The definition of observability and detectability we propose is based on the existence of at least an input–output experiment such that, after some transitions, the hybrid state is reconstructed:

**Definition 6** A GLSw–system $S$ is said to be detectable if there exist a control input $\hat{u} \in U$ and a function $\xi : \mathcal{Y}_o \times \mathcal{U} \to \Xi$ such that:

$$
\forall \varepsilon > 0, \forall \rho > 0, \exists t_0 : 

\delta(\xi(y_o|_{[t_0, t]}), \hat{u}|_{[t_0, t]}, \xi(t, j)) \leq \varepsilon,
$$

(7)

$$
\forall t \geq i, t \neq j, j = 0, 1, ..., L
$$

for any execution $\chi$ with control input $\hat{u}$ and hybrid initial state $\xi_0 = (q_0, x_0)$, where $\|x_0\|_{n_i} \leq \rho$ and $q_0 = q_i$. If condition (7) holds with $\varepsilon = 0$, then $S$ is said to be observable.

Clearly, an observable GLSw–system is also detectable.

Definition 6 can be extended to general hybrid systems since it is based only on the notion of execution and coincides with the definition proposed in [13] for a special class of linear switching systems. A comparison between observability notion of Definition 6 and some other observability notions available in the literature (e.g. [7], [38], [39]) can be found in [13].

By specializing Definition 6 to linear systems, the classical observability and detectability notions [9] are obtained. In particular, the reconstruction of the current hybrid state is required at every time $t \geq t$ and $t \neq t_j$. Time instants $t_j$ are ruled out as it is for observable linear systems, a particular instance of linear switching systems, where the current state may be reconstructed only at every time strictly greater than the initial time. However, observability and detectability for linear systems are defined independently from the control function, while here we assume to choose a suitable control law. The two definitions coincide for linear systems but not for linear switching systems. In fact, if the observability (or detectability) property were required for any input function, then any linear switching system with no discrete outputs would never be observable (or detectable). To see this, suppose that the initial continuous state is the origin and the control law is identically zero. Then, the continuous output is identically zero and it is not possible to reconstruct the current discrete state on the basis of the continuous output. This problem could be overcome by requiring the observability (or detectability) property to hold for any control input but not for all initial conditions in the state space, hence defining a local notion of observability. We adopt a global notion with respect to the initial states and we show in Section 4 that if a switching system is observable in the sense of Definition 6, then it is observable for "almost all" input functions.

Definition 6 requires the reconstruction of the current discrete and of the current continuous state from a given instant of time on. We consider those two issues separately, by analyzing in Section 3 conditions for ensuring the reconstruction of the current discrete state (location observability) and in Sections 4 and 5 conditions for ensuring the exact reconstruction and the asymptotic reconstruction of the continuous state.
3 Location observability

This section is devoted to deriving conditions under which it is possible to reconstruct the current discrete state from the knowledge of the observed output for a suitable choice of the control input. We also analyze the role of the control input and provide a characterization of the class of input functions that ensure the reconstruction of the current discrete state.

By specializing Definition 6 to the reconstruction of the discrete component of the hybrid state only, the following definition is obtained.

**Definition 7** A GLSw–system \( S \) is location observable for an input function \( \hat{u} \in \mathcal{U} \), if there exists a function \( \tilde{q} : \mathcal{Y}_o \times \mathcal{U} \rightarrow Q \) such that:

\[
\forall \rho > 0, \exists \bar{t} > t_0 : \\
\tilde{q}(y_0|_{\{t_0, \bar{t}\}}, \hat{u}|_{\{t_0, \bar{t}\}}) = q(j),
\]

\[
\forall t \geq \bar{t}, t \neq t_j, j = 0, 1, ..., L,
\]

for any execution \( \chi \) with control input \( \hat{u} \) and hybrid initial state \( \xi_0 = (q_0, x_0) \), where \( \|x_0\|_M \leq \rho \) and \( q_0 = q \). The system \( S \) is called location observable if there exists an input function \( \hat{u} \in \mathcal{U} \) for which it is location observable.

From conditions (7) and (8), it follows that location observability is a necessary condition for a GLSw–system to be observable or detectable. Definition 7 requires reconstruction of discrete state after a finite number of transitions. This notion, also known as current state observability, has been considered in [5], where conditions are derived for a hybrid system to be current state observable (see also [34]). While we assume here a non–Zeno behaviour, in [5] executions are supposed to have infinite transitions (hybrid systems are supposed to be alive).

We now characterize location observability of GLSw–systems. The symbols \( \text{Im}(M) \) and \( \text{ker}(M) \) denote respectively the range and the null space of some matrix \( M \).

Given \( i, h \in J \), define the following augmented linear system \( S_{ih} \):

\[
\dot{z} = A_{ih} z + B_{ih} u, \quad y_{ih} = C_{ih} z,
\]

where:

\[
A_{ih} = \left( \begin{array}{cc} A_i & 0 \\ 0 & A_h \end{array} \right), \quad B_{ih} = \left( \begin{array}{c} B_i \\ B_h \end{array} \right), \quad C_{ih} = \left( \begin{array}{cc} C_i & -C_h \end{array} \right).
\]

Let \( \mathcal{V}_{ih} \subset \mathbb{R}^{n_i+n_h} \) be the maximal controlled invariant subspace [4] for system \( S_{ih} \) contained in \( \text{ker}(C_{ih}) \), i.e. the maximal subspace \( F \subset \mathbb{R}^{n_i+n_h} \) satisfying the following sets inclusion:

\[
A_{ih} F \subset F + \text{Im} B_{ih}, \quad F \subset \text{ker} C_{ih}.
\]

