Stochastic Control

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April 3, 2009
PREFACE

Teaching stochastic processes to students whose primary interests are in applications has long been a problem. On one hand, the subject can quickly become highly technical and if mathematical concerns are allowed to dominate there may be no time available for exploring the many interesting areas of applications. On the other hand, the treatment of stochastic calculus in a cavalier fashion leaves the student with a feeling of great uncertainty when it comes to exploring new material. Moreover, the problem has become more acute as the power of the differential equation point of view has become more widely appreciated. In these notes, an attempt is made to resolve this dilemma with the needs of those interested in building models and designing algorithms for estimation and control in mind. The approach is to start with Poisson counters and to identify the Wiener process with a certain limiting form. We do not attempt to define the Wiener process per se. Instead, everything is done in terms of limits of jump processes. The Poisson counter and differential equations whose right-hand sides include the differential of Poisson counters are developed first. This leads to the construction of a sample path representations of a continuous time jump process using Poisson counters. This point of view leads to an efficient problem solving technique and permits a unified treatment of time varying and nonlinear problems. More importantly, it provides sound intuition for stochastic differential equations and their uses without allowing the technicalities to dominate. In treating estimation theory, the conditional density equation is given a central role. In addition to the standard additive white noise observation models, a number of other models are developed as well. For example, the wide spread interest in problems arising in speech recognition and computer vision has influenced the choice of topics in several places.
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Chapter 1

Probability Spaces

We recall a few basic ideas from probability theory and, in the process, establish some of the notation and language we will use. This is mostly background information.

1.1 Sets and Probability Spaces

A set consisting of a finite collection of elements is said to be finite. In this case the cardinality of the set is the number of elements in the set. If a set is finite, or if its elements can be put into one-to-one correspondence with the positive integers, it is said to be countable. Sets that are not countable, such as the set of real numbers between zero and one, are said to be non-denumerably infinite. If $A$ and $B$ are subsets of a set $S$ we use $A \cup B$ and $A \cap B$ to denote the union and intersection of $A$ and $B$, respectively. We denote the empty set by $\emptyset$.

We use the notation $\{0,1\}^S$ to indicate the set of all subsets of $S$ with $\emptyset$ and $S$ included. This is often written as $2^S$ but we discourage this notation. In any case, $\{0,1\}^S$ is called the power set. It is easy to see that if $S$ is a finite set with cardinality $s$ then the cardinality of $\{0,1\}^S$ is $2^s$ and hence $\{0,1\}^S$ is also finite. On the other hand, if $S$ is countably infinite then, as was discussed by Cantor, $\{0,1\}^S$ is non-denumerably infinite. This is a strong suggestion that one must exercise care in attempting to reason about the set of all subsets of $S$ when $S$ is infinite.

In 1933 the mathematician A. N. Kolmogorov described a precise mathematical model for the subject of probability. This model has come to be widely used because it is both elegant and self contained. It is not, however, necessarily easy to relate the Kolmogorov axioms to real world phenomena, nor are the axioms he used the only ones that deserve consideration. For example, the use of probability amplitudes in quantum mechanics calls for a rather different set of ideas. Although we will not give the arguments in detail, everything in these notes is consistent with the Kolmogorov point of view. The idea behind the Kolmogorov formalism is that one associates to the set $S$ a collection of subsets $\mathcal{P}$, called the events, such that each $P \in \mathcal{P}$ has a well defined probability, $\mu(P)$. Our intuitive ideas about probability
are mostly based on the analysis of simple situations for which \( S \) is a finite set and \( \mathcal{P} \) is the set of all subsets of \( S \). In this case we ask that \( \mu(S) = 1, \mu(\emptyset) = 0 \) and that \( \mu \) be additive on disjoint sets. This amounts to asking that
\[
\mu(P_1 \cup P_2) = \mu(P_1) + \mu(P_2) - \mu(P_1 \cap P_2)
\]
Sometimes such \( \mu \)'s are called additive set functions.

If the cardinality of \( S \) is infinite, and this is the situation that occurs most frequently in our applications, then it is usually impossible to develop a satisfactory theory in which \( \mathcal{P} \) is the set of all subsets of \( S \). The problem stems from the fact that although the set of all subsets of a finite set is finite, the cardinality of the set of all subsets of an infinite set is not only infinite but is essentially larger than that of the original set in the sense that the elements of \( S \) cannot be placed in one-to-one correspondence with those of \( \{0,1\}^S \). One needs to be content with a more restricted theory. In this case one asks that \( \mu \) be additive on countable disjoint unions i.e., that
\[
\mu \left( \bigcup_{i=1}^{\infty} P_i \right) = \sum_{i=1}^{\infty} \mu(P_i)
\]
if the \( P_i \) are pair-wise disjoint but makes no claim about more general unions.

Collections of subsets, finite or infinite, that are closed under finite unions and intersections are said to form a field of sets. A collection \( \mathcal{P} \) of subsets of a set \( S \) is said to be a \( \sigma \)-field if it is closed under countably infinite unions and intersections. The natural setting for probability theory is \((S, \mathcal{P}, \mu)\) with \( \mathcal{P} \) being a \( \sigma \)-field and \( \mu \) being additive on countable disjoint unions.

The basic construct in the Kolmogorov system is a triple \((S, \mathcal{P}, \mu)\), called a probability space. The elements have the following properties:

1. \( S \) = set
2. \( \mathcal{P} \) = a collection of subsets of \( S \) that includes \( S \) and the empty set and is closed under complementation, countable unions and countable intersections.
3. \( \mu \) = a map of \( \mathcal{P} \) into \([0,1]\) such that if \( \{A_i\} \) is a disjoint collection then
\[
\mu \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i)
\]

In order to describe a specific probability space it is necessary to specify \( S, \mathcal{P} \) and \( \mu \). If \( S \) is a finite set then we can take \( \mathcal{P} \) to be the set of all subsets of \( S \) and can specify \( \mu \) by giving its value on each element of \( S \) with the understanding that its value on arbitrary subsets is just the sum of its values on the constituents of the subset.

**Example:** Let \( S \) be the set of \( n \)-tuples, \( S = \{(a_1, a_2, \ldots, a_n)\} \), with the \( a_i \) taking on the value zero or the value one. This set has cardinality \( 2^n \). We take \( \mathcal{P} \) to be the set of all
subsets of \( S \), a set with cardinality \( 2^n \). Let \( \alpha \) be a real number between zero and one and let the probability of the occurrence of \((a_1, a_2, \ldots, a_n)\) be

\[
p((a_1, a_2, \ldots, a_n)) = \alpha^{\sum a_i} (1 - \alpha)^{n - \sum a_i}
\]

In view of the identity \( 1 = (\alpha + (1 - \alpha))^n \) we see that \( \sum_{k=0}^{n} \binom{n}{k} \alpha^k (1 - \alpha)^{n-k} = 1 \). We extend the definition of \( p \) from \( S \) to \( P \) using \( p(A \cup B) = p(A) + p(B) \) for \( A \cap B = \phi \). In this way we get a probability measure on the set of all subsets of the set of all binary \( n \)-tuples.

## 1.2 Probability Distributions on Vector Spaces

We often base our models of real world phenomena on variables taking on values in the set of real numbers or, more generally, on vector valued variables whose components are real numbers. This works well for phenomena described by differential equations, linear algebra, etc. It is, however, the source of some technical problems in probability theory because of the necessity of defining a suitable collection of subsets of \( \mathbb{R} \). More specifically, one needs to define a collection that is big enough to be useful but not so big as to cause set theoretic difficulties. It may be of some comfort to consider the fact that the rational numbers, like the integers, are countable and the set of \( n \)-tuples with rational entries are countable as well.

The standard topology on \( \mathbb{R} \) can be explained this way. We say that a subset \( P \subset \mathbb{R} \) is open if for each \( x_0 \in P \) there exists an \( \epsilon > 0 \) such that \( \{x | |x - x_0| < \epsilon \} \subset P \). It is not hard to show that finite unions and finite intersections of open sets are open. The countably infinite cases are different. The infinite union of open sets is open, but it can happen that an infinite intersection of open sets is closed. For example

\[
\bigcap_{n=1}^{\infty} \{x | (-1/n \leq x \leq 1 + 1/n) \} = [0, 1]
\]

Thus the smallest \( \sigma \)-field that contains all the open intervals \((a/b, c/d)\) with \( a, b, c, d \) being integers (a countable family) contains all the closed intervals as well.

There is a rich collection of subsets of \( \mathbb{R}^n \) called the Borel sets which are generated from the open subsets of \( \mathbb{R}^n \) by taking countable intersections and countable unions, countable intersections of countable unions, etc. The development of measure theory, which we do not undertake here, often begins with a focus on Borel sets before moving on to a larger collection called Lebesgue measurable sets. For many purposes the Borel sets are adequate. It is often useful to consider the probability triple \((S, \mathcal{P}, \mu)\) where \( S \) is \( \mathbb{R}^n \), \( \mathcal{P} \) is the Borel sets and \( \mu \) is the measure one gets by extending the ordinary idea of length to the Borel sets.

If \( S = (-\infty, \infty) \) and if \( \mathcal{P} \) is the Borel \( \sigma \)-field then we can specify \( \mu \) by giving a nondecreasing function \( \phi \) which maps \((-\infty, \infty)\) onto \([0, 1]\) with

\[
\phi(\hat{x}) = \text{probability } x \leq \hat{x}
\]
In this case $\phi$ is said to be the probability distribution function. We see immediately that

$$\mu((a,b]) = \phi(b) - \phi(a)$$

and, using the axioms, we can extend the definition of $\mu$ to countable unions and countable intersections. If it happens that $\phi$ is a differentiable function of $x$ then $d\phi/dx$ is called the probability density associated with $(\mathcal{S},\mathcal{P},\mu)$.

**Example 1:** One-dimensional Gaussian densities. The integral $I = \int_{-\infty}^{\infty} e^{-x^2} dx$ can be evaluated by noticing that $I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy$ and then transferring to polar coordinates to get $I^2 = \int_{0}^{2\pi} \int_{0}^{\infty} r e^{-r^2} d\theta dr = \pi$. By rescaling $x$ we see that

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma} dx = \sqrt{2\pi\sigma}$$

thus $(1/\sqrt{2\pi\sigma})e^{-x^2/2\sigma}$ is a probability density on $(-\infty, \infty)$. Karl Fredrich Gauss (1777-1855), in addition to his many other claims to fame, made an extensive study of this density and for this reason it is usually named after him. Generalizing slightly so as to include a possible shift of origin the Gaussians take the form

$$\rho(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-m)^2/2\sigma}$$

with $m$ being the mean and $\sigma$ the variance. An integration by parts is all that is required to show that the moments of a zero mean Gaussian random variable satisfy

$$E|x|^p = (2\pi\sigma)^{-1/2} \int_{-\infty}^{\infty} x^p e^{-x^2/2\sigma} = \sigma(p-1)E|x|^{p-2}$$

and so, while for $p$ odd $E(x-m)^p = 0$, for $p$ even we have

$$E(x-m)^p = \sigma^{p/2}(p-1)(p-3)\cdots(3)(1)$$

thus

$$E(x-m)^p = (\sigma/2)^{p/2}p!(p/2)!$$

The Gaussian density is the only probability density with these moments.

**Example 2:** $n$-dimensional Gaussian densities. Now let $x$ be an $n$-vector. Because the integral over all space of $e^{-x_1^2/2\sigma_1}e^{-x_2^2/2\sigma_2}\cdots e^{-x_n^2/2\sigma_n}$ is just the product of the integrals of the factors we see that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-x_1^2/2\sigma_1} e^{-x_2^2/2\sigma_2} \cdots e^{-x_n^2/2\sigma_n} dx_1 dx_2 \cdots dx_n = \sqrt{(2\pi)^n\sigma_1\sigma_2\cdots\sigma_n}$$

We may say, therefore, that if $D$ is diagonal and positive definite with eigenvalues $\{d_i\}$, then

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-x^T (2D)^{-1}x} dx_1 dx_2 \cdots dx_n = \sqrt{(2\pi)^n d_1 d_2 \cdots d_n}$$
1.3 Independence and Conditional Probability

Given any \( n \times n \) matrix \( Q \) which is symmetric and positive definite, one can find an orthogonal matrix \( \Theta \) such that \( \Theta^TQ\Theta \) is diagonal. Thus by letting \( z = \Theta^Tx \) we see that the density

\[
\rho(x) = \frac{1}{\sqrt{(2\pi)^n \det Q}} e^{-x^T(2Q)^{-1}x}
\]

becomes

\[
\rho(z) = \frac{1}{\sqrt{(2\pi)^n \det Q}} e^{-z^T\Theta^T(2Q)^{-1}\Theta z}
\]

which is a nonnegative function such that

\[
\int_{\mathbb{R}^n} \rho(x) dx = 1
\]

That is, \( \rho(x) \) is a probability density. If we make the further change of variables, \( w = x + m \) we get the still normalized, but more general, form

\[
\rho(w) = \frac{1}{\sqrt{(2\pi)^n \det Q}} e^{-(w-m)^T(2Q)^{-1}(w-m)}
\]

called the multidimensional Gaussian density of variance \( Q \) and mean \( m \). That is to say, we have the identities

\[
m = \frac{1}{\sqrt{(2\pi)^n \det Q}} \int_{\mathbb{R}^n} xe^{-(x-m)^T(2Q)^{-1}(x-m)} dx
\]

and

\[
Q = \frac{1}{\sqrt{(2\pi)^n \det Q}} \int_{\mathbb{R}^n} (x-m)(x-m)^T e^{-(x-m)^T(2Q)^{-1}(x-m)} dx
\]

valid for all \( Q = Q^T > 0 \). The latter of these equations can be verified by computing the derivative with respect to \( \alpha \) of

\[
\sqrt{(2\pi)^n \det \alpha Q} = \int_{\mathbb{R}^n} e^{-x^T(2\alpha Q)^{-1}x} dx
\]

and then evaluating the result at \( \alpha = 1 \) to get

\[
\frac{-n}{2\sqrt{(2\pi)^n \det Q}} = \int_{\mathbb{R}^n} -\frac{1}{2} x^TQ^{-1}xe^{x^T(2Q)^{-1}x} dx
\]

1.3 Independence and Conditional Probability

Given a probability space \((S, \mathcal{P}, \mu)\) with \( P_1 \) and \( P_2 \) in \( \mathcal{P} \), Consider the difference

\[
d(P_1, P_2) = \mu(P_1 \cap P_2) - \mu(P_1) \cdot \mu(P_2)
\]

In general, this need not be zero. However, if it is we will say that \( P_1 \) and \( P_2 \) are independent. Assuming independence often simplifies the analysis. On the other hand, when making
measurements one hopes that the results of the measurements will reveal something about
the state of the system being measured, i.e., that the measurements are not independent of
the other variables present. This leads to the study of conditional probabilities. We use the
notation \( \mu(P_1|P_2) \) to denote the probability of \( P_1 \) given \( P_2 \).

Because estimation theory involves estimating a random variable on the basis of ob-
servations, we are often in the situation of computing conditional probabilities. A basic
tool is Bayes’ rule which expresses the conditional probability of \( A \) given \( B \) in terms of the
conditional probability of \( B \) given \( A \) and the probabilities of \( A \) and \( B \) alone. This rule is
\[
p(A|B) = \frac{p(B|A) \cdot p(A)}{p(B)}
\]
The demonstration of this fact is completely elementary being based on the fact that one of
the four possibilities, \( A \) and \( B \), \( A \) and not \( B \), not \( A \) and \( B \), not \( A \) and not \( B \), must occur.
If they occur with probabilities \( p_1 \), \( p_2 \), \( p_3 \) and \( p_4 \) respectively, then
\[
p(A|B) = \frac{p_1}{p_1 + p_3}
\]
and
\[
p(B|A) \cdot p(A)/p(B) = \frac{p_1}{p_1 + p_2} \cdot \frac{p_1 + p_2}{p_1 + p_3}
\]
These are equal and nothing more needs to be said.