Set:

\[
J_p = \{ i \in J : \gamma(q(j)) = p \}, p \in P, \\
\bar{J} = \{ (i,h) : i, h \in J_p, i \neq h, p \in P \},
\]

and consider the set:

\[
\mathcal{U} = \left\{ u \in \mathcal{U} : u \neq \tilde{u}, \text{ a.e.}, \, \forall \tilde{u} \in \tilde{\mathcal{U}} \right\},
\]

where:

\[
\tilde{\mathcal{U}} = \bigcup_{(i,h) \in \bar{J}} \mathcal{U}_{ih}, \\
\mathcal{U}_{ih} = \left\{ u_{|_{[\tau, \infty)}} : u \in \mathcal{U}, \, \tau \in \mathbb{R}, \, u(t) = K_{ih} z(t) + v_{ih}(t), \, t \geq \tau \right\},
\]

the gain \( K_{ih} \) is such that \((A_{ih} + B_{ih} K_{ih}) \mathcal{V}_{ih} \subset \mathcal{V}_{ih}, v_{ih}(t) \in B_{ih}^{-1}(\mathcal{V}_{ih}), \forall t \geq \tau \) and \( z(t) \) is the state of system \( S_{ih} \) at time \( t \), starting from some \( z(\tau) \in \mathcal{V}_{ih} \), under control \( u_{|_{[\tau, \infty)}} \). Note that, if \( \mathcal{V}_{ih} \) is a...
proper subspace of $\mathbb{R}^{n_i+n_h}$, there exists a unique matrix $K_{ih}$ with the above requirement and for a
given $z(\bar{t}) \in \mathcal{V}_{ih}$ the functions in $\mathcal{U}_{ih}$ represent all the inputs which maintain the state evolution in
$\mathcal{V}_{ih}$, starting from $z(\bar{t})$. In fact the set $\mathcal{U}_{ih}$ is composed by all and nothing but the control inputs
that ensure zero dynamics [23] to system $S_{ih}$.

The following result shows that, under appropriate conditions on the system parameters, the set $\mathcal{U}^*$ is nonempty.

**Theorem 8** Given a GLSw—system, the set $\mathcal{U}^*$ is nonempty if

$$\forall (i,h) \in \tilde{J}, \exists k \in \mathbb{N}, k < n_i + n_h : C_i A_i^k B_i \neq C_h A_h^k B_h.$$ (12)

**Proof.** see Appendix. ■

We are now able to characterize location observability for a switching system.

**Theorem 9** The following statements are true:

i) A GLSw—system $S$ is location observable if and only if condition (12) holds.

ii) If a GLSw—system $S$ is location observable then it is location observable for any input function $u \in \mathcal{U}^*$.

**Proof.** i) (Necessity). Suppose by contradiction, that $\exists (i,h) \in \tilde{J}$ such that condition (12) is not satisfied and consider the executions:

$$\chi_1 = (((q_i,0), \tau, \sigma, u, \xi_1, \eta_1)), \quad \chi_2 = (((q_h,0), \tau, \sigma, u, \xi_2, \eta_2)),$$

where $\tau = \{I_0\}$ and $I_0 = [0, \infty)$. It is readily seen that $\eta_1 = \eta_2$ for any input and therefore the current discrete state cannot be reconstructed. (Sufficiency). If $\text{card}(J_p) \leq 1$, $\forall p \in P$, then the current discrete state can be reconstructed, on the basis of the discrete component of the observed output. Therefore assume $\exists p \in P$ such that $\text{card}(J_p) > 1$. Consider any execution with $L \geq 0$. By induction consider any $j \leq L$, let $q(j) = q_i$ and set $\gamma_Q(q_i) = p$. By denoting by $y_i(t,t_j, x_{i,t_j}, u(t,t_j))$ and $y_{ih}(t,t_j, x_{ih,t_j}, u(t,t_j))$, $h \in J_p$, respectively, the continuous output evolution at time $t$ of systems $S(q_i)$ and $S_{ih}$ with initial states $x_{i,t_j}$ and $x_{ih,j} = (x_{ih,t_j}, x_{h,t_j})$ at initial time $t_j$ and control law $u(t,t_j)$, one obtains:

$$y_{ih}(t,t_j, x_{ih,t_j}, u(t,t_j)) = y_i(t,t_j, x_{i,t_j}, u(t,t_j)) - y_h(t,t_j, x_{h,t_j}, u(t,t_j)), \forall t \geq t_j.$$

By Theorem 8, condition (12) implies that $\mathcal{U}^* \neq \emptyset$; choose any $u \in \mathcal{U}^*$. By definition of $\mathcal{U}_{ih}$ as in (11), $\exists \varepsilon > 0$, such that $y_{ih}(t,t_j, x_{ih,t_j}, u(t,t_j)) \neq 0$, for any $t \in (t_{j+1}, t_{j+1} + \varepsilon)$, for any $x_{ih,t_j}$, for any $h \in J_p$, $h \neq i$. This implies that the current discrete state can be reconstructed for any $t \in (t_{j+1}, t_{j+1} + \varepsilon)$, on the basis of the observed output. By induction the first statement follows. The second statement is a direct consequence of the proof above. ■

Note that if $u \in \mathcal{U}^*$, then by definition of $\mathcal{U}^*$, it is possible to reconstruct the discrete component of the hybrid state for any $t \neq t_j$, $j = 0,1,\ldots,L$. Hence, Theorem 9 implies that for GLSw—systems Definition 7 is equivalent to the following simpler definition:

**Definition 10** A GLSw—system $S$ is location observable for an input function $\tilde{u} \in \mathcal{U}$, if there exists a function $\tilde{q} : \mathcal{Y}_0 \times \mathcal{U} \to Q$ such that for any execution $\chi$ with control input $\tilde{u}$:

$$\tilde{q}(y_0(t_0,t), \tilde{u}(t_0,t)) = q(j),$$

$$\forall t \neq t_j, j = 0,1,\ldots,L.$$

The system $S$ is called location observable if there exists an input function $\tilde{u} \in \mathcal{U}$ for which it is location observable.
The next result shows that a stronger result can be obtained if we restrict the space of input functions $\mathcal{U}$ to the class $C^\infty(\mathbb{R}^m)$ of smooth control functions $u : \mathbb{R} \to \mathbb{R}^m$. In fact, if $\mathcal{U} = C^\infty(\mathbb{R}^m)$, then $\bar{\mathcal{U}} \subset C^\infty(\mathbb{R}^m)$, and $\bar{u} \notin \mathcal{U}^*$ if and only if $\bar{u} \in \bar{\mathcal{U}}$. Hence, by definition of $\bar{\mathcal{U}}$, the system $\mathcal{S}$ is not location observable for $\bar{u}$. Therefore:

**Corollary 11** Let $\mathcal{U} = C^\infty(\mathbb{R}^m)$. A GLSw-system $\mathcal{S}$ is location observable for $u \in \mathcal{U}$, if and only if $u \notin \mathcal{U}^*$.

Condition (12) of Theorem 9 allows the reconstruction of the switching times, whenever a switching occurs between two different discrete states, but not in general if the transition resets the discrete state into itself, as the following example shows.

**Example 12** Consider a LSw-system $\mathcal{S} = (\Xi, \Theta, \Upsilon, S, E, \gamma, R)$, where:

\[
\Xi = \{q\} \times \mathbb{R}^3, \quad \Theta = \{\sigma\} \times \mathbb{R}, \quad \Upsilon = \{\epsilon\} \times \{\epsilon\} \times \mathbb{R}^2, \\
E = \{e = (q, \sigma, q)\}, \quad \gamma = (\gamma_Q, \gamma_E), \quad \gamma_Q(q) = \gamma_E(e) = \epsilon.
\]

Let the dynamical system $S(q)$ associated to $q$ be described by the following dynamical matrices:

\[
A = \begin{pmatrix}
a_{11} & a_{12} & 0 \\
0 & a_{22} & 0 \\
0 & 0 & a_{33}
\end{pmatrix}, \quad B = \begin{pmatrix} 1 \\
0 \\
0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \end{pmatrix},
\]

with $(A, C)$ detectable, $a_{22} > 0$, $a_{12} \neq 0$, and reset matrix:

\[
R(e) = \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0.1 \end{pmatrix}.
\]

The system $\mathcal{S}$ is trivially location observable for any input function $u$. However, since for any $x \in \mathbb{R}^3$, $(R(e) - I)x$ belongs to the kernel of the observability matrix associated with $S(q)$, for any choice of the input function $u$ it is not possible to reconstruct the switching times.

The pathology illustrated in the example above plays an important role in the characterization of detectability, as discussed later on.

## 4 Characterizing Observability and Detectability

In this section, we derive conditions for assessing observability and detectability of LSw-systems. Since location observability is a necessary condition for a LSw-system to be observable or detectable, we assume that LSw-systems satisfy this property and we derive conditions for reconstructing the continuous state.

We first give a necessary and sufficient checkable condition for a LSw-system to be observable.

**Theorem 13** A location observable LSw-system $\mathcal{S}$ is observable if and only if linear system $S(q_i)$ is observable for any $q_i \in Q$.

**Proof.** Necessity comes from the fact that no switching in the discrete dynamics could occur. As for the sufficiency, location observability ensures the exact reconstruction of the current discrete state for any $t \neq t_j$, $j = 0, 1, ..., L$; once the current discrete state is known, observability of systems $S(q_i)$, $q_i \in Q$ ensures the reconstruction of the current continuous state for any $t \neq t_j$, $j = 0, 1, ..., L$. It is readily seen that Theorem 13 also holds for the class of GLSw-systems.
Remark 14 Theorem 13 shows that observability of a LSw-system $S$ is equivalent to the property of exactly reconstructing the initial hybrid state of $S$, as addressed for example in [3]. However, the concept of observability, based on the reconstruction of the initial hybrid state, and the one based on the reconstruction of the current hybrid state as required in Definition 6, do not coincide in the general case of hybrid systems, or in the case of LSw-systems with finite maximum dwell times\(^1\) (see e.g. [13]).

Theorem 13 completely characterizes observability of LSw-systems. In the following we suppose that

Assumption 15 $S$ is not observable

and we derive conditions that ensure asymptotic reconstruction of the continuous state of location observable LSw-systems. More precisely, we show how to characterize detectability of a location observable LSw-system $S$ by means of detectability of a suitable autonomous LSw-system associated with $S$. Given a LSw-system

$$S = (\Xi, \Theta, \Upsilon, S, E, \gamma, R),$$

define the autonomous LSw-system $S'$ with full discrete evolution information:

$$S' = (\Xi, \Theta, \Upsilon', S, E, \gamma', R),$$

where:

- $\Theta' = \Sigma \times \{0\}$, i.e. there is no (continuous) control input acting on $S'$;
- The output space $\Upsilon'$ and the output functions $\gamma' = (\gamma'_Q, \gamma'_E)$ are such that for any $q_i, q_h \in Q$, $\gamma'_Q (q_i) = \gamma'_Q (q_h)$ implies $q_i = q_h$, and for any $e = (q, \sigma, q) \in E$, $\gamma'_E (e) \neq \epsilon$.

By definition of $S'$, detectability of $S$ implies detectability of $S'$. Moreover the following result shows that under appropriate conditions the converse implication is true as well.

Given the LSw-system $S$ let $E^\ominus$ be the subset of silent in-loop transitions, i.e. $E^\ominus = \{ e = (q_i, \sigma, q_h) \in E : q_i = q_h \text{ and } \gamma_E (e) = \epsilon \}$. For any $q_i \in Q$, let $O_i$ be the observability matrix associated with the linear system $S(q_i)$.

Proposition 16 Given a location observable LSw-system $S$, if $S'$ is detectable and $S$ satisfies the following property:

$$E^\ominus = \emptyset \quad \text{or} \quad \text{Im} (R(e) - I) \cap \ker(O_i) = \{0\}, \forall e = (q_i, \sigma, q) \in E^\ominus, \quad (14)$$

then $S$ is detectable.