When considering continuous random variables, or sets of stochastic processes, the study
of conditioning can involve rather subtle ideas from measure theory. However in the case
of \( \mathbb{R}^n \)-valued random variables with smooth probability densities, there is an infinitesimal
version of Bayes’ rule in which the probability \( p \) is replaced by the density \( \rho \), i.e.,
\[
\rho(a|b) = \frac{\rho(b|a) \cdot \rho(a)}{\rho(b)}
\]
Consider a density which is a continuous function of two variables, \( \rho(\cdot, \cdot) \). Given this density
there are related densities formed by integration. The marginal density
\[
\rho_{x\text{m}}(x) = \int_Y \rho(x, y) dy
\]
is obtained by integrating over the domain of the second argument, \( Y \), and there is a second
marginal density obtained by integrating over the domain of the first variable, \( X \)
\[
\rho_{y\text{m}}(y) = \int_X \rho(x, y) dx
\]
In terms of this notation Bayes’ rule for densities takes the form
\[
\rho(x|y) = \frac{\rho(y|x) \rho_{x\text{m}}(x)}{\rho_{y\text{m}}(y)}
\]
Thus in this situation we have a probability density version of Bayes’ rule which is of
the same form as the finite set version with the probabilities being replaced by probability
densities.
Example 1: If we are given \( x + n = y \) and if we know that \( x, n \) and \( y \) are real valued random variables with densities \( \rho_1, \rho_2 \) and \( \rho_3 \), respectively, then, assuming that \( n \) and \( x \) are independent, the probability density for \( x \), conditioned on the fact that \( x + n = y \), is

\[
\rho(x|y) = \frac{\rho(y|x) \cdot \rho_1(x)}{\rho_3(y)} = \frac{\rho_2(y - x) \cdot \rho_1(x)}{\int_{-\infty}^{\infty} \rho_2(y - \eta) \cdot \rho_1(\eta) d\eta}
\]

Notice that we used a convolution formula to express the density of \( y = x + n \) in terms of the densities of \( x \) and \( n \).

\[
\rho_3(y) = \int_{-\infty}^{\infty} \rho_2(y - \eta) \rho_1(\eta) d\eta
\]

Example 2: If in the previous example \( x \) and \( n \) have Gaussian distributions with means \( \bar{x} \) and 0, respectively, and variances \( \sigma_x \) and \( \sigma_n \), respectively, then having observed that \( x + n = y \) the conditional density for \( x \) changes from its a priori form

\[
\rho_1(x) = \frac{1}{\sqrt{2\pi\sigma_x}} e^{-\frac{(x - \bar{x})^2}{2\sigma_x}}
\]

to

\[
\rho(x|y) = \frac{c \cdot e^{-\frac{(y-x)^2}{2\sigma_n}} e^{-\frac{(x - \bar{x})^2}{2\sigma_x}}}{\sqrt{2\pi\sigma_3}} e^{-\frac{(x - \alpha)^2}{2\sigma_3}}
\]

with \( \sigma_3 = \sigma_x\sigma_n/(\sigma_x + \sigma_n) \) and \( \alpha \) being \((\sigma_x y + \sigma_n\bar{x})/(\sigma_x + \sigma_n)\).

### 1.4 Moments and their Generating Function

If \( x \) is a real valued random variable distributed with density \( \rho \) and if \( \psi : \mathbb{R} \to \mathbb{R} \) is a map then one can investigate the existence of

\[
\mathcal{E}\psi(x) = \int_{-\infty}^{\infty} \psi(x) \rho(x) dx
\]

Of course such integrals can fail to exist in a variety of ways. If \( p \) is a positive integer and if

\[
\mathcal{E}x^p = \int_{-\infty}^{\infty} x^p \rho(x) dx
\]

exists then we say that the density \( \rho \) has a \( p^{th} \) moment. The first moment is usually called the expected value and the second is often expressed in terms of the variance.

\[
\mathcal{E}(x - \mathcal{E}x)^2 = \int_{-\infty}^{\infty} x^2 \rho(x) dx - (\mathcal{E}x)^2
\]

assuming these integrals exist.
In probability theory the Fourier transform of the density is usually defined as
\[ \mathcal{F}(\rho) = \int_{-\infty}^{\infty} e^{-i\omega x} \rho(x) dx \]

Notice that if the \( p \)th moment exists then
\[ \frac{d^p}{d\omega^p} \mathcal{F}(\rho) = \int_{-\infty}^{\infty} (-ix)^p e^{-i\omega x} \rho(x) dx \]
which establishes a relationship between the \( p \)th moment and the \( p \)th term in the Taylor series expansion of the Fourier transform about 0. The Fourier transform of \( \rho \) is sometimes called the moment generating function because when all moments exist
\[ \mathcal{F}(\rho(\omega)) = \sum_{p=0}^{\infty} m_p \frac{(-i)^p}{p!} \omega^p \]

Probability densities that approach zero sufficiently rapidly as \( |x| \) goes to infinity are uniquely determined by their moments but not all densities have this property. As we will see, a zero mean Gaussian density with variance \( a \) has as its moments
\[ \mathbb{E} x^p = \frac{p!}{(p/2)!} \left( \frac{a}{2} \right)^{p/2} \quad ; \quad p \text{ even} \]
with the odd moments being zero. In this case, and in all cases where the moments grow more slowly with \( p \), the moments uniquely determine the density.

### 1.5 Transformation of Densities

If \( y \) is a random variable that takes on values in \( \mathbb{R}^n \) and if \( \phi: \mathbb{R}^n \to \mathbb{R}^m \) is a differentiable function, then \( x = \phi(y) \) is also a random variable. Many problems involve making computations that characterize the way \( x \) is distributed, given the distribution of \( y \). In the situation where \( y \) has a smooth density and \( \phi: \mathbb{R}^1 \to \mathbb{R}^1 \) the matter can be dealt with rather easily. Let \( S \subset \mathbb{R}^1 \) denote the set \( S = \{ x | x = \phi(y) \text{ has a solution} \} \). If \( x \) has a probability density \( \tilde{\rho} \) it follows that \( \tilde{\rho} \) is zero for any \( x \) not in \( S \). On the other hand, at a point \( x_0 \) in \( S \) there is an inverse image \( y_0 \) such that \( x_0 = \phi(y_0) \). If we linearize \( \phi \) near \( y_0 \) then
\[ \phi(y_0 + \delta) = x_0 + \frac{\partial \phi}{\partial y} \bigg|_{y_0} \delta + \text{higher order terms} \]
and the measure of the set of points \( T = \{ y | |y - y_0| \leq \epsilon \} \) is approximately \( \rho(y_0) \cdot \epsilon \). Thus the measure of the set that \( T \) maps into is \( \rho(y_0) \cdot \epsilon \). However, the length of this interval in \( x \)-space is the absolute value of \( \frac{\partial \phi}{\partial y} \bigg|_{y_0} \epsilon \). Thus, we see that \( \tilde{\rho}(x) = \frac{1}{|\partial \phi/\partial y|_{y_0}} \rho(\phi^{-1}(x)) \). However, a given \( x \) may have several inverse images. In such cases we need to express matters as
\[ \tilde{\rho}(x) = \sum_{\{ y | \phi(y) = x \}} \frac{1}{|\partial \phi/\partial y|_{y_0}} \rho(\phi^{-1}(x)) \]
with the understanding that $\partial \phi / \partial y$ is to be evaluated at the appropriate inverse image of $x$. Suppose now that $m = n > 1$. If $\rho_1(\cdot)$ and $\rho(\cdot)$ are the probability densities of the random variables $x$ and $y$ with $x$ and $y$ taking on values in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, then the appropriate formula involves the determinant of the Jacobian

$$
\rho_1(x) = \sum_{\{y|\phi(y) = x\}} \frac{1}{|\det(\partial \phi / \partial y)|} \rho(\phi^{-1}(y))
$$

where the sum is to be taken over all inverse images of the given value of $y$. Of course this only works if $x = \phi(y)$ has a finite number of solutions.

**Example 1:** Suppose that $y$ has Gaussian density $\rho(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma}$ and that $r = y^2$. Thus $dr = 2ydy$. Let $\tilde{\rho}$ denote the density with respect to $r$. Then $\tilde{\rho}$ is zero for negative arguments. There are two inverse images for $r > 0$ and the above formula yields

$$
\tilde{\rho}(r) = \frac{1}{\sqrt{2\pi}\sigma} \left( \frac{1}{2\sqrt{r}} e^{-r/2\sigma} + \frac{1}{2\sqrt{r}} e^{-r/2\sigma} \right) = \frac{1}{\sqrt{2\pi}\sigma} \frac{1}{\sqrt{r}} e^{-r/2\sigma}
$$

**Example 2:** Let $x_1$ and $x_2$ be jointly Gaussian with density

$$
\rho(x_1, x_2) = \frac{1}{2\pi\sigma} e^{-(x_1^2 + x_2^2)/2\sigma}
$$

Consider a transformation to polar coordinates with $r = \sqrt{x_1^2 + x_2^2}$ and let $\theta = \tan^{-1}(x_2/x_1)$. Find the density with respect to $(r, \theta)$. In this case $r$ is non negative and each $(r, \theta)$ for $r > 0, 0 \leq \theta < 2\pi$ has a unique inverse image. The Jacobian of the map from $(x_1, x_2)$ to $(r, \theta)$ is given by

$$
J = \begin{vmatrix}
\frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial x_2} \\
\frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2}
\end{vmatrix} = \begin{vmatrix}
\frac{x_1}{\sqrt{x_1^2 + x_2^2}} & \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \\
\frac{x_2}{x_1 + x_2} & \frac{x_1}{x_1 + x_2}
\end{vmatrix}
$$

The determinant of the Jacobian is $1/\sqrt{x_1^2 + x_2^2} = 1/r$, and so

$$
\tilde{\rho}(r, \theta) = \frac{r}{2\pi\sigma} e^{-r^2/2\sigma}
$$

**Example 3:** Suppose that $(x, y)$ is a Gaussian random vector in $\mathbb{R}^2$, with zero mean and variance

$$
\Sigma = \begin{bmatrix}
a & b \\
b & a
\end{bmatrix}
$$

We want to find the probability that the product $xy$ is positive. Begin by observing the identity

$$
\begin{vmatrix}
\frac{1}{\sqrt{a-b}} & \frac{1}{\sqrt{a+b}} \\
\frac{1}{\sqrt{a+b}} & \frac{1}{\sqrt{a-b}}
\end{vmatrix} \begin{bmatrix}
a & b \\
b & a
\end{bmatrix} = \begin{bmatrix}
a + b & 0 \\
0 & a - b
\end{bmatrix}
$$

This shows that the change of variables

$$
\begin{bmatrix}
u \\
w
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\sqrt{a+b}} & 0 \\
0 & \frac{1}{\sqrt{a-b}}
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}
$$
CHAPTER 1. PROBABILITY SPACES

results in a pair of variables that are Gaussian with zero mean and unity variance. The next step is to determine which values of \((u, v)\) correspond to a positive value of \(xy\). To this end, observe that \(xy\) changes sign if and only if either \(x\) or \(y\) changes sign. However,

\[
\sqrt{2\sqrt{a} + bu} = x + y
\]

Thus \(x = 0\) implies that

\[
\sqrt{2\sqrt{a} + bu} = \sqrt{2\sqrt{a} - bv}
\]

and \(y = 0\) implies that

\[
\sqrt{2\sqrt{a} + bu} = -\sqrt{2\sqrt{a} - bv}
\]

The straight lines defined by these equations pass through the origin and the angle between them is

\[
\theta = \tan^{-1} \frac{\sqrt{a + b}}{\sqrt{a - b}}
\]

Because the density in \((u, v)\)-space is circularly symmetric, this angle, divided by \(\pi\) is the value of the integral of the density over the region where \(xy\) is positive. Thus the probability that \(xy\) is positive is given by

\[
p = \frac{2}{\pi} \tan^{-1} \frac{\sqrt{a + b}}{\sqrt{a - b}}
\]

**Example 4:** Suppose that \(x\) is a real valued random variable taking on values in the interval \([0, \infty)\) with density \(\rho(x) = \lambda e^{-\lambda x}\). If \(y = \lfloor x \rfloor\), i.e., if \(y\) is the largest integer less than or equal to \(x\) then \(y\) takes on values in \(\mathbb{Z} \cup 0\). Let \(\tilde{p}(y = k)\) be the probability of the event \(y = k\). What is \(\tilde{y}\) and what are its moments? An integration over the intervals of the form \([n, n + 1)\) shows that

\[
p_n = \int_n^{n+1} \lambda e^{-\lambda x} dx = e^{-\lambda n}(1 - e^{-\lambda})
\]

To evaluate the expected value of \(y^p\) we need to evaluate

\[
\mathcal{E}y^p = \sum_{n=0}^{\infty} n^p e^{-\lambda n}(1 - e^{-\lambda}) = \sum_{n=0}^{\infty} n^p (-1)^p(1 - e^{-\lambda}) \frac{d^p}{d\lambda^p} \sum_{n=0}^{\infty} e^{-\lambda n}
\]

Summing the geometric series, we see that

\[
\mathcal{E}y^p = (-1)^p(1 - e^{-\lambda}) \frac{d^p}{d\lambda^p} \frac{1}{e^\lambda - 1}
\]

and, in particular, that \(\mathcal{E}y = 1/(e^\lambda - 1)\)

**Example 5:** Let \(H = H^T\) be an \(n\) by \(n\) matrix. We denote its \(ij^{th}\) entry by \(h_{ij}\) and of course \(h_{ij} = h_{ji}\). We assume that the \(h_{ij}\) are random and are distributed according to zero mean Gaussian distributions. The \(h_{ij}\) are independent except for the requirement that \(h_{ij} = h_{ji}\). Given this distribution on the entries of \(H\) we want to use the change of coordinates formula to find the distribution law for the eigenvalues of \(H\). To begin with, we
recall that it is possible to represent an arbitrary symmetric matrix $H$ as $\Theta^T \Lambda \Theta$ with $\Lambda$ being diagonal and $\Theta$ being orthogonal. The transformation from $H$ to $(\Theta, \Lambda)$ is not one to one because for a given $H$, there is more than one orthogonal matrix such that $\Theta H \Theta^T$ is diagonal. In fact, if the eigenvalues of $H$ are distinct then there $n!^2(n-1)$ possible choices of $(\Theta, \Lambda)$ for each $H$. within multiplication on the left by a diagonal matrix $D$ whose diagonal entries are $\pm 1$ and a further multiplication on the left by a permutation matrix $P$. Taking the derivative of $\Theta^T \Lambda \Theta = H$ we get $d\Omega H - H d\Omega = dH$ with $d\Omega = d\Theta * \Theta^T$ being skew symmetric. Adding in the change of $\Lambda$ we get $H d\Omega - d\Omega H + \Theta^T (d\Lambda) \Theta = dH$ where $d\Omega$ is skew-symmetric and equal to $d\theta \Theta^T$. Write $ad_H(d\Omega) = H d\Omega - dH$. We see that the inverse image in $H$ space associated with the set of eigenvalues $|\lambda_1 - \lambda_a| \leq \epsilon, |\lambda_2 - \lambda_b| \leq \epsilon \ldots, |\lambda_n - \lambda_c| \leq \epsilon$ is just $ad_H^{-1}(dH)$. We can use this to show that the eigenvalues of $H$ are distributed according to the law $\rho(\lambda_1, \lambda_2, \ldots, \lambda_n) = Ne^{\sum \lambda^2} \prod_{i,j} (\lambda_i - \lambda_j)$ where $N$ is the normalization factor needed to make the area under the function one.

### 1.6 Empirical Determination of Densities

When applying probability theory to problems arising in the real world, densities must be determined from data. If there is adequate reason to think that the density belongs to a particular parametrized family, then a procedure based on maximizing a likelihood is often used to select the values of these parameters. We briefly explain this here.

Having decided to use a density belonging to a particular parametrized family, say $\rho_{\theta_1, \theta_2, \ldots, \theta_k}(\cdot)$ and having observations $\{x_1, x_2, \ldots, x_k\}$, the parameters, $\theta$ are selected so as to make the observations as probable as possible. A standard assumption is that the process of generating observations are the result of sampling from the, as yet unknown distribution, and can be treated as independent events. Thus the joint probability of having observed the given set of $x$’s is just the product

$$L = \prod_{i=1}^{k} \rho_{\theta_1, \theta_2, \ldots, \theta_k}(x_i)$$

and the most probable values of the parameters is obtained by maximizing this product. Because of the product form it is often convenient to work with the logarithm, the so called log likelihood function. We illustrate with three standard examples.

**Example 1:** Consider the problem of determining a possible bias associated with tossing a coin. Assume that the probability for the number of heads is given by the binomial distribution

$$p(k) = \frac{n!}{k!(n-k)!} \theta^k (1-\theta)^{n-k}$$
and that we want to determine the parameter \( \theta \). If the result of \( n \) tosses is \( n_1 \) heads and \( n - n_1 \) tails. The maximum likelihood method selects \( \theta \) by maximizing

\[
L = \frac{n!}{n_1!(n-n_1)!} \theta^{n_1} (1-\theta)^{n-n_1}
\]

Not surprisingly, the maximum occurs when \( \theta = n_1/n \).

**Example 2:** The exponential density on the half-line \([0, \infty)\) takes the form \( \rho(x) = \theta e^{-\theta x} \).

Suppose we are given observations \( x_1, x_2, \ldots, x_k \) and wish to determine \( \theta \). The likelihood function is

\[
L = \prod_{i=1}^{k} \theta e^{-\theta x_i}
\]

and the log likelihood can be expressed as

\[
LL = k \ln \theta - \theta \sum x_i
\]

To maximize this we equate to zero the derivative with respect to \( \theta \). This shows that the probability of having obtained these samples is maximized by letting \( \theta = k/\sum x_i \).

**Example 3:** Given \( k > n \) points in \( \mathbb{R}^n \), suppose we want to find the most likely Gaussian probability density. Of course a Gaussian is parametrized by its mean and variance so and so the problem can be phrased as that of finding \( x_0 \) and \( \Sigma \) so as to maximize

\[
L = \prod \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2} (x_i - x_0)^T (2\Sigma)^{-1} (x_i - x_0)}
\]

The Log likelihood is

\[
LL = c - \frac{n}{2} \ln (\det \Sigma) - \sum_{i=1}^{k} (x_i - x_0)^T (2\Sigma)^{-1} (x_i - x_0)
\]

where \( c \) is a constant. Equating to zero the the derivative with respect to \( x_0 \) gives

\[
x_0 = \frac{1}{k} \sum x_i
\]

Optimizing the value of \( \Sigma \) requires a bit more effort. The details of the calculations are explored in the homework problems. They led to the conclusion that the maximum likelihood solution is the sample covariance,

\[
\Sigma = \frac{1}{k} (x_i - x_0)(x_i - x_0)^T
\]

### 1.7 Exercises Chapter 1

1. Consider a coin which comes up heads with probability \( \alpha \) and tails with probability \( 1 - \alpha \). Suppose the coin is tossed \( n \) times and the tosses are independent.
i) Show that the probability that the \( n \) tosses will yield exactly \( k \) heads is
\[
p(k) = \frac{n!}{k!(n-k)!} \alpha^k (1 - \alpha)^{n-k}
\]

ii) Show that the average is
\[
\frac{1}{n} \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} \alpha^k (1 - \alpha)^{n-k} = \alpha
\]
and that the variance is
\[
\frac{1}{n} \sum_{k=0}^{n} (k - \alpha)^2 \frac{n!}{k!(n-k)!} \alpha^k (1 - \alpha)^{n-k} = \frac{\alpha(1-\alpha)}{n}
\]

iii) Show that the most probable number of heads is the integer closest to \( \alpha p \), provided \( \alpha p - 1/2 \) is not an integer.