Proof. Condition (14) ensures, for any control law $u \in \mathcal{U}^*$, the reconstruction of the switching time $t_j$ for $S$, for any silent in-loop transition, whenever the hybrid states of $S$ at $t_j$ and $t_{j+1}$ are not equal. Therefore condition (14) allows the computation of the forced response of $S$: thus the free response of $S$ can be computed as well. Consider now any two executions $\chi$ and $\chi'$ of system $S$ and respectively system $S'$, sharing the same continuous component of the free response. Since $S$ is location observable, then, by assuming that for any silent transition $e \in E^\ominus$, the hybrid state at $t'_j$ is not equal to the hybrid state at $t_j + 1$, the information available up to time $t$ on system $S$ is the same as that available on system $S'$, for any $t \in (t_j, t'_j)$. Let us suppose that in the execution of $S$ there is a silent transition $e \in E^\ominus$ with the hybrid state at $t'_j$ equal to the hybrid state at $t_j + 1$. Since the information $\gamma'_E (e) \neq \epsilon$ available for system $S'$ has no role in reconstructing the continuous component of the hybrid state, then we can conclude that detectability of $S'$ implies detectability of $S$. $\blacksquare$

\(^1\)Given a LSw-system $S$, the maximum dwell time $\delta_M$ is a positive real (possibly infinite) number such that for any execution, the corresponding hybrid time basis $\tau$ is such that $t'_j - t_j < \delta_M$, for any $j = 0, 1, ..., L$.
Remark 17 We point out that condition (14) ensures that whenever a transition occurs from hybrid state \((q, x^-)\) to hybrid state \((q, x^+)\) with \(x^- \neq x^+\), the switching time related to that transition can be reconstructed. Hence condition (14) prevents phenomena like the one illustrated in Example 12, where condition (14) is not satisfied.

We now investigate detectability of the switching system \(S'\) as defined in (13).

An execution \(\chi = (\xi_0, \tau, \sigma, 0, \xi, \eta)\) of \(S'\) is said to be a hidden execution if the continuous component of the hybrid output \(\eta\) is identically zero, i.e. if:

\[
C_t x(t, j) = 0, \forall t \in I_j, \forall j = 0, 1, ..., L - 1,
\]

where for any \(j = 0, 1, ..., L - 1, C_t\) is the output matrix associated with the current discrete state \(q(j) = q_i\). Let be \(B = \bigcup_{q_i \in Q} \{q_i\} \times B_t\) and \(B_t = \{x \in \mathbb{R}^n : \|x\|_{\infty} \leq 1\}\). Given an execution \(\chi\) of \(S\), the hybrid state \(\xi\) of \(\chi\) is said to asymptotically converge to the origin if

\[
\forall \varepsilon > 0, \exists \delta > 0 : \forall \xi(t, j) \in \varepsilon B, \forall t \geq \delta, \forall j = j, j + 1, ..., L, j = \min\{j : \hat{t} \in I_j\}.
\]

The next result characterizes detectability of \(S'\) in terms of dynamical properties of hidden executions.

Theorem 18 The \(LS_w\)-system \(S'\) is detectable if and only if the hybrid state of any hidden execution of \(S'\) asymptotically converges to the origin.

Proof. (Necessity.) Assume by contradiction that there exists a hidden execution \(\chi = ((q_0, x_0), \tau, \sigma, 0, \xi, \eta)\) such that the continuous component of \(\xi\) does not asymptotically converge to the origin.

Consider an execution \(\chi_0 = ((q_0, 0), \tau, \sigma, 0, \xi', \eta')\) with the same discrete state evolution \((\tau, q)\) of \(\chi\). Executions \(\chi\) and \(\chi_0\) are characterized by the same observed output. Moreover \(\xi\) does not asymptotically converge to the origin while \(\xi'\) coincides with the origin. Thus \(S'\) is not detectable.

(Sufficiency.) Consider an execution \(\chi = (\xi_0, \tau, \sigma, 0, \xi, \eta)\). At any time \(t \in (t_j, t'_j]\) the execution of \(S\) up to \(t\), denoted \(\chi_{[0,t]}\), can be decomposed as \(\chi_{[0,t]} = \tilde{\chi}_t \oplus \chi^0_t\), where \(\tilde{\chi}_t\) is a finite execution of \(S'\), compatible with the observed output, and \(\chi^0_t\) is an unknown finite hidden execution. Let \(\tilde{\chi}_t\) be the hybrid state evolution of \(\tilde{\chi}_t\). Since there exists an infinite hidden execution \(\chi^0\) such that \(\chi^0_{[0,t]} = \tilde{\chi}_t\), then condition in the statement implies that:

\[
\forall \varepsilon > 0, \exists \delta > 0 : \delta(\tilde{\chi}_t(t, j), \xi(t, j)) = \delta(\tilde{\chi}_t(t, j), \tilde{\chi}_t(t, j) + \xi^0(t, j)) \leq \varepsilon, \forall t \geq \delta, t \neq t_j,
\]

where \(\xi^0\) is the hybrid state evolution of execution \(\chi^0\). Since there exists a function \(\tilde{\chi}\) such that \(\tilde{\chi}(y_0, [0,t]), 0 = \tilde{\chi}(t, j), t \geq 0, j = 0, 1, ..., \), then the result follows. □

Remark 19 The result above can be seen as a generalization to linear switching systems of well-known results in linear systems theory. In fact, a hidden execution of an autonomous linear system \(S\) is a state trajectory \(x(\cdot)\) with initial condition \(x(0) \in \ker(O)\), where \(O\) is the observability matrix of \(S\); therefore \(S\) is detectable if and only if any hidden execution of \(S\) asymptotically converges to the origin.

Theorem 18 completely characterizes detectability of \(LS_w\)-systems. The following section is devoted to finding conditions that are equivalent to the ones established in Theorem 18 but that may be checked more easily.

5 Checking detectability: a Kalman–like approach

This section is devoted to the characterization of detectability of an autonomous linear switching system \(S'\) with full discrete evolution information, as defined in (13). Once conditions for checking detectability of \(S'\) have been derived, Theorem 9 and Proposition 16 allow the detectability characterization of the associated \(LS_w\)-system \(S\).
Conditions that we develop in this section link to classical Kalman decomposition of linear systems, since they reduce the detectability of $S'$ to the asymptotic stability of a suitable GLSw-system associated with $S'$. Moreover, they also allow the use of a wealth of existing results on the stability of switching and hybrid systems (see e.g. [8, 40, 27, 22, 33, 26, 1, 32, 20]).

First of all we formally introduce a notion of asymptotic stability of autonomous GLSw-systems as follows. Recall $\mathcal{B} = \bigcup_{q_i \in Q} \{ q_i \} \times \mathcal{B}_i$ and $\mathcal{B}_i = \{ x \in \mathbb{R}^n_i : \|x\|_{n_i} \leq 1 \}$.