2. Let \( \ln \) denote the natural logarithm. Using \( \ln(1) = 0 \) and the monotonicity of the logarithm we see that,
\[
\int_2^n \ln(x-1)dx \leq \sum_{k=2}^{n} \ln(k) = \sum_{k=1}^{n} \ln(k) \leq \int_1^n \ln(x)dx
\]
Moreover, an integration-by-parts shows that
\[
\int_1^n x \ln x \, dx = x \ln x|_1^n - \int_1^n 1 \, dx = n \ln(n) + 1 - n
\]
Putting these together we have
\[
(n-1) \ln(n-1) - n \leq \sum_{k=2}^{n} \ln k = \ln(n!) \leq n \ln(n) + 1 - n
\]
Thus
\[
(n-1)^{n-1}e^{-n} \leq n! \leq en^n e^{-n}
\]
Use the fact that the derivative of \( \ln x \) is monotone decreasing to show that
\[
\sum_{k=1}^{n} \ln(k) \leq \int_2^n \ln(x-1)dx + \sum_{k=1}^{n-1} \frac{1}{2k}
\]
Develop a bound on \( \ln(x) \) that yields
\[
n! \approx e \sqrt{n} n^{n} e^{-n}
\]
If you are more ambitious you can try to get the still better Stirling’s approximation
\[
n! \approx \sqrt{2\pi} n^n e^{-n}
\]
Hint: For the latter analysis you may wish to consider the gamma function
\[
\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \, dt
\]
which, as can be verified by repeated integration-by-parts, for positive integer values of \( x \) satisfies \( \Gamma(x) = (x-1)! \).
3. If $x_1$ and $x_2$ are scalar Gaussian random variables with zero mean and variance $\sigma$ and if $y_1 = \max\{x_1, x_2\}$, and $y_2 = \min\{x_1, x_2\}$, show that the probability density for $y_1$ is

$$
\rho(y_1) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{y_1} e^{-\eta^2/2\sigma} d\eta
$$

4. (Compare with example 5 of section 1.5.) Let $H$ be an $n$ by $n$ symmetric matrix with real entries. Suppose that the entries $h_{ij}$ for $i \geq j$ are independent and identically distributed random variables, distributed according to a Gaussian distribution with zero mean and variance $\sigma$. Let $\beta$ denote the sum of the squares of the eigenvalues of $H$. Determine the probability density for $\beta$. Hint: Recall, or prove for yourself, that the sum of the squares of the eigenvalues of a symmetric matrix equals the sum of the squares of the entries.

5. Let $\rho_1: (-\infty, \infty) \to [0, \infty)$ and $\rho_2: (-\infty, \infty) \to [0, \infty)$ be such that

$$
\int_{-\infty}^{\infty} \rho_i(x) dx = 1; \ i = 1, 2
$$

Show that if $\rho_3$ is defined by

$$
\rho_3(x) = \int_{-\infty}^{\infty} \rho_1(x - \eta) \rho_2(\eta) d\eta
$$

then

$$
\int_{-\infty}^{\infty} \rho_3(x) dx = 1
$$

6. Let $\rho$ be a probability density on the real line with mean $\mu$ and variance $\alpha$. Show that the variance of

$$
\rho_2(x) = \int_{-\infty}^{\infty} \rho(x - y) \rho(y) dy
$$

is $2\alpha$.

7. Let $\Sigma$ be a square matrix. Show that the first two terms in the Taylor series expansion of $\det(\Sigma + \Delta)$ are given by

$$
\det(\Sigma + \Delta) = \det(\Sigma) + \frac{\text{tr} (\Delta \Sigma^{-1})}{\det \Sigma}
$$

provided that $\Sigma$ is nonsingular.

8. If $\mu_1: \mathcal{P} \to [0, 1]$ and $\mu_2: \mathcal{P} \to [0, 1]$ are two different probability measures corresponding to the same choice of $S$ and $\mathcal{P}$ then we can define the distance between $\mu_1$ and $\mu_2$ as

$$
d(\mu_1, \mu_2) = \max_{P \in \mathcal{P}} |\mu_1(P) - \mu_2(P)|
$$

If $S$ is a finite set and $\mathcal{P}$ is the set of all subsets of $S$ then show that

$$
d(\mu_1, \mu_2) = \sum_{s_i \subseteq S} |\mu_1(s_i) - \mu_2(s_i)|
$$

show that $d(\mu_1, \mu_2) + d(\mu_2, \mu_3) \geq d(\mu_1, \mu_3)$.

9. Let $S$ be a finite set and let $\mathcal{P}$ be the set of all subsets of $S$. Find the probability measure $\mu: \mathcal{P} \to [0, 1]$ that maximizes the sum

$$
H = \sum_{s_i \subseteq S} \mu(s_i) \ln(\mu(s_i))
$$
10. If \( S = (-\infty, \infty) \) and \( \mathcal{P} \) is set of all Borel subsets of \( (-\infty, \infty) \) find the probability density function \( \mu \) that maximizes

\[
H = \int_{-\infty}^{\infty} \mu(x) \ln \mu(x) \, dx
\]

subject to \( \mathbb{E}x = 0 \) and \( \mathbb{E}x^2 = \sigma \).

11. Suppose that \( a_1, a_2, \ldots, a_n \) are independent Gaussian random variables with mean 0 and variance \( \sigma \). Consider

\[
x_n(t) = a_0 + \sqrt{2} \sum_{k=1}^{n} a_k \cos kt
\]

(i) Show that \( \mathbb{E}x_n(t) = 0 \)

(ii) Show that \( \mathbb{E}x_n^2(t) \leq n \)

(iii) Compute the probability density for \( x(t) \)

If we integrate this sum from \( t = 0 \) we get

\[
y_n(t) = a_0 t + \sqrt{2} \sum_{k=1}^{n} \frac{a_k}{k} \sin kt
\]

Repeat the above calculations for \( y_n(t) \). What about taking the limit as \( n \) goes to infinity?

12. Let \( Q = Q^T \) be positive definite and let \( x \) be a vector-valued Gaussian random variable with zero mean and variance \( Q \). Assume \( R = R^T \). Show that

\[
\mathbb{E}e^{x^T (2R)^{-1} x}
\]

exists if \( Q^{-1} - R^{-1} \) is positive definite and that in this case

\[
\mathbb{E}e^{x^T (2R)^{-1} x} = \frac{\sqrt{\det(Q^{-1} - R^{-1})}}{\sqrt{\det Q}}
\]

13. If \( x \) is a random variable that takes on values in a finite set \( x = \{x_1, x_2, \ldots, x_n\} \) then the possible ways to assign probabilities to this set can be identified with the set

\[
S = \{ (p_1, p_2, \ldots, p_n) | p_i \geq 0 ; \sum p_i = 1 \}
\]

\( S \) is a closed bounded subset of \( \mathbb{R}^n \) called the standard simplex. (a) Show that if \( T \) is a linear transformation of \( \mathbb{R}^n \) into \( \mathbb{R}^n \) then \( T \) maps the standard simplex into itself if and only if the matrix representation of \( T = (t_{ij}) \) has nonnegative entries and

\[
\sum_{j=1}^{n} t_{ij} = 1
\]

14. Consider a random vector \( x \) in \( \mathbb{R}^2 \) with a Gaussian distribution having mean 0 and variance

\[
\Sigma = \begin{bmatrix}
a & b \\
b & c
\end{bmatrix}
\]

Modify the analysis given in example 3 of section 1.5 to determine the probability that \( x_1 x_2 \) is positive.

15. Let \( A \) be a nonsingular matrix and \( D \) an arbitrary matrix, both square. Show that the first two terms of the Taylor series expansion of \( \det(A + \epsilon D) \) is

\[
\det(A + \epsilon D) = \det(A)(1 + \epsilon \text{tr}(A^{-1}D) + e)
\]

where \( \text{tr} \) denotes the trace and \( e \) is an error term involving terms of order \( \epsilon^2 \) and higher.
16. Let $A$ be a nonsingular matrix and $D$ an arbitrary matrix, both square. Show that the first two terms of the Taylor series expansion of $(A + \epsilon D)^{-1}$ are given by

$$(A + \epsilon D)^{-1} = A^{-1}(I + \epsilon \text{diag}(D)) + \text{h.o.t.}$$

where $\text{diag}$ denotes the diagonal terms and $\epsilon$ is an error term involving terms of order $\epsilon^2$ and higher.

1.8 Notes and References

1. The basic elements of set theory are covered in many books introducing topology, measure theory, etc. In the 19th century Cantor showed that the study of infinite sets leads to nontrivial problems, and took important steps toward clarifying their properties. With the formalization of various aspects of combinatorics it is now widely appreciated that there are decidedly nontrivial questions about finite sets as well. A standard reference is Hausdorff [1]

Basic works on probability include Kolmogorov [2] and Feller [3]

2. Gaussian densities are standard fare in large parts of probability theory
Chapter 2

Poisson Counters and Differential Equations

A stochastic process is, from one point of view, just a random variable whose events are collections of functions of one or more independent variables. We are mostly interested in the cases where there is just one independent variable and we think of it as being time. It is not our intention to give a precise, general definition at this point. We prefer to discuss some specific cases instead. Indeed, we begin our discussion of a very flexible and interesting family of Markov processes based on a simple counting model of the type which describes many natural phenomena. We introduce stochastic differential equations, discuss statistical properties of their solutions, and derive characterizations of the corresponding probability laws.

2.1 Poisson Counters

By a continuous time, countable state Markov process we understand an ensemble of sample paths \( \{ x(\cdot) \} \) each of which is a function of time with the functions taking on values in a countable set \( X \). (See Figure 2.1.) We impose the condition that for all \( \tau > 0 \), the probability of the event \( x(t+\tau) = x_i \), given the values of \( x(\sigma) \) for all \( \sigma \leq t \) is the same as the probability that \( x(t+\tau) = x_i \) given only the value of \( x(t) \).

We let \( Z \) denote the integers \( \{ 0, \pm 1, \pm 2, \ldots \} \) and let \( Z^+ \) denote the positive integers \( \{ 1, 2, \ldots \} \). Let \( x_t \) be a stochastic process which takes on values in the nonnegative integers \( Z^+ \cup \{ 0 \} \), and let it be characterized by the transition rule,

\[
\dot{p}_n(t) = -\lambda p_n(t) + \lambda p_{n-1}(t) \quad ; \quad p_0(0) = 1 \quad ; \quad p_i(0) = 0 \text{ for } i > 0
\]

where \( \lambda \) is a constant called the counting rate and \( p_i(t) \) is the probability that \( x_t = i \). We
call this the Poisson counter of rate $\lambda$. We can write these equations as an infinite system

\[
\begin{bmatrix}
p_0 \\
p_1 \\
p_2 \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
-\lambda & 0 & 0 & \cdots \\
\lambda & -\lambda & 0 & \cdots \\
0 & \lambda & -\lambda & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
p_0 \\
p_1 \\
p_2 \\
\vdots
\end{bmatrix}

p(0) = \begin{bmatrix} 1 \\
0 \\
0 \\
\vdots
\end{bmatrix}
\]

and solve them one at a time, starting with the equation for $p_0$. This yields

\[
p_0(t) = e^{-\lambda t} \\
p_1(t) = \lambda te^{-\lambda t} \\
p_2(t) = \frac{\lambda^2 t^2}{2!} e^{-\lambda t} \\
... \quad = \quad ...
\]

\[
p_n(t) = \frac{\lambda^n t^n}{n!} e^{-\lambda t}
\]

Notice that the sum $p_0(t) + p_1(t) + p_2(t) + \cdots$ is one as it should be. If $x$ is a real valued random variable we let $E_x$ denote its expected value. We justify our use of the term “counting rate” with the calculation

\[
E_x t = \sum_{n=0}^{\infty} n p_n(t) = \sum_{n=0}^{\infty} (n(\lambda t)^n/n!)e^{-\lambda t}
\]

\[
= e^{-\lambda t}(\lambda te^{\lambda t})
\]

\[
= \lambda t
\]

Thus the expected value of $x_t$ increases by one every $1/\lambda$ units of time. More generally, we can use this same method of computation to evaluate $E x^p_t$ in terms of the lower order moments. That is,

\[
E x^p_t = e^{-\lambda t} \sum_{n=0}^{\infty} n^p(\lambda t)^n/n!
\]

\[
= e^{-\lambda t} \lambda t \sum_{n=0}^{\infty} (n^p (\lambda t)^{n-1}/(n)!
\]

\[
= e^{-\lambda t} \lambda t \sum_{n=0}^{\infty} (n^{p-1} (\lambda t)^{n-1}/(n-1)!
\]

\[
= e^{-\lambda t} \lambda t \sum_{r=0}^{p-1} (r + 1)^{p-1} (\lambda t)^r/(r)!
\]

\[
= e^{-\lambda t} \lambda t \sum_{r=0}^{p-1} \sum_{k=0}^{r-1} \binom{p-1}{k} r^k (\lambda t)^r/r!
\]

\[
= \lambda t \sum_{r=0}^{p-1} \binom{p-1}{r} E x^r_t
\]

Thus, for example, $E x^3_t = (\lambda t)(E x^0_t + 2E x^1_t + E x^2_t)$
We will also have occasion to use a bidirectional Poisson counter of rate $\lambda$. This is a process that can be defined as the difference between two independent Poisson counters of rate $\lambda$. It takes on values in $\mathbb{Z}$ and starts at zero at $t = 0$. It’s probability law is defined implicitly by the family of ordinary differential equations.

$$\dot{p}_i = +\lambda p_{i-1} - 2\lambda p_i + \lambda p_{i+1} ; \quad i = 0, \pm 1, \pm 2, \ldots$$

subject to the initial conditions $p_0(0) = 1$ and $p_i(0) = 0$ for $i \neq 0$. One sees easily that the sum of the $p$’s is always one. In order to develop the properties of the bidirectional counter it is useful to introduce the generating function

$$g(t, z) = \sum_{i=-\infty}^{\infty} z^i p_i(t)$$

Because the $p_i$ are nonnegative and sum to one, this series converges uniformly for $z$ on the unit circle, i.e., for $z = e^{i\theta}$ and $\theta$ real. Clearly $\partial g(t, z)/\partial t = \lambda(z - 2 + z^{-1})g(t, z)$ and so

$$g(t, z) = e^{-2\lambda t} e^{(z+z^{-1})\lambda t}$$

In terms of $z = e^{i\theta}$ we have

$$g(t, e^{i\theta}) = e^{(2\cos\theta - 2)\lambda t}$$

Using the binomial formula to expand $e^{(z+z^{-1})\lambda t}$ we see that $g(t, z)$ can be expressed as

$$g(t, z) = e^{-2\lambda t} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} z^{n-2k} (\lambda t)^n / n! = e^{-2\lambda t} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} z^{n-2k} (\lambda t)^n / n!$$

The partial sum over the set $n - 2k = i$ gives $p_i$ as the coefficient of $z^i$. This is

$$p_i(t) = e^{-2\lambda t} \sum_{m=0}^{\infty} \frac{t^{2m}}{2^{2m}m!(n+m)!}$$

It is well known that this implies that $p_n$ can be expressed in terms of the Bessel functions of the second kind and of integral order. (These are ordinary Bessel functions with purely imaginary argument.) In fact

$$I_n(t) = \sum_{m=0}^{\infty} \frac{t^{2m}}{2^{2m}m!(n+m)!}$$
is the series definition of the \( n^{th} \) modified Bessel function of the first kind. Thus

\[ p_n(t) = e^{-2\lambda t}I_n(2\lambda t) \]

It may also be shown that

\[ \ddot{I}_n(t) + \frac{1}{t}\dot{I}_n(t) + \left(1 + \frac{n^2}{t^2}\right)I_n(t) = 0 \]

with \( I_0(0) = 1, \dot{I}_0(0) = -1 \) and \( I_n(0) = 0; n \neq 0 \).

![Figure 2.2: The plots of \( p_1, p_2 \) and \( p_3 \) for the bidirectional counter.](image)

The utility of counters is greatly extended by combining them with some ideas from differential equations. Consider a differential equation in \( \mathbb{R}^n \) written as

\[ \dot{x}(t) = f(x(t), t) \]

or as

\[ x(t) = x(0) + \int_0^t f(x(\sigma), \sigma) \, d\sigma \]

Suppose that \( f(\cdot, \cdot) \) is continuous in both arguments and suppose that

\[ ||f(x, \sigma) - f(y, \sigma)|| \leq k||x - y|| \]

for some \( k \) and all \( \sigma, x \) and \( y \). It is known that such an equation has a unique solution corresponding to a given \( x(0) \). We are interested in a stochastic version of this, i.e., stochastic differential equations. Consider

\[ x(t) = x(0) + \int_0^t f(x(\sigma), \sigma) \, d\sigma + \int_0^t g(x(\sigma), \sigma) \, dN_\sigma \quad (2.1) \]

where \( N \) is a Poisson counter. How can we give meaning to such an object?

**Definition:** A function \( x(\cdot) \) is a solution of (2.1) in the Itô sense if, on an interval where \( N \) is constant, \( x \) satisfies \( \dot{x} = f(x, t) \) and if, \( N \) jumps at \( t_1 \), \( x \) behaves in a neighborhood of \( t \) according to the rule

\[ \lim_{t \to t_1^-} x(t) = g(\lim_{t \to t_1^-} x(t), t_1) + \lim_{t \to t_1^-} x(t) \]

and \( x(\cdot) \) is taken to be continuous from the left.
2.1. POISSON COUNTERS

Figure 2.3: Showing a sample path of a solution.

**Notation:** When this definition is in force it is common to rewrite equation (2.1) as

\[ dx = f(x,t)dt + g(x,t)dN \]

**Example:** Consider

\[ dx = xdt + xdN \ ; \ x(0) = 1 \]

where \( N \) is a Poisson counter of rate \( \lambda \). Then \( x(t) \) will be of the form

\[
x(t) = \begin{cases} 
e^t & 0 \leq t \leq t_1 \\
2e^t & t_1 < t \leq t_2 \\
4e^t & t_2 < t \leq t_3 \\
\ldots 
\end{cases}
\]

if \( t_1, t_2, \ldots \) are the times at which the jumps of \( N \) occur.

**Beware:** An important part of the definition of the solution of an Itô equation is that the equation must be solved for \( dx \). You cannot manipulate an Itô equation with \( dN \)'s on the right-hand side before solving it. For example, one cannot replace the equation \( dx = xdt + xdN \) by the equation

\[
\frac{dx}{x} = dt + dN
\]

and then integrate, as one would do in the case of ordinary differential equations, to get

\[ x(t) = Me^{N(t)+t} \]

This is not the solution we defined above for the equation \( dx = xdt + xdN \). **DO NOT FAIL TO GRASP THIS POINT.**

How can one simulate such a differential equation on a computer? Most implementations of the better known computer languages allow the program to call a random number generator that returns a “random” number that is, to the accuracy of the real numbers being used, uniformly distributed between zero and one. If \( r_i \) for \( i = 0, 1, 2, \ldots \) are random variables that are independent and uniformly distributed on the interval \([0,1]\) and if we chose a small constant \( h \) as a step size, we can simulate (with an approximation that improves as \( h \to 0 \)) a Poisson counter via a difference equation. Let

\[ a((n+1)h) = a(nh) + m(n) \ ; \ n = 0, 1, 2, \ldots \]
with \( m(n) \) being one if \( r_n \) is smaller than \( h\lambda \) and zero otherwise. This means that in \( h \) units of time the probability of jump is \( h\lambda \). A more accurate version would equate this probability to \( 1 - e^{-\lambda h} \); this could be achieved by equating \( m(n) \) to one if \( r_n \) is smaller than \( 1 - e^{-\lambda h} \) and zero otherwise.

### 2.2 Finite-State, Continuous-Time Jump Processes

Consider a finite set \( X = \{x_1, x_2, \ldots, x_n\} \) and consider a stochastic process \( x_t \) which takes on values in \( X \). Let \( p_i(t) \) be the probability that \( x_t = x_i \) and suppose that

\[
\dot{p}_i(t) = \sum_{j=1}^{n} a_{ij}p_j(t)
\]

The matrix

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

is variously called the *infinitesimal generator* of the process or the *intensity* matrix of the process. Conservation of probability, (i.e., the condition \( \dot{p}_1 + \dot{p}_2 + \cdots + \dot{p}_n = 0 \)) requires that the entries in the columns of \( A \) sum to zero, i.e.,

\[
\sum_{i=1}^{n} a_{ij} = 0 \quad ; \quad j = 1, 2, \ldots, n
\]

Non-negativity of the \( p_i \) requires that we impose the condition

\[
a_{ij} \geq 0 \quad ; \quad i \neq j
\]

for otherwise we would get negative value for \( p_i(t) \) if \( p_j(0) = 1 \) and \( a_{ij} \) were negative for some \( i \neq j \). These two conditions and no others are required for \( A \) to be an intensity matrix. Of course these imply \( a_{ii} \leq 0 \). Because the columns sum to zero the set of all \( n \) by \( n \) intensity matrices is parameterized by \( n(n - 1) \) parameters which we may take to be the \( (a_{ij}) \) for \( i \neq j \).

**Example 1:** Consider a continuous time jump process taking on values in the set \( \{4, 2, -5\} \). Let \( p_1 \) be the probability that \( x = 4 \), \( p_2 \) be the probability that \( x = 2 \) and \( p_3 \) be the probability that \( x = -5 \). Let the probabilities evolve according to

\[
\frac{d}{dt} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}
\]

find the steady state probability distribution. Find the steady state mean of \( x \). Find the steady state value of \( \mathcal{E}(x - \bar{x})^2 \). Make a quantitative statement about how fast \( p \) approaches its steady state value.
2.2. FINITE-STATE, CONTINUOUS-TIME JUMP PROCESSES

**Solution:** If the steady state eigenvector has components \([a, b, c]\) then from the first component we see \(a = b\) and from the second \(b = c\) and so the steady state is \([1/3, 1/3, 1/3]\). Thus the expectation of \(x\) is \(4/3 + 2/3 - 5/3 = 1/3\). Similarly, \(\mathcal{E}(x - \bar{x})^2 = 1/3(121/9 + 25/9 + 196/9) = 402/27 = 134/9\). The real parts of the two nonzero eigenvalues are \(-1.5\) and the rate of approach is \(e^{-1.5t}\).

We can give a sample path realization of such a process in terms of Poisson counters. Suppose we consider

\[
\dot{x} = -2xdN; \quad x(0) = 1
\]

then \(x\) jumps first to \(-1\), then back to \(+1\), etc., so we have defined a finite state process. (If \(x(0) = a\) then \(x(t)\) takes on the values \(a\) and \(-a\).) If we let \(p_1\) be the probability that \(x(t) = 1\) and let \(p_2\) be the probability that \(x(t) = -1\), then the corresponding description in terms of an intensity matrix is expressible in terms of the rate of the counter \(N\) as

\[
\begin{bmatrix}
\dot{p}_1 \\
\dot{p}_2
\end{bmatrix} =
\begin{bmatrix}
-\lambda & \lambda \\
\lambda & -\lambda
\end{bmatrix}
\begin{bmatrix}
p_1 \\
p_2
\end{bmatrix}
\]

(2.3)

To see this notice that the probability that there will be a jump in \(\Delta t\) seconds is \(\lambda \Delta t + \) higher order terms in \(\Delta t\). The following language will be used. We speak of (2.2) as being a sample-path description of the process and speak of (2.3) as being a probabilistic description.

We now describe two different general procedures for constructing sample path realizations for finite state, continuous time Markov processes using Poisson counters. These are but two of many possibilities.

**Unit Vector Representations:** Suppose we have a process that takes values in a set \(S\) with \(n\) elements, \(s_1, s_2, ..., s_n\) and which satisfies the probability law \(\dot{p} = Ap\) with \(p_i(t)\) being the probability that the state at time \(t\) is \(s_i\). We represent the states as the \(n\) standard basis vectors in \(\mathbb{R}^n\), i.e., \(S\) is mapped to \(\{e_1, e_2, ..., e_n\}\). The sample path description is constructed using at most \(n(n-1)\) Poisson counters \(N_{ij}\) with \(i \neq j\). The Itô equation is

\[
dx = \sum_{i=1}^{n} \sum_{j=1}^{n} G_{ij} xdN_{ij}; \quad i \neq j
\]

with \(G_{ij} = (e_i - e_j)e_j^T\). The rates of the counters are given by

\[
\lambda_{ij} = a_{ij}; \quad i \neq j
\]

The resulting Itô equation generates a Markov process whose transition probabilities are \(a_{ij}\) as required. We refer this representation of the process as a unit vector representation. Insofar as finite state Markov processes are concerned, this representation is completely general although it may not be the most economical in terms of minimizing the number of independent Poisson counters. Note that \(\mathcal{E}x(t) = p(t)\).

**Example 2:** Suppose that we have a three state process whose probability law is

\[
\begin{bmatrix}
\dot{p}_1(t) \\
\dot{p}_2(t) \\
\dot{p}_3(t)
\end{bmatrix} =
\begin{bmatrix}
-3 & 0 & 8 \\
3 & -2 & 0 \\
0 & 2 & -8
\end{bmatrix}
\begin{bmatrix}
p_1(t) \\
p_2(t) \\
p_3(t)
\end{bmatrix}
\]
Coding the states as elements of $e_1, e_2, e_3 \in \mathbb{R}^3$, we see that if $x(0) \in \{e_1, e_2, e_3\}$ and have
\[
\begin{bmatrix}
  dx_1 \\
  dx_2 \\
  dx_3
\end{bmatrix} =
\begin{bmatrix}
  -dN_{21} - dN_{31} & 0 & dN_{13} \\
  dN_{21} & -dN_{32} & 0 \\
  dN_{31} & dN_{32} & -dN_{13}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
\]
with the rates of the counters being $\lambda_{13} = 8, \lambda_{21} = 3, \lambda_{32} = 2$, then we have a sample path realization.

**Real Number Representations:** In this scheme we code the states $\{s_1, s_2, \ldots, s_n\}$ of the finite-state, continuous-time jump process as distinct real numbers, say $z_1, z_2, \ldots, z_n$. Let $\phi_{ij}$ for $i, j = 1, 2, \ldots, n$ and $i \neq j$ be such that
\[
\phi_{ij}(z) = \begin{cases} 
0 & \text{if } z \neq z_j \\
 z_i - z_j & \text{if } z = z_j
\end{cases}
\]
Let $N_{ij}$ be a Poisson counter with rate $\lambda_{ij}$. Consider
\[
d z = \sum_{i \neq j=1}^n \phi_{ij}(z) dN_{ij} ; \quad z(0) \in \{z_1, z_2, \ldots, z_n\}
\]
(a) If this process starts in the set $\{z_1, z_2, \ldots, z_n\}$ it remains in the set $\{z_1, z_2, \ldots, z_n\}$
(b) If $p_i(t)$ is the probability that $x(t) = z_i$, then
\[
\dot{p}_i = \sum_{j=1}^n \lambda_{ij} p_j - \left( \sum_{j=1}^n \lambda_{ij} \right) p_i
\]
(Notice there is no dependence on $\lambda_{ii}$ because the terms cancel.) Thus to obtain a realization of $\dot{p} = Ap$ choose the rates $\lambda_{ij} = a_{ij}$ for $i \neq j$. Notice that the Lagrange interpolation formula assures that one can always take the $\phi_{ij}$ to be polynomials of degree $n$ or less. This gives a second general method for constructing sample path descriptions for continuous-time jump process.

These constructions illustrate that we may think of Poisson counters as having a “universal” property insofar as their ability to represent continuous time jump processes.

**Example 3:** Consider again the process defined in example 1. Let the states be coded as $-1, 0, 1$. We need to find functions $\phi_1(x), \phi_2(x)$ and $\phi_3(x)$ and counting rates $\lambda_1, \lambda_2, \lambda_3$ such that if $x(0) \in \{-1,0,1\}$ then for
\[
dx(t) = \phi_1(x(t))dN_1 + \phi_2(x(t))dN_2 + \phi_3(x(t))dN_3
\]
it happens that $x(t)$ belongs to $\{-1,0,1\}$ for all future time and the transition probabilities are as above, with $p_1(t)$ being the probability that $x(t)$ is $-1$, $p_2(t)$ being the probability that $x(t) = 0$ and $p_3(t)$ being the probability that $x(t)$ is $1$. To get the appropriate $\phi$’s we see that
\[
\phi_1(x) = (x)(x-1)/2
\]
2.3. The Itô Rule for Jump Processes

If we have a stochastic differential equation

\[ dx = f(x)dt + \sum_{i=1}^{m} g_i(x)dN_i ; \quad x \in \mathbb{R}^n \]

and if \( \psi : \mathbb{R}^n \rightarrow \mathbb{R} \) is a given function, then of course \( \psi(x(t)) \) is a stochastic process. Using our definition of the solution of an Itô equation, we see that

\[ d\psi(t,x) = \frac{\partial \psi}{\partial t} dt + \left( \frac{\partial \psi}{\partial x} , f(x) \right) dt + \sum_{i=1}^{n} [\psi(t,x + g_i(x)) - \psi(t,x)] dN_i \]

This is the so-called Itô rule for jump processes. In writing it down we have used the fact that the probability that two counters will jump at the same time is zero and therefore such a possibility can be ignored. Notice that in this context the Itô rule is a completely trivial consequence of the definition of the solution of the differential equation.

As discussed above when the solution for the Itô equation was defined, it is important to keep in mind that the value of \( x \) is “frozen” at its left limit until the jump is completed. This means that the differential equation can be given a different meaning if it is manipulated using some of the ordinary rules available for deterministic differential equations.

**Example**: Given that

\[ dx(t) = -x(t)dt + dN_1(t) - dN_2(t) \]

where \( N_1 \) and \( N_2 \) are Poisson counters of rates \( \lambda_1 \) and \( \lambda_2 \), respectively, what is the equation for \( x^2 \)? An application of the Itô rule gives

\[ dx^2(t) = -2x^2(t)dt + (x(t) + 1)^2 - x^2(t) dN_1(t) + (x(t) - 1)^2 - x^2(t) dN_2(t) \]

\[ = -2x^2(t)dt + (2x(t) + 1) dN_1(t) + (1 - 2x(t)) dN_2(t) \]
2.4 Computing Expectations

If $x$ satisfies the differential equation

$$dx = f(x, t)dt + \sum_{i=1}^{m} g_i(x, t)dN_i$$

how can we get an equation for the expectation of $x$? The key observations here are that

(a) $(\mathcal{E}N(t) - \lambda t) = 0$ (This was established in section 2.1) and

(b) the probability that $N(t)$ will jump in the interval $[t, t + \Delta]$ is independent of the value of $x(t)$.

Thus

$$\mathcal{E}(x(t + \Delta) - x(t)) = \mathcal{E} \int_{t}^{t+\Delta} f(x(\sigma), \sigma) d\sigma + \sum_{i=1}^{m} \mathcal{E} \int_{t}^{t+\Delta} g_i(x(\sigma), \sigma)dN_i(\sigma)$$

Expanding in a Taylor series and taking the limit as $\Delta$ goes to zero we get

$$\frac{d}{dt} \mathcal{E} x(t) = \mathcal{E} f(x(t), t) + \sum_{i=1}^{m} \left( \mathcal{E} g_i(x(t), t) \right) \lambda_i$$

Thus the general rule for computing expectations is to replace $dN_i$ by $\lambda_i dt$, divide by $dt$ and take expectations.

**Example 1:** Given

$$dx = -x(t)dt + dN_1 - dN_2$$

the above rule implies that

$$\frac{d}{dt} \mathcal{E} x = -\mathcal{E} x + \lambda_1 - \lambda_2$$

Using the Itô rule for $x^2$ and the calculating the expectation we get

$$\frac{d}{dt} \mathcal{E} x^2 = -2\mathcal{E} x^2 + \mathcal{E} (2x + 1)\lambda_1 + \mathcal{E} (1 - 2x)\lambda_2$$

which expands to give

$$\frac{d}{dt} \mathcal{E} x^2 = -2\mathcal{E} x^2 + 2\mathcal{E} x(\lambda_1 - \lambda_2) + \lambda_1 + \lambda_2$$

If $\lambda_1 = \lambda_2$ we see that

$$x^2(t) = \left( \mathcal{E} x^2(0) - \frac{\lambda_1 + \lambda_2}{2} \right) e^{-2t} + \frac{\lambda_1 + \lambda_2}{2}$$

**Example 2:** Let $N$ be a Poisson counter of rate $\lambda$ and let $x$ and $z$ satisfy $z(t) \in \{-1, 1\}$

$$dx = -xdt + zdt$$
We wish to find an equation for the variance of \(x\). We begin by using the differentiation rule to write

\[ dx^2 = -2x^2 dt + 2xz dt \]

\[ d(xz) = (-xz + z^2) dt - 2xz dN \]

Taking expectations we get

\[ \frac{d}{dt} \mathcal{E} x^2 = -2 \mathcal{E} x^2 + 2 \mathcal{E} xz \]

\[ \frac{d}{dt} \mathcal{E} xz = -(1 + 2\lambda) \mathcal{E} xz + 1 \]

These linear equations can then be solved by standard means. Notice that steady-state value is

\[ \lim_{t \to \infty} \mathcal{E} x^2(t) = 1/(1 + 2\lambda) \]

2.5 The Equation for the Density

In order to gain a better understanding of the class of problems introduced in 2.2 we now turn to the problem of finding an evolution equation for the probability law associated with the state. This will only be partly successful, however, because the equation we derive will only be useful under suitable conditions. The situation is complicated for a variety of reasons, one of which is that the solution of stochastic differential equations involving Poisson counters need not have a continuous density. For example, if \(x\) satisfies

\[ dx = xdN_1 - 2xdN_2 \; ; \; x(0) = \text{rational} \]

it will never take on an irrational value. However, when the stochastic equation has a smooth drift term and an initial density which is smooth, such difficulties are often absent and in this situation there is a useful evolution equation for the density.