**Definition 20 (Asymptotic Stability)** An autonomous GLSw-system $S$ is asymptotically stable if:

$$\forall \varepsilon > 0, \forall \rho > 0, \exists \bar{t} > t_0 : \xi(t, j) \in \mathcal{B}, \forall t \geq \bar{t}, \forall j = j, j + 1, \ldots, L, \ j = \min\{ j : \check{t} \in I_j \},$$

for any execution $\chi$ with hybrid initial state $\xi_0 = (q_0, x_0)$, where $\|x_0\|_{n_i} \leq \rho$ and $q_0 = q_i$.

For analyzing detectability of $S'$ it is useful to first perform a discrete state space decomposition. Given $S' = (\Xi, \Theta', \Upsilon', S, E, \gamma', R)$ as in (13) and a set $\hat{Q} \subset Q$ let

$$S'|_{\hat{Q}} = (\Xi|_{\hat{Q}}, \Theta', \Upsilon', S|_{\hat{Q}}, E|_{\hat{Q}}, \gamma'|_{\hat{Q}}, R|_{\hat{Q}}),$$

be the switching sub-system of $S'$ obtained by restricting the discrete state space $Q$ of $S$ to $\hat{Q}$, i.e. such that:

$$\forall \xi |_{\hat{Q}} = \bigcup_{q_i \in Q \cap \hat{Q}} \{ q_i \} \times \mathbb{R}^{n_i}; \quad S|_{\hat{Q}}(q_i) = S(q_i), \forall q_i \in \hat{Q};$$

$$E|_{\hat{Q}} = \{ (q_i, \sigma, q_j) \in E : q_i, q_j \in \hat{Q} \}; \quad \gamma|_{\hat{Q}} = (\gamma|_{\hat{Q}}, \gamma'|_{\hat{Q}});$$

$$\forall \gamma|_{\hat{Q}}(q_i), \forall q_i \in \hat{Q}; \quad \gamma'|_{\hat{Q}}(e) = \gamma_E(e), \forall e \in E|_{\hat{Q}};$$

$$R|_{\hat{Q}}(e) = R(e), \forall e \in E|_{\hat{Q}}.$$

The following result can be obtained as a by-product of recent results [11] on discrete state space decompositions for hybrid systems. It shows that the system $S'$ is detectable if and only if a linear switching system, obtained by skipping all discrete states associated with observable linear systems of $S'$, is detectable.

**Proposition 21** The switching system $S'$ as in (13) is detectable if and only if the linear switching system $S'|_{\hat{Q}}$, where $\hat{Q} = \{ q_i \in Q : S(q_i) \text{ is not observable} \}$, is detectable.

In view of Assumption 15 and Theorem 13, $\hat{Q} \neq \emptyset$ and hence $S'|_{\hat{Q}}$ is well defined. By Proposition 21, we can assume that for any discrete state $q_i \in Q$ of $S'$, system $S(q_i)$ is not observable. Moreover, we assume that the dynamical systems $S(q_i), q_i \in Q$ are in observability canonical form [9], i.e. that dynamical matrices associated with $S(q_i)$ are of the form:

$$A_i = \begin{pmatrix} A_{i}^{(11)} & 0 \\ A_{i}^{(21)} & A_{i}^{(22)} \end{pmatrix}, \quad C_i = \begin{pmatrix} C_{i}^{(11)} & 0 \end{pmatrix},$$

where $A_{i}^{(22)} \in \mathbb{R}^{d_i \times d_i}, 0 < d_i \leq n_i$, matrices $A_{i}^{(11)}, A_{i}^{(21)}$ are of appropriate dimensions and $(A_{i}^{(11)}, C_{i}^{(1)})$ is an observable matrix pair, $\forall i \in J$. This assumption is made without loss of generality since if dynamical systems $S(q_i), q_i \in Q$ are not in this form there always exists a suitable hybrid state space transformation that takes the linear switching system in the desired form. More precisely, define $T : \Xi \to \Xi$ such that for any $(q_i, x) \in \Xi$, $T(q_i, x) := (q_i, T_i x)$ and $T_i$ is an invertible linear transformation $T_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ such that $T_i A_i T_i^{-1}$ and $C_i T_i^{-1}$ are in the desired observability canonical form.
The continuous component \( x \) of the hybrid state \((q, x)\) can be then partitioned as \( x = (x_1', x_2')'\), where \( x_1 \in \mathbb{R}^{n_i-d_i}, x_2 \in \mathbb{R}^{d_i}\), and for any \( e = (e, q_i, q_h) \in E \), the reset matrix \( R(e) \) can be partitioned as:

\[
R(e) = \begin{pmatrix}
R_{e}^{(11)} & R_{e}^{(12)} \\
R_{e}^{(21)} & R_{e}^{(22)}
\end{pmatrix},
\]

where \( R_{e}^{(22)} \in \mathbb{R}^{d_h \times d_i} \) and matrices \( R_{e}^{(11)}, R_{e}^{(12)}, R_{e}^{(21)} \) are of appropriate dimensions.

We can now associate a suitable GLSw—system to switching system \( S_0 \). Given \( S_0 \) as in (13), define the following autonomous GLSw—system:

\[
S_0 = (\Xi_0, \Theta', \Upsilon_0, S_0, E, \gamma', G_0, R_0),
\]

where:

- \( \Xi_0 = \bigcup_{i \in J} \{q_i \} \times \mathbb{R}^{d_i} \);
- \( S_0(q_i) \) is described by dynamics \( \dot{z}(t) = A_i^{(22)} z(t), \) for any \( q_i \in Q \);
- \( G_0(e) = \ker(R_{e}^{(12)}) \), for any \( e \in E \);
- \( R_0(e) = R_{e}^{(22)} \), for any \( e \in E \).

The following theorem reduces detectability of \( S' \) to asymptotic stability of \( S_0 \).

**Theorem 22** The switching system \( S' \) is detectable if and only if the autonomous GLSw—system \( S_0 \) is asymptotically stable.