The derivation of the equation for the density proceeds by computing the time rate of change of the expected value of a “test” function in two different ways and then equating these.

Let \(\mathcal{A}\) denote the set of all subsets of \(\mathbb{R}^n\) which are parallepipeds; i.e., sets that are the intersections of \(n\) slices of the form \(\{x|a_i \leq x \leq b_i\}\). Given an equation

\[ dx = f(x(t)) dt + \sum_{i=1}^{m} g_i(x(t)) dN_i \]

suppose that there exists a differentiable function \(\rho\) such that for all sets \(A \subset \mathcal{A}\)

\[ p(x(t) \in A) = \int_A \rho(t,x) dx \]

In this case it is of interest to find an equation for the evolution of \(\rho\). The device to be used here is worth noting because it is a basic tool which will also appear in the next chapter.
First of all introduce a smooth “test function” $\psi$, a map of $\mathbb{R}^n$ into $\mathbb{R}$. Use the Itô rule to get

$$d\psi = \frac{\partial \psi}{\partial x} f dt + \sum_{i=1}^{m} \left( \psi(x + g_i(x)) - \psi(x) \right) dN_i$$

Next we compute the expected value of $\psi$ using the results of section 2.4. This gives

$$\frac{d}{dt} E\psi(x(t)) = E\frac{\partial \psi}{\partial x} f + E\sum_{i=1}^{m} \left( \psi(x(t) + g_i(x(t))) - \psi(x(t)) \right) \lambda_i$$

Now notice that if $\rho$ exists then we can also compute the expectation of $\psi$ by integrating against $\rho$. That is,

$$E\psi(x) = \int \psi(x) \rho(t, x) dx$$

Differentiating this expression with respect to time gives

$$\frac{\partial}{\partial t} \int \psi(x) \rho(t, x) dx = \int \left[ \frac{\partial \psi(x)}{\partial x} f + \sum_{i=1}^{m} \left( \psi(x + g_i(x)) - \psi(x) \right) \lambda_i \right] \rho(t, x) dx$$

If $\rho(t, x)$ is smooth function of $x$ we may use integration-by-parts to get

$$\int \psi(x) \frac{\partial \rho(t, x)}{\partial t} dx = -\int \psi \frac{\partial}{\partial x} (f\rho) - \sum_{i=1}^{m} \lambda_i \psi \rho dx + \sum \int \lambda_i \psi(x + g_i(x)) \rho(t, x) dx$$

provided $\psi(x) = 0$ for $|x|$ sufficiently large.

In order to go further it is necessary to make some assumptions about the function defined by

$$\tilde{g}_i(x) = x + g_i(x)$$

If, for example, $x$ takes on values in $\mathbb{R}^1$ and $\tilde{g}$ defines a map of $\mathbb{R}^1$ onto $\mathbb{R}^1$ which is one to one, then letting $z = \tilde{g}(x)$ we have

$$dz = |\det(I + \frac{\partial g_i}{\partial x})| dx$$

This allows us to use a change of variables to get

$$\int \psi(x + g_i(x)) \rho(t, x) dx = \int \psi(z) \rho(t, \tilde{g}_i^{-1}(z)) \left| \det(I + \frac{\partial g_i}{\partial x}) \right|^{-1} dz$$

In this case we can argue that because $\psi$ is arbitrary this integral equation can be replaced by the differential difference equation.

$$\frac{\partial \rho(t, x)}{\partial t} = -\frac{\partial}{\partial x} \left( f(x) \rho(t, x) \right) + \sum_{i=1}^{m} \lambda_i \left( \rho(t, \tilde{g}_i^{-1}(x)) \left| \det(I + \frac{\partial g_i}{\partial x}) \right|^{-1} x + g_i(x) \right) - \rho(t, x)$$

This is, then, an evolution equation for the density, provided a smooth density exists. If $\tilde{g}$ is not one to one but yet has a finite number of solutions for each point in the range, this argument can be modified to yield a density equation.

**Example 1:** Consider the linear equation

$$dx = -x dt + dN_1 - dN_2$$
2.5. **THE EQUATION FOR THE DENSITY**

with \( N_1 \) and \( N_2 \) being standard Poisson counters of rate \( \lambda \). It can be thought of as a stabilized version of the bidirectional counter. If a smooth density exists it satisfies

\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} x \rho + \lambda \rho(t, x - 1) - 2\lambda \rho(t, x) + \lambda \rho(t, x + 1)
\]

This is an example of a functional differential equation. In spite of the linearity and simple appearance, explicit solutions of this type of equation are rare. On the other hand, qualitative information about the solution can often be obtained. In this case there is a steady state solution satisfying

\[
0 = \frac{\partial}{\partial x} x \rho + \lambda \rho(t, x - 1) - 2\lambda \rho(t, x) + \lambda \rho(t, x + 1)
\]

and this latter equation can be solved using Fourier transforms.

**Example 2:** Consider a model for a *queueing problem* in which customers arrive for service and service is provided as long as there is a customer to be served. Assume that the arrival of customers is modeled by a Poisson process of rate \( \lambda \) and that each customer requires \( \mu \) amount of time to be served. (We could also let \( \mu \) be random at the expense of some extra work.) Let \( f_p(x) \) denote the function whose value is 1 if \( x \) is positive and zero if \( x \) is non positive. The variable \( x \) in the differential equation

\[
dx = -\frac{1}{\mu} f_p(x) dt + dN
\]

then describes the total effort that the server must provide to meet the demands of those presently in the queue, including the customer currently being served. The probability density satisfies

\[
\frac{\partial \rho(t, x)}{\partial t} = \frac{\partial f_p(x) \rho(t, x)}{\mu \partial x} + \lambda \rho(t, x - 1) - \lambda \rho(t, x)
\]

We expect that there will be a steady state condition if the arrival rate is smaller than the service rate, i.e. if \( \lambda \mu < 1 \). The steady state density must satisfy

\[
\frac{\partial f_p(x) \rho(x)}{\mu \partial x} - \lambda \rho(x) = -\lambda \rho(x - 1)
\]

Of course there is a nonzero probability that the queue will be empty. Taking expectations of both sides of the sample path equation we see that in steady state \( E f_p(x) = \lambda \mu \). Thus in steady state the probability that the queue is empty is just \( 1 - \lambda \mu \). There is a steady state solution of the density equation having delta function of this strength and taking the form of a delta function plus a density

\[
\rho_{ss}(x) = (1 - \lambda \mu) \delta(x) + \psi(x)
\]

We can solve for \( \psi \) by dividing the positive half-line into intervals of unit length. In the segment \((0, 1)\) the \( \rho(x - 1) \) term is zero and thus we need only solve the homogenous equation

\[
\frac{1}{\mu} \frac{d \rho}{dx} - \lambda \rho = 0
\]

Thus on \((0, 1)\) we have a segment of the growing exponential, \( \rho(x) = \beta e^{\lambda \mu x} \). In order to evaluate \( \beta \), we observe that the drift term adds probability to the point \( x = 0 \) at the rate
\[ \rho(0^+)/\mu = \beta/\mu \] and the counter removes probability from this point at the rate \((1 - \lambda \mu)\lambda\). In equilibrium these must be equal so \(\beta = (1 - \lambda \mu)\lambda \mu\). At the point \(x = 1\) the impulse at 0 makes itself felt through the contribution of the \(\rho(x-1)\) term in

\[ \frac{1}{\mu} \frac{d\rho(x)}{dx} + \lambda \rho(x) = +\lambda(1 - \lambda \mu)\delta(x - 1) \]

Thus the solution jumps down by \(\lambda \mu(1 - \lambda \mu)\) at \(x = 1\) and on the interval \((1, 2)\) satisfies the equation

\[ \frac{1}{\mu} \frac{d\rho}{dx} - \lambda \rho = (1 - \lambda \mu)\lambda \mu e^{-\mu x} \quad ; \quad x(1^+) = \lambda \mu(1 - \lambda \mu)(e^{\lambda \mu} - 1) \]

This is a linear differential equation with a confluence between the forcing term and the homogeneous response so the solution takes the form

\[ \rho(x) = (a x + b)e^{\lambda \mu x} \quad ; \quad (a + b)e^{\lambda \mu} = \lambda \mu(1 - \lambda \mu)(e^{\lambda \mu} - 1) \]

With \(a\) and \(b\) being constrained by fact that

\[ \lim_{x \to 1, x<1} \rho(x) = (1 - \lambda \mu)\lambda \mu e^{\lambda x} \]

Continuing on, for the interval \([2, 3]\) we have the differential equation

\[ \frac{1}{\mu} \frac{d\rho}{dx} - \lambda \rho = (a(x - 1) + b)e^{\lambda \mu(x - 1)} \]

which has a double confluence and hence a solution of the form of an exponential times a quadratic polynomial, etc. It is of interest to note that while the exponential involved grows with \(x\), the polynomials are such that \(\rho\) decays to zero. There is an impulse at \(x = 0\), a discontinuity at \(x = 1\), a discontinuous first derivative at \(x = 2\), etc. with the function being continuous for \(x > 1\) and becoming more regular as \(x\) gets larger.

![Figure 2.4: A sketch of the steady state probability density for the queueing problem.](image-url)

**Example 3.** Consider the coupled equations of example 2 of the previous section. In this case it is convenient to write the equation for the density as a pair of coupled equations. Let \(\rho_+(t, x)\) be the probability density associated with \(x\), given that \(z = +1\) and let \(\rho_-(t, x)\) be the probability density associated with \(x\) given that \(z = -1\). In terms of this notation we have

\[ \frac{\partial \rho_+(t, x)}{\partial t} = \frac{\partial}{\partial x}(x - 1)\rho_+(t, x) + \lambda(\rho_-(t, x) - \rho_+(t, x)) \]

\[ \frac{\partial \rho_-(t, x)}{\partial t} = \frac{\partial}{\partial x}(x + 1)\rho_-(t, x) + \lambda(\rho_+(t, x) - \rho_-(t, x)) \]
Example 4: Let \( f \) be a function \( f : [0, \infty) \rightarrow \{0, 1\} \) such that \( f(0) = 0 \) and \( f(x) = 1 \) for \( x > 0 \). Consider the Itô equation
\[
dx = dN_1 - f(x)dN_2 ; \quad x(0) = 0
\]
with \( N_1 \) being a standard Poisson counter of rate \( \lambda \) and \( N_2 \) being a standard Poisson counter of rate \( \mu > \lambda \). In this case \( x \) will only assume integer values. Thus the steady state probability distribution can be characterized as follows. To begin, notice that by taking expectations of both sides we see that in steady state
\[
0 = \lambda - \mu \mathcal{E} f ; \quad \mathcal{E} f = \lambda/\mu
\]
Moreover, in steady state we have \( 0 = \lambda(p_{n-1} - p_n) - \mu(p_{n-1} - p_n) \) provided that \( n \geq 1 \). Thus we see that
\[
\frac{p_{n+1} - p_n}{p_{n-1} - p_n} = \frac{\mu}{\lambda}
\]
and so if we let \( a = \lambda/\mu \) then
\[
p_n = (1 - a)a^n
\]
As for the moments,
\[
\mathcal{E}x = \sum_{n=0}^{\infty} n(1-a)a^n = (1-a) \sum_{n=0}^{\infty} a \frac{d}{da}a^n = (1-a)a \frac{d}{da} 1 - a = \frac{a}{1-a} = \lambda/(\mu - \lambda)
\]
and one can continue in this way to get the higher moments.

It is a little easier to note that when steady state is reached the equation
\[
\frac{d}{dt} \mathcal{E}x^p = \mathcal{E}((x + 1)^p - x^p)\lambda + \mathcal{E}(x - 1)^p - x^p)f \mu
\]
implies
\[
0 = \mathcal{E}((x + 1)^p - x^p)\lambda + \mathcal{E}(x - 1)^p - x^p)f \mu
\]
Then observe that \( \mathcal{E}x^p f = \mathcal{E}x^p \) and \( \mathcal{E} f dN_2 = \frac{\lambda}{\mu} \mu = \lambda \). Thus with \( p = 2 \) we have
\[
0 = \mathcal{E}(2x + 1)\lambda - 2\mathcal{E}x\mu + \mathcal{E} f \implies 2\mathcal{E}x(\mu - \lambda) = 2\lambda
\]
so that \( \mathcal{E}x = \lambda/(\mu - \lambda) \). To get the second moment start with
\[
0 = \frac{d}{dt} \mathcal{E}x^3 = \mathcal{E}((x + 1)^3 - x^3)\lambda + \mathcal{E}(x - 1)^3 - x^3)f \mu
\]
and proceed as above, etc.

2.6 The Backwards Evolution Equation

Although there are many problems in which the forward evolution of the density, as described in the previous section, is the most natural evolution to focus on, there are other problems
for which an evolution in a negative direction is important. For example, if it is known that \( x(1) = x_1 \) we may wish to know the probability density at \( t = 0 \). More generally, given \( \rho(1, x) \) what governs the evolution of \( \rho \) backwards in time? Because the definition of what one means by a solution of the differential equation does not treat \( t \) and \( -t \) symmetrically, it is not to be expected that one can get the evolution of \( \rho \) backwards in time simply by changing the sign of the right-hand side of the equation.

To gain some intuition about this question it is useful to consider a finite state discrete-time process whose transition probabilities are such that

\[
p(i + 1) = Ap(i)
\]

Given that \( x(1) = x_k \), we see that the probability vector at time 0 is determined by the \( k^{th} \) row of \( A \). In fact, if the entries in the \( k^{th} \) row of \( A \) sum to \( s_k \) then

\[
p(0) = \frac{1}{s_k} A^k; \quad A^k = k^{th} \text{ row of } A
\]

We can, therefore, express the backwards equation as

\[
p(i - 1) = A^T D p(i)
\]

with \( T \) denoting the transpose of \( A \) and \( D \) being a diagonal matrix that normalizes the columns of \( A^T \) so that the sum of their entries is one.

The corresponding equation situation in continuous time relates the forward equation \( \dot{p} = Ap \) to the backwards equation

\[
\dot{p} = (A^T + D)p
\]

with \( D \) being a diagonal matrix chosen so as to make the sum of the entries in a column of \( A^T + D \) equal to zero.

**Example:** Consider a continuous-time jump process whose probability law satisfies

\[
\begin{bmatrix}
\dot{p}_1 \\
\dot{p}_2 \\
\dot{p}_3
\end{bmatrix} = \begin{bmatrix}
-1 & 1 & 2 \\
1 & -1 & 1 \\
0 & 0 & -3
\end{bmatrix} \begin{bmatrix}
p_1 \\
p_2 \\
p_3
\end{bmatrix}
\]

To get the backward propagation equation we observe that

\[
A = \begin{bmatrix}
-1 & 1 & 2 \\
1 & -1 & 1 \\
0 & 0 & -3
\end{bmatrix}, \quad A^T = \begin{bmatrix}
-1 & 1 & 0 \\
1 & -1 & 0 \\
2 & 1 & -3
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{p}_1 \\
\dot{p}_2 \\
\dot{p}_3
\end{bmatrix} = (A^T + D)p = \begin{bmatrix}
-3 & 1 & 0 \\
1 & -2 & 0 \\
2 & 1 & 0
\end{bmatrix} p = Bp
\]

\[
D = \begin{bmatrix}
-2 & -1 \\
& 3
\end{bmatrix}
\]
2.7. COMPUTING TEMPORAL CORRELATIONS

\[ p(0) = e^B p(1) \]

The eigenvalues of \( B = A^T + D \) are \( \lambda = 0, -1, -3 \). Some work shows that

\[
e^Bt = \begin{bmatrix}
e^{-2t} \cosh t & e^{-2t} \sinh t & 0 \\
e^{-2t} \sinh t & e^{-2t} \cosh t & 0 \\
1 - e^{-t} & b(t) & 1
\end{bmatrix}
\]

Given that \( p_1(1) = 1, p_2(1) = 0, p_3(1) = 0 \), what is the value of \( p(0) \)? It is simply the first column of this matrix evaluated at \( t=1 \).

### 2.7 Computing Temporal Correlations

Let \( x_t \) be a finite-state, continuous-time jump process which takes on values in the set \( \{x_1, x_2, \ldots, x_n\} \subset \mathbb{R} \). To compute the expected value of \( \mathcal{E}(x_t x_{t+\tau}) \) we let \( \psi_{ij}(t, \tau) \) denote the probability that \( x(t) = x_i \) and \( x(t+\tau) = x_j \) and use the formula

\[
\mathcal{E}(x_t x_{t+\tau}) = \sum_{i,j} \psi_{ij}(t, \tau) x_i x_j
\]

Now the probability that \( x_t = x_i \) is just \( p_i(t) \). The probability that \( x_{t+\tau} = x_j \) given that \( x_t = x_i \) is just the \( ji^{\text{th}} \) element of \( e^{A\tau} \), provided \( \tau \geq 0 \). Thus we see that for \( \tau \geq 0 \).

\[
\mathcal{E}(x_t x_{t+\tau}) = \sum_{i,j} p_i(t) \phi_{ij}(\tau) x_i x_j
\]

where \( \phi_{ij}(\tau) \) is the \( ij^{\text{th}} \) entry in \( e^{A\tau} \). Of course if we are given \( p_i(0) \), then this is

\[
\mathcal{E}(x_t x_{t+\tau}) = \sum_{i,j,k} \phi_{ik}(t)p_k(0) \phi_{ji}(\tau) x_i x_j
\]

If we want to write this in matrix notation, it is

\[
\mathcal{E}(t) x(t + \tau) = [x_1(t), x_2(t), \ldots, x_n(t)] e^{A\tau} = \begin{bmatrix}
p_1(t) & 0 & 0 \\
0 & p_2(t) & 0 \\
0 & 0 & \ddots
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
\vdots \\
x_n(t)
\end{bmatrix}
\]

An application of the same methodology yields formulas for more complicated statistical properties such as

\[
\phi(t_1, t_2, \ldots, t_n) = \mathcal{E}(x_{t_1} x_{t_2} \ldots x_{t_n})
\]

There is a second approach which begins with a sample path description. Consider

\[
dx_t = f(x_t) dt + \sum g_i(x_t) dN_{it}
\]
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\[
\frac{d}{d\tau} \mathcal{E} x_t x_{\tau} = \mathcal{E} x_t f(x_{\tau}) + \sum \mathcal{E} x_t g_i(x_{\tau}) \lambda_i
\]

Now suppose that \( f \) is linear and \( g \) is constant. Then

\[
\frac{d}{d\tau} \mathcal{E} x_t x_{\tau} = \mathcal{E} \alpha x_t x_{\tau} + \mathcal{E} x_t \beta \lambda_i
\]

To use this we must obtain previously \( \mathcal{E} x^2(t) \) because this will be the initial data for the computation of the \( \tau \) dependence. Again the sample path approach is very effective.