**Proof.** Given any hidden execution \( \chi \) of \( S \), the continuous component \( x(t,j) \) of the hybrid state \( \xi(t,j) = (q(j), x(t,j)) \), \( q(j) = q_i \), belongs to the subspace \( \ker(\mathcal{O}_i) \) for any \( j = 0, 1, ..., L \) and therefore, by Theorem 18 and by definition of observability canonical form matrices, the result follows. 

**Remark 23** Consider the switching system \( S \) of Example 12 and the corresponding switching system \( S' \). It is readily seen that the autonomous GLSw—system \( S_0 \) extracted from \( S' \) is asymptotically stable. Hence, by Theorem 22, \( S' \) is detectable. However condition (14) is not satisfied and therefore detectability of \( S' \) cannot be assessed.

The following theorem summarizes results characterizing detectability and highlights relationships between detectability of \( S \) and asymptotic stability of \( S_0 \).

**Theorem 24** A linear switching system \( S \) is detectable if the following conditions are satisfied:

i) \( S \) is location observable,

ii) \( S \) satisfies condition (14), and

iii) \( S_0 \) is asymptotically stable.

Conversely, if \( S \) is detectable then conditions i) and iii) are satisfied.

How complex is to check the conditions given in Theorem 24 depends strongly on the size of the switching system. A procedure for reducing the size of the hybrid state space of a switching system, while preserving the detectability property, can be found in [15].

While conditions i) and ii) can be checked by using the results of the previous sections, condition iii) requires the analysis of the asymptotic stability of an autonomous GLSw—system: this task is
in general not easy (see e.g. [40] for a review on this issue) and this is mainly due to the presence of guard conditions. In the last part of this section, we derive sufficient conditions for the asymptotic stability of $S_0$, by "abstracting" $S_0$ with a linear switching system (with no guards). For linear switching systems, several results have been developed in the literature for characterizing asymptotic stability, see e.g. [8, 33, 26, 32, 20].

We recall that:

**Definition 25** [2] Given a pair of autonomous GLSw–systems $S_1$ and $S_2$, $S_1$ is said to be an abstraction of $S_2$ if the set $E(S_1)$ of all the executions of $S_1$ contains the set $E(S_2)$ of all executions of $S_2$, i.e. if $E(S_2) \subset E(S_1)$.

This property is very important because:

**Proposition 26** An autonomous GLSw–system $S_1$ is asymptotically stable if there exists an autonomous GLSw–system $S_2$, that is an abstraction of $S_1$ and that is asymptotically stable.

We propose two different abstractions of the autonomous GLSw–system $S_0$:

- a LS–system $S_1$, whose tuple formally coincides with that one of $S_0$, except for the guards that are assumed to be the whole continuous state space for any transition;
- a LS–system $S_2$, whose tuple formally coincides with that one of $S_1$, except for the reset matrix that takes into account guard condition of $S_0$.

Formally, given the autonomous GLSw–system $S_0$ as in (15):

$$S_0 = (\Xi_0, \Theta_0, \Upsilon_0, S_0, E, \gamma_0, G_0, R_0),$$

define the following autonomous LS–systems:

$$S_1 = (\Xi_0, \Theta_0, \Upsilon_0, S_0, E, \gamma_0, R_0),$$

$$S_2 = (\Xi_0, \Theta_0, \Upsilon_0, S_0, E, \gamma_0, R_2),$$

where $R_2(e) = R_2^{(22)} \Pi_{\ker(R_2^{12})}$, being $\Pi_F$ the projector on the subspace $F$, i.e. $\Pi_F x$ is the Euclidean orthogonal projection of $x$ on $F$.

By construction, $S_1$ and $S_2$ are abstractions of $S_0$ and therefore by Proposition 26:

**Theorem 27** The autonomous GLSw–system $S_0$ is asymptotically stable if one of the following statements are true:

- $S_1$ is asymptotically stable;
- $S_2$ is asymptotically stable.

The following examples show an application of the result above and point out that asymptotic stability of $S_1$ does not imply asymptotic stability of $S_2$ and vice versa.

**Example 28** Consider an autonomous LS–switching system:

$$S' = (\Xi, \Theta', \Upsilon', S, E, \gamma', R)$$

with full discrete evolution information, where:

$$\Xi = \{q\} \times \mathbb{R}^2, \quad \Theta' = \{\sigma\} \times \{0\}, \quad \Upsilon' = \{p\} \times \{p\} \times \mathbb{R},$$

$$E = \{e = (q, \sigma, q)\}, \quad \gamma' = (\gamma_Q', \gamma_E'), \quad \gamma_Q'(q) = \gamma_E'(e) = p \neq e,$$
and \( S(q) \) and \( R(e) \) are defined by the following dynamical matrices:

\[
A = \begin{pmatrix} a_{11} & 0 \\ a_{21} & -1 \end{pmatrix}, \quad C = \begin{pmatrix} c & 0 \end{pmatrix}, \quad R(e) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & 10 \end{pmatrix},
\]

where \( m_{12} \neq 0 \). The switching system \( S_1 \) is characterized by dynamical matrix \( A_1 = -1 \) and reset matrix \( R_{e_1}^{(22)} = 10 \) and hence it is not asymptotically stable. On the other hand, the switching system \( S_2 \) is characterized by dynamical matrix \( A_2 = -1 \) and reset matrix \( R_{e_2}(e) = R_{e_2}^{(22)} \Pi_{\ker(R_{e_2}^{(12)})} = 0 \) and therefore it is asymptotically stable. By Theorem 27, the autonomous switching system \( S' \) as in (16), is detectable.

**Example 29** Consider an autonomous \( LSw-\)switching system:

\[
S' = (\Xi, \Theta', \Upsilon', S, E, \gamma', R)
\]

with full discrete evolution information, where:

\[
\Xi = \{q\} \times \mathbb{R}^3, \quad \Theta' = \{\sigma\} \times \{0\}, \quad \Upsilon' = \{p\} \times \mathbb{R}, \quad E = \{e \mid (q, \sigma, q), \gamma' = (\gamma'_q, \gamma'_E), \gamma'_q(q) = \gamma'_E(e) \neq e, \}
\]

and \( S(q) \) and \( R(e) \) are defined by the following dynamical matrices:

\[
A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & -1/10 & 1 \\ a_{31} & -10 & -1/10 \end{pmatrix}, \quad C = \begin{pmatrix} c & 0 & 0 \end{pmatrix}, \quad R(e) = \begin{pmatrix} m_{11} & 1 & -1 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{pmatrix},
\]

where \( a_{11} \neq 0 \) and \( c \neq 0 \). The \( LSw-\)system \( S_1 \), characterized by the following dynamical matrix and reset matrix:

\[
A_{1e}^{(22)} = \begin{pmatrix} -1/10 & 1 \\ -10 & -1/10 \end{pmatrix}, \quad R_{e_2}^{(22)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

is asymptotically stable, since \( A_{1}^{(22)} \) is Hurwitz and the reset matrix is the identity. Thus by Theorem 27, the autonomous switching system \( S' \) as in (17), is detectable. The \( LSw-\)system \( S_2 \) is characterized by dynamical matrix \( A_{2e}^{(22)} \) as in (18) and by reset matrix:

\[
R_{e_2}(e) = R_{e_2}^{(22)} \Pi_{\ker(R_{e_2}^{(12)})} = 0.5 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, e \in E.
\]

Consider an execution of \( S_2 \) with \( \text{card}(\tau) = \infty \) and \( t'_j - t_j = 0.5 \), for any \( j \geq 0 \). Then by setting,

\[
F = R_{e_2}^{(22)} \Pi_{\ker(R_{e_2}^{(12)})} \exp(A_{1e}^{(22)}(t'_j - t_j)) = \begin{pmatrix} -1.5089 & 0.1455 \\ -1.5089 & 0.1455 \end{pmatrix},
\]

the value of the continuous state after the \( k \)-th transition is given by \( x(t_k, k) = F^k x(t_0, 0) \). Since the eigenvalues of \( F \) are \( \lambda_1 = -1.3634, \lambda_2 = 0 \) the switching system \( S_2 \) is unstable.

Finally, we can generalize Theorem 27 as follows. Given a function:

\[
\theta : E \rightarrow \{0, 1\},
\]

consider a linear switching system:

\[
S_\theta = (\Xi_0, \Theta', \Upsilon', S_0, E, \gamma', R_\theta),
\]

where for any \( e = (q_0, \sigma, q_h) \in E, \)

\[
R_\theta(e) = R_{e_2}^{(22)} \left( \Pi_{\ker(R_{e_2}^{(12)})} + \theta(e) \left( I - \Pi_{\ker(R_{e_2}^{(12)})} \right) \right).
\]
Note that:

\[ S_\theta = \begin{cases} S_1, & \text{if } \theta(e) = 1, \forall e \in E \\ S_2, & \text{if } \theta(e) = 0, \forall e \in E. \end{cases} \]

Moreover, it is easily seen that for any choice of function \( \theta \), \( S_\theta \) is an abstraction of \( S_0 \) and therefore by Proposition 26:

**Proposition 30** The autonomous GLSw–switching system \( S_\theta \) is asymptotically stable if there exists a function \( \theta \) as defined in (19), such that the autonomous LSw–system \( S_\theta \) is asymptotically stable.

6 Conclusions

We addressed observability and detectability for linear switching systems. We proposed a definition of observability, and a weaker notion of detectability, related to the possibility of reconstructing the system state. To the best of our knowledge, detectability has not been addressed as yet in the literature on hybrid systems. The study of detectability is a fundamental step towards the design of a hybrid observer. We obtained a computable necessary and sufficient condition for a switching system to be observable. Further, we derived a Kalman–like decomposition of the switching system that reduces the detectability of linear switching systems to the asymptotic stability of suitable linear switching systems with guards associated with the original systems.

Future work will focus on the actual implications of detectability on the construction of a hybrid observer.

References


7 Appendix

Proof of Theorem 8.
We first need two preliminary technical lemmas.

Lemma 31 Given \((i, h) \in \hat{J}\), if there exists \(k \in \mathbb{N}\), \(k < n_i + n_h\), such that \(C_i A_i^k B_i \neq C_h A_h^k B_h\), then \(B_{ih}^{-1} (V_{ih}) \neq \mathbb{R}^m\).

Proof. By contradiction, suppose that \(B_{ih}^{-1} (V_{ih}) = \mathbb{R}^m\). Then \(\text{Im} (B_{ih}) \subset V_{ih}\). Since \(V_{ih}\) is controlled invariant, by (10), \(A_{ih} V_{ih} \subset V_{ih} + \text{Im} (B_{ih})\), and therefore the above condition implies that \(A_{ih} V_{ih} \subset V_{ih}\), i.e., \(V_{ih}\) is \(A_{ih}\)-invariant and contains \(\text{Im} (B_{ih})\). Recall that the minimal \(A_{ih}\)-invariant subspace containing \(\text{Im} (B_{ih})\) is \(\text{Im} \left( B_{ih} A_{ih} B_{ih} \ldots A_{ih}^{n-1} B_{ih} \right)\), \(n = n_i + n_h\). Therefore
\[
\text{Im} \left( B_{ih} A_{ih} B_{ih} \ldots A_{ih}^{n-1} B_{ih} \right) \subset V_{ih} \subset \ker (C_{ih})
\]

that implies
\[
C_{ih} \left( B_{ih} A_{ih} B_{ih} \ldots A_{ih}^{n-1} B_{ih} \right) = 0.
\]

and therefore a contradiction holds. \(\blacksquare\)

Lemma 32 Let \(\{M_i \in \mathbb{R}^{m \times n_i}, i \in J\}\) be a family of nonzero matrices. There exists \(z \in \mathbb{R}^n\) and \(\lambda \in \mathbb{R}\) such that \(M_i z \neq 0\), \(\forall i \in J\), where

\[
\begin{pmatrix}
    z \\
    \lambda z \\
    \lambda^2 z \\
    \vdots \\
    \lambda^{T-1} z
\end{pmatrix}
\]

(20)

Proof. Set
\[
M_i = \begin{pmatrix} M_{i0} & M_{i1} & M_{i2} & \ldots & M_{iT-1} \end{pmatrix}, \quad M_i(\lambda) = M_{i0} + \lambda M_{i1} + \lambda^2 M_{i2} + \ldots + \lambda^{T-1} M_{iT-1}.
\]

Then for any \(z \in \mathbb{R}^n\), \(M_i z = M_i(\lambda) z\). Given \(i \in J\), since \(M_i \neq 0\), there are a finite number of values \(\lambda\) such that \(M_i(\lambda) = 0\). Let \(\vec{X}\) be such that \(M_i(\vec{X}) \neq 0\), \(\forall i \in J\). Then there exists \(z \notin \bigcup_{i \in J} \ker (M_i(\vec{X}))\) and the result follows. \(\blacksquare\)

Using the results above, we now give the proof of Theorem 8.