**Example:** Consider the stochastic process \( y \) defined by

\[
y(t) = \begin{cases} 
\alpha & \text{if } x(t) = 1 \\
\beta & \text{if } x(t) = -1 
\end{cases}
\]

where the transition probabilities for the \( x \) process are such that

\[
\begin{bmatrix} \hat{p}_1 \\
\hat{p}_{-1} \end{bmatrix} = \begin{bmatrix} -a & b \\
\alpha & -b \end{bmatrix} \begin{bmatrix} p_1 \\
p_{-1} \end{bmatrix}
\]

\((p_1 = \text{probability } x = 1; p_{-1} = \text{probability } x = -1)\).

(a) Find conditions such that

\[
\lim_{t \to \infty} \mathcal{E} y(t) = 0
\]

(b) Find conditions such that (a) holds and

\[
\lim_{t \to \infty} \mathcal{E} y(t) y(t + \tau) = e^{-|\tau|}
\]

Begin by finding a sample-path description for \( y(t) \); i.e., by finding appropriate \( \phi_{ij} \) as in section 2.2. Consider the specification

\[
\phi_{\alpha\beta}(y) = \begin{cases} 
0 & \text{if } y \neq \beta \\
\alpha - \beta & \text{if } y = \beta
\end{cases}
\]

for which \( \phi_{\alpha\beta}(y) = \alpha - y \) is suitable. By the same token, we may choose \( \phi_{\beta\alpha}(y) = \beta - y \). Now consider the process whose sample-path description is

\[
dy = (\alpha - y)dN_{\alpha\beta} + (\beta - y)dN_{\beta\alpha},
\]

with counters of rates \( \lambda_{\alpha\beta}, \lambda_{\beta\alpha} \) respectively; if \( y = \alpha \) this reduces to \( dy = (\beta - \alpha)dN_{\beta\alpha} \), so that \( \hat{p}_\beta = \lambda_{\beta\alpha} \) and hence \( \hat{p}_\alpha = -\lambda_{\beta\alpha} \). Repeating this argument assuming \( y = \beta \) and combining, we get

\[
\begin{align*}
\hat{p}_\alpha &= -\lambda_{\beta\alpha}p_\alpha + \lambda_{\alpha\beta}p_\beta \\
\hat{p}_\beta &= \lambda_{\beta\alpha}p_\alpha - \lambda_{\alpha\beta}p_\beta.
\end{align*}
\]
2.7. COMPUTING TEMPORAL CORRELATIONS

Now if we set $\lambda_{\alpha\beta} = b$ and $\lambda_{\beta\alpha} = a$ we recover the probabilistic description given in the problem and hence a sample-path description of $y(t)$:

$$dy = (\alpha - y)dN_b + (\beta - y)dN_a,$$

where the subscripts denote the counting rates.

a) Using the sample-path description we see immediately that

$$\frac{d}{dt}E[y(t)] = a\beta + b\alpha - (a + b)E[y(t)],$$

so that $\lim_{t \to \infty} E[y(t)] = \frac{a\beta + b\alpha}{a + b}$. Thus the required condition is $a\beta + b\alpha = 0$.

b) To find the correlation $E[y(t)y(t + \tau)]$ we first need to find $E[y^2(t)]$. The Itô formula gives

$$\frac{d}{dt}E[y^2(t)] = b(\alpha^2 - E[y^2(t)]) + a(\beta^2 - E[y^2(t)])$$

thus

$$E[y^2(t)] = \frac{a\beta^2 + b\alpha^2}{a + b} + ke^{-(a+b)t}$$

To compute the correlation we rewrite the original Itô equation using $\tau$ as the independent variable:

$$\frac{d}{d\tau}y(\tau) = (\alpha - y(\tau))dN_b(\tau) + (\beta - y(\tau))dN_a(\tau)$$

$$\frac{d}{d\tau}y(t)y(\tau) = (\alpha - y(\tau))y(t)dN_b(\tau) + (\beta - y(\tau))y(t)dN_a(\tau).$$

Assuming $\tau > t$ we take the expectation, e.g., $E[y(t)dN_a(\tau)] = aE[y(t)]$

$$\frac{d}{d\tau}E[y(t)y(\tau)] = -(a + b)E[y(t)y(\tau)]$$

or

$$E[y(t)y(\tau)] = E[y^2(t)]e^{-(a+b)(\tau-t)}$$

for $\tau \geq t$. Using symmetry arguments, changing from $\tau$ to $t + \tau$ and substituting for the variance,

$$E[y(t)y(t + \tau)] = \frac{a\beta^2 + b\alpha^2}{a + b}e^{-(a+b)|\tau|} + ke^{-(a+b)(t+|\tau|)}.$$
In the limit as $t \to \infty$ the second term vanishes; in order to have the first term be the desired quantity we must have $a + b = 1$ and $a\beta^2 + b\alpha^2 = 1$. We can make this a little simpler by noting that $a\beta^2 + b\alpha^2 = (a\beta + b\alpha)(\alpha + \beta) - (a + b)\alpha\beta = 1$.

### 2.8 Linear Systems with Jump Process Coefficients

The polynomial interpolation formula of Lagrange identifies a polynomial $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(z_i) = a_i$ for a given set of pairs $(z_1, a_1), (z_2, a_2) \ldots (z_n, a_n)$ (no $z$’s repeated) and $\phi$ may be taken to be of degree $n$. The explicit formula is

$$
\phi(z) = a_1 \frac{(z - z_2)(z - z_3) \ldots (z - z_n)}{(z_1 - z_2)(z_1 - z_3) \ldots (z_1 - z_n)} + \cdots + a_n \frac{(z - z_1)(z - z_2) \ldots (z - z_{n-1})}{(z_n - z_1)(z_n - z_2) \ldots (z_n - z_{n-1})}
$$

Thus if we wish to find a sample path description of a finite state continuous time jump process we may take it to be of the form

$$
dz = \sum_{i,j} \phi_{ij}(z) dN_{ij}
$$

with $\phi_{ij}$ being polynomials of degree $n$ where $n$ is the number of states.

Now consider the linear stochastic differential equation

$$
\dot{x} = A(z)x + b(z)
$$

where $z$ is a finite state continuous time jump process. We can use the elementary device of appending 1 to $x$ to rewrite this as

$$
\frac{d}{dt} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b(z) & A(z) \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}
$$

thus bringing it to a form in which there is no inhomogeneous term

$$
\frac{d}{dt} \tilde{x} = \tilde{A}(z)\tilde{x}
$$

(The reason for wanting to do this will appear shortly.)

If $z$ is a FSCT jump process, then model it as above with the $\phi_{ij}$ polynomial. Then $A(z)$ can be expressed as

$$
A(z) = A_0 + zA_1 + \ldots z^nA_n
$$

In fact, the Lagrange formula does the job for us again

$$
A(z) = A(z_1) \frac{(z - z_2)(z - z_3) \ldots (z - z_n)}{(z_1 - z_2)(z_1 - z_3) \ldots (z_1 - z_n)} + \cdots + A(z_n) \frac{(z - z_1)(z - z_2) \ldots (z - z_{n-1})}{(z_n - z_1)(z_n - z_2) \ldots (z_n - z_{n-1})}
$$

Thus we can write, dropping the $\tilde{\cdot}$ over $x$,

$$
dz = \sum \phi_{ij}(z) dN_{ij}
$$

$$
\dot{x} = (A_0 + zA_1 + \ldots + z^{n-1}A_{n-1})x
$$
We now take the major step which allows us to put these into a form which permits us to compute the statistical properties of the solutions. It goes like this. Consider, along with the given \( x \) equation, an equation for \( z_2, \ldots, z_n \). Since \( z \) takes on the values of \( z_1, z_2, \ldots, z_n \) and no others, we see that

\[
p(z) = (z - z_1)(z - z_2) \ldots (z - z_n) = 0
\]

and thus \( z^n \) can be expressed as \( z^n = p_0 + p_1 z + \cdots + p_{n-1} z^{n-1} \) for some choice of \( p_i \). Using the Itô rule we then obtain (with \( A \)'s above)

\[
d\begin{bmatrix} x \\ z \\ \vdots \\ z^{n-1} x \end{bmatrix} = \begin{bmatrix} A_0 & A_1 & \cdots & A_{n-1} \\ A_{n-1} & A_0 & \cdots & A_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \begin{bmatrix} x \\ z \\ \vdots \\ z^{n-1} x \end{bmatrix} dt + \sum \begin{bmatrix} \phi^{21}_{ij} & \phi^{22}_{ij} & \cdots & \phi^{2n}_{ij} \\ \phi^{n1}_{ij} & \phi^{n2}_{ij} & \cdots & \phi^{nn}_{ij} \end{bmatrix} \begin{bmatrix} x \\ z \\ \vdots \\ z^{n-1} x \end{bmatrix} dN_{ij}
\]

where \( \phi^{kl}_{ij} \) are such that

\[
dz^{k-1} = \sum_{l,j,i} z^{l-1} \phi^{kl}_{ij} dN_{ij}
\]

What we have accomplished with this device is the reduction of the original problem to one of the form (new notation)

\[
dx = Ax dt + \sum B_i x dN_i
\]

The advantage of this is that we can compute statistical properties of this equation using the calculus developed earlier.

**Example:** Consider the linear equation with an additive jump process term

\[
dx = Ax dt + bz dt
\]

where

\[
dz = -2z dN \quad ; \quad z(0) \in \{-1, 1\}
\]

This set of equations is linear as it stands.

\[
d\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A & b \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} dN
\]

\[
\frac{d}{dt} \mathbb{E} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A & b \\ 0 & -2 \end{bmatrix} \mathbb{E} \begin{bmatrix} x \\ z \end{bmatrix}
\]

\[
\frac{d}{dt} \mathbb{E} x(t) = Ax + b \mathbb{E} z
\]

\[
\frac{d}{dt} \mathbb{E} z(t) = -2 \lambda \mathbb{E} z(t)
\]
To compute the variance notice that \( x \) is continuous so that the Itô rule in this case simplifies to

\[
\begin{align*}
\frac{d}{dt}xx^T &= (Ax + xx^TA^T)dt + (bx + xb^T)zdt \\
\frac{d}{dt}xz &= (Ax + bz^2)dt - 2zxdN \\
\frac{d}{dt}z^2 &= 0
\end{align*}
\]

Solving the second of these we get (note \( E z^2 = 1 \))

\[
\frac{d}{dt}Ezx = A(Ezx) + b - 2EzxA^T
\]

so

\[
Ezx = e^{(A-2\lambda)t}a
\]

\[
\frac{d}{dt}E(xx^T) = A(E(xx^T)) + (E(xx^T))A^T + ba^Te^{(A-2\lambda)t} + e^{(A-2\lambda)t}ab^T
\]

### 2.9 Exercises Chapter 2

1. Consider the stochastic differential equation

\[
dx = \frac{1}{2}x(x-1)dN_1 - \frac{1}{2}x(x+1)dN_2 + (x^2 - 1)dN_3 - (x^2 - 1)dN_4
\]

Suppose that \( x(0) \in \{1, 0, -1\} \) and that the \( N_i \) are Poisson counters. Show that \( x \) evolves in the set \( \{1, 0, -1\} \). If \( p_1(t) \) is the probability that \( x(t) = 1, p_2(t) \) is the probability that \( x(t) = 0 \) and \( p_3(t) \) is the probability that \( x(t) = -1 \) and if \( EN_i = \lambda t, \) find \( A \) such that

\[
\frac{d}{dt} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}
\]

2. Consider the scalar Itô equation

\[
dx = adN_1 - bdN_2
\]

with \( N_1 \) and \( N_2 \) being Poisson counting process of rate \( \lambda \) and \( \mu \), respectively.

(a) Compute \( E x(t) \) in terms of \( a, b, \lambda, \mu \) and \( E x(0) \)

(b) Suppose that we chose \( a, b, \lambda, \mu \) so that \( E x(0) = 0 \) implies \( E x(t) \equiv 0 \). Under this hypothesis compute \( E x^2(t) \).

(c) Under this hypothesis and under certain further conditions on the parameters it will happen that we can let the rates \( \lambda \) and \( \mu \) go to infinity and the amplitudes \( a \) and \( b \) go to zero and get a limiting value for \( E x^2(t) \). What is the most general condition on the coefficients such that this is true?

3. Consider the scalar Itô equation

\[
dx = -1(x-1)(x-2)dN_1 + (x-1)(x-3)dN_2 + 1(x-2)(x-3)dN_3
\]
Assume that the rates of the counters are \( f, g, h \), respectively. Show that it evolves in the set \{1, 2, 3\}. In this case we have a transition matrix in \( \dot{p} = Ap \) of the form
\[
A = \begin{bmatrix}
-h & g & f \\
0 & -g & 0 \\
h & 0 & -f
\end{bmatrix}
\]
Let \( y \) be given by \( y = x_1 - 2x_2 + x_3 \). Find conditions on the rates such that in steady state the mean of \( y \) is zero. Compute the autocorrelation function of \( y \) in steady state assuming that the conditions for the mean to be zero are met.

4. Let \( \alpha, \beta, \gamma, \delta \) be four real numbers with \( 0 < \alpha < \beta < \gamma < \delta \). Let \( N \) be a standard Poisson counter of rate \( \lambda \). Show that the probability that \( N(\alpha) = 0, N(\beta) = 1, N(\gamma) = 1, N(\delta) = 2 \) is given by
\[
p = \lambda^2(\beta - \alpha)(\delta - \gamma)e^{-\lambda\delta}
\]
5. Consider again the scalar Itô equation
\[
dx = adN_1 - bdN_2 \quad x(0) = 0
\]
with \( N_1 \) and \( N_2 \) being Poisson counting process of rate \( \lambda \) and \( \mu \), respectively. Explain in qualitative terms the difference between the properties of the solution when \( a \) and \( b \) are integers, when \( a \) and \( b \) are rational numbers, and \( a \) is rational and \( b \) is irrational.

6. In a certain mixture of radioactive isotopes there are two main reactants having mass \( x \) and \( y \), respectively. In suitable units, these decay according to the stochastic equations
\[
dx = -0.01xdN_1 \quad dy = xdN_1 - 0.001ydN_2
\]
where \( N_1 \) and \( N_2 \) are Poisson counters of rate 1 and 5, respectively. If \( x(0) = 10 \) and \( y(0) = 2 \) what is the smallest value of \( T \) such that we can assert that for all times greater than \( T \)
\[
E|y(t)| \leq 1
\]
7. Consider the unidirectional Poisson counter \( N \) of rate \( \lambda \). As we have seen, the expected value of \( N(t) \) is \( \lambda t \) and
\[
E(N(t) - \lambda t)^2 = \lambda t
\]
Thus if we fix \( t \), say \( t = 1 \) then the process \( (N(t)/\lambda - t) \) has a variance \( 1/\lambda \) and an expected value of zero. Show that
\[
\lim_{\lambda \to \infty} \text{prob}\{N(t)/\lambda > 1\} = \begin{cases} 1 & t > 1 \\ 0 & t \leq 1 \end{cases}
\]
This implies that we can use Poisson counters to model events that are arbitrarily close to certain events.

8. Let \( \dot{p} = Ap \) describe the evolution of the probabilities associated with a continuous time jump process. If \( a_{ij} \) is the \( ij \)th entry of \( A \), show that the the probability that the system has remained in state \( i \) on the interval \([0, T]\), given that it was in state state \( i \) at \( t = 0 \) is \( e^{a_{ii}T} \). By way of contrast, what does the corresponding element of \( e^{At} \) represent?

9. A discrete time, a Markov process has a probability law governed by \( p(k+1) = Tp(k) \) with \( T \) being any matrix with nonnegative entries and columns which sum to one. Denote the set of all such matrices by \( \mathcal{T} \). This stands in contrast with the description of continuous time Markov processes which satisfy \( \dot{p} = Ap \) with the entries of \( A \) being such that the off-diagonals are nonnegative and the columns sum to zero. Denote the class of all such infinitesimaly stochastic matrices by \( \mathcal{A} \). Although \( e^{At} \) with \( A \in \mathcal{A} \) is necessarily in \( \mathcal{T} \), not
every element of $T$ can be expressed this way, or even as a product $e^{A_{1}}e^{A_{2}} \cdots e^{A_{n}}$. In particular, show that
\[
T = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]
cannot be expressed as a product of the form
\[
T = e^{A_{1}}e^{A_{2}} \cdots e^{A_{n}}
\]
with the $A_{i} \in \mathcal{A}$.