Proof. By contradiction, suppose \(U^*\) empty and let \(n = n_i + n_h\). Then
\[
\forall u \in U, \exists t', t'' \in \mathbb{R}, t' < t'', (i, h) \in \hat{J} \text{ and } \tilde{u} \in U_{ih} \text{ s.t. } u(t) = \tilde{u}(t), \forall t \in [t', t''].
\]

(21)

Let \(V_{ih}\) be the set of smooth functions \(v : \mathbb{R} \rightarrow B_{ih}^{-1} (V_{ih})\). Let \(\hat{U} \subset U\) be the set of smooth, not identically zero functions. By definition of \(U_{ih}\), condition (21) implies

\[
\forall u \in \hat{U}, \exists t', t'' \in \mathbb{R}, t' < t'', (i, h) \in \hat{J}, \vec{v} \in V_{ih} \text{ and } v_{ih} \in V_{ih} \text{ s.t. } u(t) = K_{ih} z(t) + v_{ih}(t), \forall t \in [t', t''],
\]

(22)

where \(\hat{z}(t) = \hat{A}_{ih} z(t) + B_{ih} v_{ih}(t), \hat{A}_{ih} = A_{ih} + B_{ih} K_{ih}\) and \(z(t') = \vec{v} \in V_{ih}\). Condition (22) implies:

\[
\forall u \in \hat{U}, \exists t' \in \mathbb{R}, \exists (i, h) \in \hat{J} \text{ s.t. } \forall N \geq 0
\]

(23)

\[
\begin{pmatrix}
    u(t') \\
    \dot{u}(t') \\
    \ddot{u}(t') \\
    \vdots \\
    u^{(N)}(t')
\end{pmatrix} \in M_{ih}^N V_{ih} + F_{ih}^N (F_{ih} \times F_{ih} \times \ldots \times F_{ih}),
\]
where:

\[
F_{ih} = B_{ih}^{-1}(V_{ih}), \quad M_{ih}^{N} = \begin{pmatrix}
K_{ih} \\
K_{ih}A_{ih} \\
\vdots \\
K_{ih}A_{ih}^{N}
\end{pmatrix} \in \mathbb{R}^{m(N+1) \times n},
\]

\[
F_{ih}^{N} = \begin{pmatrix}
I & 0 & 0 & \cdots & 0 \\
K_{ih}B_{ih} & I & 0 & \cdots & 0 \\
K_{ih}A_{ih}B_{ih} & K_{ih}B_{ih} & I & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
K_{ih}A_{ih}^{N-1}B_{ih} & K_{ih}A_{ih}^{N-2}B_{ih} & K_{ih}A_{ih}^{N-3}B_{ih} & \cdots & I
\end{pmatrix} \in \mathbb{R}^{mN \times mN}.
\]

The matrix \( F_{ih}^{N} \) is nonsingular. Denote by \( \dim(H) \) the dimension of some subspace \( H \). By setting \( \dim(F_{ih}) = \nu \), one obtains \( \dim(F_{ih}^{N}(F_{ih} \times F_{ih} \times \cdots \times F_{ih})) = \nu (N+1) \) and since (12) holds, \( \dim(M_{ih}^{N}V_{ih}) < n \); thus \( \dim(M_{ih}^{N}V_{ih} + F_{ih}^{N}(F_{ih} \times F_{ih} \times \cdots \times F_{ih})) \leq \nu (N+1) + n \). Therefore, since by Lemma 31, \( \nu < m \), for \( N > \frac{m - n}{\nu} - 1 \) we have that \( \nu (N+1) + n < m (N+1) \), and hence \( M_{ih}^{N}V_{ih} + F_{ih}^{N}(F_{ih} \times F_{ih} \times \cdots \times F_{ih}) \) is a proper subspace of \( \mathbb{R}^{m(N+1)} \). Hence there exists a sufficiently large \( N \) such that the set \( M_{ih}^{N}V_{ih} + F_{ih}^{N}(F_{ih} \times F_{ih} \times \cdots \times F_{ih}) \) is a proper subspace of \( \mathbb{R}^{m(N+1)} \), for any \( (i, h) \in \tilde{J} \). Let be \( u(t) = z \exp(\lambda t) \in \tilde{U} \). It follows that

\[
\begin{pmatrix}
u(t) \\
\dot{u}(t) \\
\ddot{u}(t) \\
\vdots \\
u^{(N)}(t)
\end{pmatrix} = \begin{pmatrix}
z \\
\lambda z \\
\lambda z \\
\vdots \\
\lambda^{N-1} z
\end{pmatrix} \exp(\lambda t).
\]

Set \( M_{ih}^{N}V_{ih} + F_{ih}^{N}(F_{ih} \times F_{ih} \times \cdots \times F_{ih}) = \ker(G_{ih}) \), for some matrix \( G_{ih} \). By applying Lemma 32 there exist \( z \) and \( \lambda \) such that \( G_{ih}z \neq 0 \), \( \forall (i, h) \in \tilde{J} \) where \( z \) is as in (20). This implies that

\[
\begin{pmatrix}
u(t) \\
\dot{u}(t) \\
\ddot{u}(t) \\
\vdots \\
u^{(N)}(t)
\end{pmatrix} \notin M_{ih}^{N}V_{ih} + F_{ih}^{N}(F_{ih} \times F_{ih} \times \cdots \times F_{ih}), \quad \forall (i, h) \in \tilde{J}, \forall t \in \mathbb{R}
\]

and hence condition (23) is false; thus the result follows.