10. The discrete time Markov process $p(k+1) = Tp(k)$ with $T$ as defined in the previous problem whose transition law is actually deterministic in the sense that a knowledge of the state of process at any one moment in time determines the state (not just its probability distribution) at any other point in time. The only continuous time analog of such “non-mixing” processes is the trivial process $x(t) = \text{constant}$. If $p(t)$ is a 3-vector, and if
\[
\dot{p}(t) = Ap(t)
\]
with $A$ being infinitessimally stochastic, and if $p(0) = e_{1}$, the standard basis vector, show that
\[
\min_{t > 0} \|e^{At}e_{1} - e_{2}\| + \min_{t > 0} \|e^{At}e_{1} - e_{3}\| \geq \sqrt{2}
\]
where $\|p\| = \sqrt{p_{1}^{2} + p_{2}^{2} + p_{3}^{2}}$. (This describes a sense in which a continuous time finite state jump process can not be too close to a deterministic process.)

11. A person finds that on average he/she receives 30 e-mail messages per day and that there seems to be a tendency for the messages to come in bunches. After collecting suitable statistics it was determined that if the time of arrival is expressed in hours then
\[
\mathcal{E}(t_{i+1} - t_{i}) = 30/24
\]
and
\[
\mathcal{E}(t_{i+1} - t_{i} - \frac{30}{24})(t_{k+i+1} - t_{k+i} - \frac{30}{24}) = ae^{-\gamma|k|}
\]
Build a model for the process that represents the number of e-mails that have accumulated over time, starting at the begining of some particular day. Is there such a model, consistent with the given autocorrelation function that takes the form
\[
dx = \sum \phi_{i}(x)dN_{i}
\]

12. A matrix is said to be doubly stochastic if it is stochastic and its rows sum to one.
(a) Show that every permutation matrix is doubly stochastic.
(b) Show that the set of all doubly stochastic matrices is a convex set characterized by $(n - 1)^{2}$ parameters.
(c) Show that if $\Theta$ is a real $n$ by $n$ orthogonal matrix then the matrix $\Psi$ whose $ij^{th}$ entry is $\psi_{ij} = (\theta_{ij})^{2}$ is doubly stochastic. Such matrices are said to be orthostochastic.
(d) Show that not every doubly stochastic matrix is orthostochastic.
(e) Show that every permutation matrix is orthostochastic.

13. Let $\{T_{1}, T_{2}, \ldots, T_{k}\}$ be a sequence of transition times associated with a two state continuous time jump process with intensity matrix
\[
A = \begin{bmatrix}
-p & q \\
p & -q
\end{bmatrix}
\]
Suppose that $\epsilon_{1}, \epsilon_{2}, \ldots, \epsilon_{k}$ are small and sum to zero. Compute to first order in $\epsilon$ the probability that the process will have transition times $\tilde{T}_{1}, \tilde{T}_{2}, \ldots, \tilde{T}_{n}$ with $T_{1} - \epsilon \leq \tilde{T}_{1} \leq T_{1} + \epsilon$, $T_{2} - \epsilon \leq \tilde{T}_{2} \leq T_{2} + \epsilon$, $\ldots$, $T_{k} - \epsilon \leq \tilde{T}_{k} \leq T_{k} + \epsilon$?
14. In the text we have taken the counting rates to be constant. If the counting rates are state dependent then it is often possible to proceed in a similar way but of course the formula for the expectation will need to reflect the dependence. As a specific example, consider the pair of equations
\[ dx = (-x + z)dt \]
\[ dz = -2zdN \]
with \( N \) being a Poisson counter whose rate is \( 1 + |x| \). Are the following equations for \( E_x \) and \( E_z \) correct?
\[ \frac{d}{dt} E_x = -E_x + E_z \]
\[ \frac{d}{dt} E_z = -E(1 + |x|)z \]

15. Let \( N \) be a Poisson Counter of rate \( \lambda \). Derive the Fokker-Planck equation for
\[ dx = -2xdt + 5xdN \]

16. Let \( N_1 \) and \( N_2 \) be Poisson counters of rate \( \lambda \). For
\[ dx = -xdt + dN_1 - dN_2 \]
the probability density, if it exists, satisfies
\[ \frac{\partial p}{\partial t}(t, x) = \frac{\partial}{\partial x} x \rho(t, x) + \left( \rho(t, x + 1) + \rho(t, x - 1) - 2\rho(t, x) \right) \lambda \]
(a) When will \( \rho(t, x) \) be continuous? (Discuss at an intuitive level.)
(b) Describe the qualitative behavior of \( \rho(t, x) \) as \( t \) becomes large - make a sketch.
(c) Back up your analysis by computing the differential equation for \( E_x^n(t) \)
and using it to solve for \( \lim_{t \to \infty} E_x^2(t) \).

17. Suppose that \( x \) satisfies the linear differential equation
\[ \dot{x} = Ax + bu \quad ; \quad x(0) = 0 \]
with \( u \) being a finite state continuous time jump process taking on the values 1, 0, and -1 with transition probabilities
\[ \frac{d}{dt} \begin{bmatrix} p_{+1} \\ p_0 \\ p_{-1} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} p_{+1} \\ p_0 \\ p_{-1} \end{bmatrix} \]
Find a differential equation for the expected value of \( xx^T \). (Observe that if \( z \) only takes on the values 1, 0, -1 then \( z^3 = z \). Thus although you will need to consider equations for \( xx^T z \) and \( xx^T z^2 \), because \( xx^T z^3 = xx^T z \) one can get a closed set of equations involving no powers of \( z \) higher than 2.)

18. Find a set of ordinary (deterministic) differential equations (some in \( t \) and some in \( \tau \)) whose solution will determine \( E_x(t) x(t + \tau) \).

19. Let \( N \) be a Poisson counter with rate \( \lambda \). Compute the steady state covariance for
\[ dx_1 = -2x_1dN \quad ; \quad x_1(0) = 1 \]
\[ dx_2 = -x_2dt + x_1dt \quad ; \quad x_2(0) = 0 \]
i.e., compute
\[ M(\tau) = \lim_{t \to \infty} E \begin{bmatrix} x_1(t)x_1(t + \tau) & x_1(t)x_2(t + \tau) \\ x_2(t)x_1(t + \tau) & x_2(t)x_2(t + \tau) \end{bmatrix} \]
20. A Markov renewal process is a generalization of the continuous time jump processes considered here. In this case one has a finite or countable state space $X$ and the process jumps from one state to another as in the continuous time jump process but in this case there is a distribution $q(i, j, t)$ which is the probability distribution for the time spent in state $j$ before jumping to state $i$. Clearly

$$\lim_{t \to \infty} q(i, j, t) = p_{ij}$$

must define a matrix with nonnegative entries whose columns sum to one. Show that if $q(i, j, t)$ takes the special form

$$q(i, j, t) = p_{ij}(1 - e^{-\lambda_{ij}t})$$

with $p_{ij}$ and $\lambda_{ij}$ constant then the Markov renewal process is a continuous time jump process.

21. Euler’s method for generating an approximate solution to a ordinary differential equation

$$\dot{x} = f(x)$$

with a given initial condition $x(0)$ consists of picking a step size $h$ and solving the difference equation $x(t+h) = x(t) + hf(x(t))$. It is but the simplest of many such methods and in a deterministic setting it is used less often than the more accurate higher order methods. However, it is used for stochastic differential equations because the higher order methods involve difficulties stemming from issues involving statistical independence. Study the following Matlab code and run it.

```matlab
x(1)=1; a=.95;
for i=2:50 f(i)=rand; n(i)=1-(-a+f(i) -abs(a-f(i)))/(2*(f(i)-a)); x(i)= .9*x(i-1) +n(i); end
plot(x(:)) hold plot(n(:))
```

If it is interpreted as generating the solution to a stochastic differential equation of the form

$$dx = \alpha x dt + \beta dN$$

by means of the Euler method then what would be the rate of the Poisson counter $N$ and what would be the values of $\alpha$ and $\beta$?

22. Now suppose you want to numerically generate the steady state solution to the Fokker-Planck equation for the differential equation you found in answer to the previous problem. The general form will be

$$\frac{\partial \rho(t,x)}{\partial t} = -\frac{\partial \alpha x \rho(t,x)}{\partial x} + \lambda(\rho(x(t, x-\beta) - \rho(t, x)))$$

By generating the large time behavior for a large number of sample paths and constructing a histogram, determine an approximate solution to the steady state solution. More specifically, you can assume that for the accuracy required, steady state has been reached after 200 steps. You may also assume that the accuracy will be sufficient if you use 30 bins for the histogram.

23. Consider the function $f: [0, \infty) \to [0, 1]$ defined by

$$f(x) = \begin{cases} 
1 & 0 \leq x < 1 \\
0 & 1 \leq x < \infty 
\end{cases}$$

If we would like to find a state transition associated with a finite state continuous time jump process that has this function as the distribution of its transition time we need to find an approximation in terms of real exponentials. The function $g: [0, \infty) \to [0, 1]$ defined by

$$g(x) = 1 - e^{-x}$$

is a candidate. Find a better approximation by making use of $n$ by $n$ transition matrices of the circulant form

$$A_n = \begin{bmatrix} 
-1 & 1 & 0 & 0 & \cdots \\
0 & -1 & 1 & 0 & \cdots \\
0 & 0 & -1 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots 
\end{bmatrix}$$
Hint: The eigenvalues of $A_n$ are $-1 + r_n$ with $r_n$ being an $n^{th}$ root of unity.

24. An $n$ by $n$ matrix $A$ is said to be a circulant matrix if $a_{ij} = b_{j-i} \mod n$. For example, three by three matrices are of the circulant if they take the form

$$A = \begin{bmatrix} b_0 & b_1 & b_2 \\ b_2 & b_0 & b_1 \\ b_1 & b_2 & b_0 \end{bmatrix}$$

(a) Show that the sum of the two circulant matrices is circulant and that the product of two circulant matrices is circulant.

(b) If $A$ and $F$ are circulant show that $AF = FA$.

(c) Show that there exists a unitary matrix $U$ that diagonalizes circulant matrices.

(d) If $A$ is circulant we can associate with it a polynomial

$$\hat{A}(z) = \sum_{i=1}^{n-1} b_i z^i$$

which can be thought of as being a finite $z$-transform of $A$.

(e) Show that if $A$ and $F$ are circulant with $T = AF$ then $\hat{T}(z) = \hat{A}(z)\hat{F}(z)$ if we agree to interpret $e^{\hat{A}(z)\mod(z^{\dim A} - 1)}$.

25. Show that there exists $n$ by $n$ infinitesimally stochastic matrices whose eigenvalues are $\{-1 + e_i\}_{i=1}^n$ where $\{e_i\}_{i=1}^n$ are the $n$th roots of 1.

26. Consider an infinitesimally stochastic matrix $A$ (as in $\dot{p} = Ap$) which is $n$ by $n$. Of course it has the solution $p(t) = e^{At}p(0)$ and, because probability vectors are bounded, $A$ can not have eigenvalues with positive real parts; the eigenvalues of $A$ lie in the half-plane $\text{Re} s \leq 0$ and furthermore, the only eigenvalues on the imaginary axis lie at zero. In this sense, the solutions of $\dot{p} = Ap$ are not purely oscillatory. To see how close the probability vector can come to a pure oscillation we might investigate circumstances under which the solution $p$ contains damped oscillations which take a long time to die out. To this end, consider the special infinitesimally stochastic matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & \ldots & 0 & 1 \\ 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & -1 \end{bmatrix}$$

Show that the eigenvalues of this matrix are $-1 + e^{2\pi \sqrt{-1}/m}$ for $m$ an integer. Make a diagram showing their positions in the complex plane and verify that the eigenvalues all lie in a wedge with vertex at 0 and edges making an angle of $2\pi/n$ with the imaginary axis. This is a general property for $n$ by $n$ infinitesimally stochastic matrices.

27. Consider a two dimensional situation in which $x(t, \tau)$ takes on the values $\{0,1,2,\ldots\}$ with $x(0,0) = 0$ and $x$ being a Poisson process of rate $\lambda$ in $t$ along lines of constant $\tau$ and a Poisson process of rate $\mu$ in $\tau$ along lines of constant $t$. Does it follow that

$$x(t, \tau) = \phi(t) \cdot \psi(\tau)$$

NOTES AND REFERENCES
1. Poisson Counters are discussed in most introductory books on probability. One classic is [3].
   The book of Snyder, [4] was influential in bringing these ideas to the attention of the engineering community.
   Because we will make heavy use of systems of ordinary (deterministic) differential equations, the reader may want a reference. Everything we need and much more can be found in one of the classics of the subject [5].

2. Continuous time jump process are discussed in many books. An introductory treatment can be found in [6].

3. This is an almost trivial special case of the general form of Itô’s rule. The origin of the idea is Itô’s much cited paper [7].

4. Computing an expectation is often the key step in getting a concrete result. We have arranged the presentation so that the reader quickly arrives at a point where some interesting calculations can be done. One advantage of the differential equations approach to stochastic process is that such computations are usually rather transparent.

5. The computation of two point statistics will play an important role in the stochastic realization work to be discussed in chapter V.

6. Jump processes and linear systems are sometimes used as models for systems with failure modes.

7. These equations are of theoretical interest but they are very seldom soluble in closed form. In some cases the steady state value can be computed even though the time dependent solution is not known. In the case of finite state continuous time jump process the Peron-Frobenius theory applies and in the non finite state cases these are conditions under which one can guarantee the existence of an invariant measure as well.
Chapter 3

Wiener Processes and Differential Equations

In this chapter we introduce a second basic stochastic process, the Wiener process. The Itô calculus will be extended to include stochastic equations involving Wiener processes and the computation of expectations, the equation for the probability law, etc. will be developed.

3.1 Gaussian Distributions

From one point of view, the special significance of the Gaussian density derives from its role in solving the differential equation variously called the diffusion equation or the heat equation. In one spatial dimension, this is the partial differential equation of evolutionary type

$$\frac{\partial \rho(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2 \rho(t, x)}{\partial x^2}$$

which governs diffusion. A short calculation shows that the function $\rho(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ satisfies this equation for all $t > 0$. In fact, the general solution of this equation for twice differentiable initial data $\rho(0, x)$ is, for $t > 0$, given by

$$\rho(t, x) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} \rho(0, y) dy$$

For this reason $\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ is often called the heat kernel.

This generalizes nicely to $n$ spatial dimensions. If $Q = Q^T$ is positive definite, and if we let $q_{ij}$ denote the $ij^{th}$ entry of $Q$ then the partial differential equation

$$\frac{\partial \rho(t, x)}{\partial t} = \frac{1}{2} \sum_{i,j=1}^{n} q_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \rho(t, x)$$
is an \( n \)-dimensional generalization of the one-dimensional equation given above. It has the fundamental solution

\[
\rho(t, x) = \frac{1}{\sqrt{\det Q(2\pi t)^n}} e^{-x^T (2Qt)^{-1} x}
\]

The solution for smooth initial data \( \rho(0, x) \) is

\[
\rho(t, x) = \int_{\mathbb{R}^n} \frac{1}{\sqrt{\det Q(2\pi t)^n}} e^{-(x-y)^T (2Qt)^{-1} (x-y)} \rho(0, y) dy
\]

The Gaussian distribution can also be thought of as a limiting form of the binomial distribution in the following way, in that

\[
\lim_{n \to \infty} \left( 1 - \frac{\alpha^2}{n} \right)^n = \lim_{n \to \infty} e^{n \ln(1 - \frac{\alpha^2}{n})} = e^{-\alpha^2}
\]

As for the moments of a Gaussian, the odd moments are zero and the even ones can be computed as follows. By definition

\[
\mathcal{E} x^p = \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} x^p e^{-x^2/2a} dx
\]

An integration by parts yields

\[
\frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} x^p e^{-x^2/2a} dx = \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} \frac{1}{p+1} x^{p+1} e^{-x^2/2a} dx
\]

Proceeding from

\[
a = \mathcal{E} x^2 = \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2a} dx
\]

we see that after some algebraic manipulations we have for even values of \( p \)

\[
\mathcal{E} x^p = \frac{p!}{(p/2)!} \left( \frac{a}{2} \right)^{p/2}
\]

### 3.2 Brownian Motion

Robert Brown (1773–1858) was a Scottish botanist who became interested in stochastic processes after looking under a microscope at grains of pollen suspended in water. He published influential papers on this subject in 1827. Norbert Wiener undertook the mathematical study of stochastic processes in the 1920’s after important work earlier in the century by Bachelier, Einstein, Smoluchowski, et al. There are several ways of approaching the mathematics of this subject. We chose one that continues from our work on jump processes.

Let \( y(t) \) be a bidirectional Poisson counter of rate \( \lambda/2 \) and let \( x_\lambda(t) = y(t)/\sqrt{\lambda} \). We may realize this process as

\[
dx_\lambda = \frac{1}{\sqrt{\lambda}} (dN_1 - dN_2) ; \quad x(0) = 0
\]
3.2. BROWNIAN MOTION

where $N_1$ and $N_2$ are ordinary Poisson counters of rate $\lambda/2$. A quick calculation based on the Itô rule of chapter 2 shows that

$$dx^p_{\lambda} = \left(\left(x_{\lambda} + \frac{1}{\sqrt{\lambda}}\right)^p - x^p_{\lambda}\right) dN_1 + \left(\left(x_{\lambda} - \frac{1}{\sqrt{\lambda}}\right)^p - x^p_{\lambda}\right) dN_2$$

For reasons of symmetry, if $p$ is an odd positive integer $E x^p_{\lambda} = 0$. If $p$ is an even positive integer then the expectation rule of chapter 2 yields

$$\frac{d}{dt} E x^p_{\lambda} = \frac{1}{2} E \left(\left(x_{\lambda} + \frac{1}{\sqrt{\lambda}}\right)^p + \left(x_{\lambda} - \frac{1}{\sqrt{\lambda}}\right)^p - 2x^p_{\lambda}\right)$$

$$= \left(\frac{p}{2}\right) E x^{(p-2)}_{\lambda} + \frac{1}{\lambda} \left(\frac{p}{4}\right) E x^{(p-4)}_{\lambda} + \ldots$$

where the terms not written involve powers of $\lambda^{-1}$ greater than one. Because of this, for $t$ fixed there exists a high counting rate limit for the expected value of each moment

$$\lim_{\lambda \to \infty} E x^p_{\lambda}(t) = \frac{1}{2} \int_0^t (p-1) \lim_{\lambda \to \infty} E x^{p-2}(\sigma)d\sigma$$

Solving this set of equations beginning at $p = 2$ and continuing to larger, even values of $p$ we see that the limiting values of the moments are

$$\lim_{\lambda \to \infty} E x^2_{\lambda}(t) = t$$
$$\lim_{\lambda \to \infty} E x^4_{\lambda}(t) = 3t^2$$
$$\lim_{\lambda \to \infty} E x^6_{\lambda}(t) = 5 \cdot 3 \cdot t^3$$
$$\ldots = \ldots$$
$$\lim_{\lambda \to \infty} E x^p_{\lambda}(t) = (p-1)(p-3)\ldots t^{p/2}$$

$$= \frac{p!}{(p/2)!} \left(\frac{t}{2}\right)^{p/2}$$

Thus in the limit as $\lambda$ goes to infinity we get the moments associated with a Gaussian density having zero mean and variance $t$. Likewise, we see that for $\tau \geq t$ we have the differential equation

$$\frac{d}{d\tau} E x(t)x(\tau) = E x(t) \frac{1}{\sqrt{\lambda}}(dN_1(\tau) - dN_2(\tau)) = 0$$

and so $E x(t)x(t + \tau) = E x^2(t)$ for $\tau \geq t$.

We remark that $x_{\lambda}(t)$ has three properties which are more or less obvious

1. $x_{\lambda}(0) = 0$.
2. $x_{\lambda}(t) - x_{\lambda}(\tau)$ is a random variable whose distribution is dependent only on $|t - \tau|$ and if $[t, \tau] \cap [s, \sigma]$ is empty, the random variables $x_{\lambda}(t) - x_{\lambda}(\tau)$ and $x_{\lambda}(s) - x_{\lambda}(\sigma)$ are independent. (Because $N$ on the interval $[s, \sigma]$ is independent of $N$ on the interval $[t, \tau]$)
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3. The limit as $\lambda \to \infty$ of $\mathcal{E}(x_\lambda(t) - x_\lambda(\tau))^2 = \mathcal{E}x_\lambda^2(t) - 2\mathcal{E}x_\lambda(t)x_\lambda(\tau) + x_\lambda^2(\tau)$ exists and is just $|t - \tau|$.

From the results of section 2.5 we see that if there exists a probability density for $x_\lambda$ then it is given by

$$\frac{\partial \rho(t, x)}{\partial t} = \frac{\lambda}{2} \left( \rho(t, x + \frac{1}{\sqrt{\lambda}}) - 2\rho(t, x) + \rho(t, x - \frac{1}{\sqrt{\lambda}}) \right)$$

Notice that in the limit as $\lambda$ goes to infinity, we get (formally) the diffusion equation

$$\frac{\partial \rho(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2 \rho(t, x)}{\partial x^2}$$

This analysis shows that if we use the Itô rule for jump processes, then certain limits as $\lambda$ goes to infinity exist. There is, in fact, a stochastic process known as a standard Wiener process, which has the properties we have derived for the above limiting forms and, in addition, is continuous with probability one. Continuity is, of course, not surprising in view of the fact that $\mathcal{E}(x(t) - x(\tau))^2 = |t - \tau|$. The analysis of limits given here does not, in and of itself, establish the existence of a stochastic process having the limiting properties. We will not, in these notes, prove its existence. We will, instead, appeal to limiting processes of the above type whenever we need to explore its properties. For convenience we will refer to the limiting form as being Brownian motion.

3.3 Stochastic Differential Equations

This same limiting process can be applied to give meaning to a general class of equations which we write as

$$dx = f(x)dt + \sum_{i=1}^{n} g_i(x)dw_i$$

By a solution of this equation we understand the limit as $\lambda$ goes to infinity of the solution of

$$dx = f(x)dt + \sum_{i=1}^{n} g_i(x)(dN_i - dN_{-i})/\sqrt{\lambda}$$

where $N_i$ and $N_{-i}$ are independent Poisson counters with rate $\lambda/2$.

Example 1: (The Ornstein-Uhlenbeck Process) Consider the process

$$dx = -\alpha x dt + bdw + cdt$$

The process would be Brownian motion if $\alpha$ and $c$ were zero. The equation for the expectation of $x$ is

$$\frac{d\hat{x}}{dt} = -\alpha \hat{x} + c$$
3.4. THE ITÔ RULE

The solution of this equation is

\[ \hat{x} = e^{-\alpha t}(x(0) - c/\alpha) + c/\alpha \]

If we assume \( \alpha \) is positive then, regardless of the initial condition, the expected value of \( x \) approaches \( c/\alpha \) as \( t \) goes to infinity. It is often said that \( x \) “reverts to its mean”.

3.4 The Itô Rule

Let \( \psi \) be a twice differentiable function of \( x \). Consider the evaluation of \( d\psi \) for

\[ dx = f(x)dt + \sum_{i=1}^{n} g_i(x)(dN_i - dN_{-i})/\sqrt{\lambda} \]

where \( N_i \) are Poisson counters of rate \( \lambda/2 \). Using the Itô rule of chapter 2 we get

\[
\begin{align*}
\langle \frac{d\psi}{dx} , f(x) \rangle dt + \sum_{i=1}^{n} \left[ \psi(x + \frac{1}{\sqrt{\lambda}} g_i(x)) - \psi(x) \right] dN_i \\
+ \sum_{i=1}^{n} \left[ \psi(x - \frac{1}{\sqrt{\lambda}} g_i(x)) - \psi(x) \right] dN_{-i}
\end{align*}
\]

In order to explore the limit as \( \lambda \) goes to infinity, we expand \( \psi \) in a Taylor series about \( x \). The result is

\[
\begin{align*}
\langle \frac{d\psi}{dx} , f(x) \rangle dt + \sum_{i=1}^{n} \left[ \frac{\partial^2 \psi}{\partial x^2} g_i(x) \right] (dN_i + dN_{-i})/\lambda + O(1/\lambda^{3/2})dN_i
\end{align*}
\]

Consider now the special case

\[ dz_\lambda = (dN_1 + dN_{-1})/\lambda \]

where \( N_1 \) and \( N_{-1} \) are independent Poisson counters of rate \( \lambda/2 \). Using the formula for expectation from chapter 2, we see that

\[ \mathcal{E}z_\lambda(t) = t + \mathcal{E}z(0) \]

Let \( m_\lambda = z_\lambda^2(t) \). Then from the Itô rule for jump processes we see that

\[ dm_\lambda = \left( \left( z_\lambda + \frac{1}{\lambda} \right)^2 - z_\lambda^2 \right) (dN_1 + dN_2) = \frac{2z_\lambda}{\lambda} (dN_1 + dN_2) + \frac{1}{\lambda^2} (dN_1 + dN_2) \]
A short calculation, using $E dN_i = (\lambda/2) dt$, yields

$$\frac{d}{dt} E m_\lambda(t) = E(2z\lambda) + \frac{1}{\lambda}$$

Thus if $E z(0) = 0$ we have

$$E m_\lambda(t) = t^2 + t/\lambda$$

It is remarkable that the variance of the process defined by $z$, namely

$$\sigma(t) = E(z_\lambda(t) - E z_\lambda(t))^2$$

which goes along with the initial condition $z(0) = 0$ is just

$$\sigma(t) = t^2 + t/\lambda - 2t^2 + t^2$$

$$= t/\lambda$$

and hence goes to zero as $\lambda$ goes to infinity. As $\lambda$ becomes larger the uncertainty associated with $z$ decreases and, in a sense, $z$ tends to the deterministic process defined by $z(t) = t$. As a result, the limiting form of the Itô rule takes a simple form. We may make the replacement, as $\lambda$ goes to infinity, $(dN_i + dN_{-i})/\sqrt{\lambda} = dt$. Thus for

$$dx = f(x)dt + \sum_{i=1}^{m} g_i(x) dw_i$$

we have

$$d\psi = \left< \frac{\partial \psi}{\partial x} , f(x) \right> dt + \sum_{i=1}^{m} \left< \frac{\partial \psi}{\partial x} , g_i(x) \right> dw_i + \frac{1}{2} \sum_{i=1}^{m} \left< g_i(x) , \frac{\partial^2 \psi}{\partial x^2} g_i(x) \right> dt$$

where we have abbreviated $(dN_i - dN_{-i})/\sqrt{\lambda}$ as $dw_i$. This is known as the Itô rule for Brownian motion.

**Example 1:** Consider

$$dx = -x dt + x dw$$

Suppose we want to find an equation for $z = x^2$. Using the Itô rule we have

$$dx^2 = 2x(-x dt + x dw) + x^2 dt$$

The last term results from an evaluation of $1/2(\partial^2 \psi/\partial x^2) \cdot (g(x))^2 dt$. The net result is

$$dz = -2z dt + 2z dw + z dt$$

$$= -z dt + 2z dw$$

**Example 2:** Consider the pair of Itô equations

$$dx = dw \ ; \ dy = dw$$
3.5. EXPECTATIONS

with \( w \) and \( \nu \) being independent Wiener processes. If we describe matters in polar coordinates, \( x = r \cos \theta \), \( y = r \sin \theta \), then of course \( r = \sqrt{x^2 + y^2} \) and \( \theta = \tan^{-1}(y/x) \). To obtain the differential equations for \( r \) and \( \theta \) we can use the Itô rule as follows. First of all,

\[
\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}; \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}; \quad \frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2}; \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2}
\]

The matrix of second partial derivatives is

\[
\begin{bmatrix}
\frac{\partial^2 r}{\partial x^2} & \frac{\partial^2 r}{\partial x \partial y} \\
\frac{\partial^2 r}{\partial y \partial x} & \frac{\partial^2 r}{\partial y^2}
\end{bmatrix} = \begin{bmatrix}
y^2 & -xy \\
-xy & x^2 + y^2
\end{bmatrix}; \quad \begin{bmatrix}
\frac{\partial^2 \theta}{\partial x^2} & \frac{\partial^2 \theta}{\partial x \partial y} & \frac{\partial^2 \theta}{\partial y^2} \\
\frac{\partial^2 \theta}{\partial y \partial x} & \frac{\partial^2 \theta}{\partial y^2}
\end{bmatrix} = \begin{bmatrix}
x^2 & -1 \\
1 & x^2 + y^2
\end{bmatrix}
\]

Using this in the Itô rule we have

\[
\frac{dr}{\sqrt{x^2 + y^2}} = \frac{x dw + y d\nu}{\sqrt{x^2 + y^2}} + \frac{dt}{2 \sqrt{x^2 + y^2}}
\]

A similar calculation shows that \( \theta \) satisfies

\[
\frac{d\theta}{x^2 + y^2} = \frac{x d\nu - y dw}{(x^2 + y^2)^2} + \frac{xy dt}{(x^2 + y^2)^2}
\]

Expressing the right-hand sides of these equations in terms of \( r \) and \( \theta \) we get

\[
\frac{dr}{r} = \frac{1}{2r} dt + \sin \theta d\nu + \cos \theta dw
\]

\[
\frac{d\theta}{r} = \frac{\cos \theta d\nu - \sin \theta dw}{r} + \frac{\sin \theta \cos dt}{r^2}
\]

In view of the orthogonal invariance of the two-dimensional wiener process we can simplify this to

\[
\frac{dr}{2r} dt + dw' = \frac{d\theta}{r}
\]

### 3.5 Expectations

Adopting the notation of the previous section we now consider stochastic equations of the form

\[
dx = f(x) dt + \sum_{i=1}^{m} g_i(x) dw_i
\]

with \( w_i \) being a Wiener process. What is the rule for computing expectations for such an equation? If we replace \( dw_i \) by

\[
dw_i = \frac{1}{\sqrt{\lambda}}(dN_i - dN_{-i})
\]
where \( N_i \) and \( N_{-i} \) are Poisson counters of rate \( \lambda/2 \), then it is clear that in the limit as \( \lambda \) goes to infinity we get simply

\[
\frac{d}{dt} \mathcal{E}x = \mathcal{E}f(x)
\]

Thus the rule for computing expectation is simply that \( \mathcal{E}g(x)dw = 0 \). This is a consequence of the fact that we defined the solutions of our jump-process equations to be continuous from the left and have defined the solution after the jump in terms of the left limit.

The result of this method of defining solutions is to destroy the usual time reversal symmetry of the time axis in a way that has no analogy in the study of ordinary differential equations.

**Example 1**: To illustrate, consider the model

\[
dx = xdt + \alpha xdw
\]

This might represent, for example, the growth of capital when the interest rate is a random process. It is easy to see that if we let \( z = \ln x \) then the Itô rule implies that

\[
ds = a(t) - \frac{1}{2} \alpha^2(t) + \alpha dw
\]

so that the density of \( z \) is Gaussian with mean and variance that are easily calculated. This model for \( x \) is widely used in describing the growth of capital subject to a random interest rate. Of course the density for \( z \) is Gaussian if the initial density is Gaussian and the density of \( x \) can be obtained by the standard formulae given in Chapter one. Working this out under the further assumption that \( a \) and \( \alpha \) are constant gives

\[
\rho(x) = \frac{1}{xb\sqrt{2\pi t}} e^{-((\ln(x) - at)^2/2(b)^2)}
\]

This is the so-called log normal distribution widely used in applications including finance, reliability, and communication theory.

**Example 2**: Some stochastic models have the property that we are only interested in trajectories that take on a specific value at some point in the future but which are random in the meantime. The *Brownian bridge* is a popular model in such a situation. Consider

\[
dx = -\frac{1}{1-t} x + dw \ ; \ x(0) = 0
\]

The mean and second moment satisfy the differential equations

\[
\frac{d}{dt} \mathcal{E}x = \frac{-1}{1-t} \mathcal{E}x(t)
\]

\[
\frac{d}{dt} \mathcal{E}x^2 = \frac{-2}{1-t} \mathcal{E}x^2(t) + 1
\]

Clearly \( \mathcal{E}x(t) \) is identically zero but the variance is more interesting. Observe that

\[
\mathcal{E}x^2(t) = t(1-t)
\]
satisfies the variance equation. This construction can be scaled in various ways to adjust the starting and ending values of $x$, the strength of the Brownian motion term and the starting and ending times.

Example 3: Consider system

$$
\begin{align*}
\text{dx} &= v \text{dt} \quad x(0) = 0 \\
\text{dv} &= a \text{dt} \quad v(0) = 0 \\
\text{da} &= d \text{w} \quad a(0) = 0
\end{align*}
$$

as an explicit example of $\text{dx} = Ax \text{dt} + B \text{dw}$. The equation for the expected values can be written as

$$
\frac{d}{dt} \mathbb{E} \begin{bmatrix} x \\ v \\ a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbb{E} \begin{bmatrix} x \\ v \\ a \end{bmatrix}
$$

and

$$
\mathbb{E} \begin{bmatrix} x(t) \\ v(t) \\ a(t) \end{bmatrix} = e^{At} \mathbb{E} \begin{bmatrix} x(0) \\ v(0) \\ a(0) \end{bmatrix} = 0
$$

If we let $\Sigma$ denote the covariance then

$$
\dot{\Sigma} = A\Sigma + \Sigma A^T + BB^T
$$

$$
\Sigma(t) = \int_0^t e^{As} BB^T e^{At} \sigma \ d\sigma
$$

A routine calculation establishes

$$
e^{At} = I + At + \frac{1}{2} A^2 t^2 = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}
$$

and thus

$$
\Sigma(t) = \int_0^t \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2 & t & 1 \end{bmatrix} dt
$$

$$
= \int_0^t \begin{bmatrix} 0 & 0 & \frac{t^2}{2} \\ 0 & 0 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2 & t & 1 \end{bmatrix} dt = \begin{bmatrix} \frac{1}{3} T^5 & \frac{1}{5} T^4 & \frac{1}{7} T^3 \\ \frac{1}{3} T^4 & \frac{1}{4} T^3 & \frac{1}{2} T^2 \\ \frac{1}{3} T^3 & \frac{1}{2} T^2 & T \end{bmatrix}
$$

Example 5: Consider a pair of stochastic differential equations with independent Wiener processes $w$ and $v$

$$
\text{dx} = \frac{-1}{1-t} x \text{dt} + d \text{w} \quad x(0) = 0
$$
CHAPTER 3. WIENER PROCESSES AND DIFFERENTIAL EQUATIONS

\[ dy = \frac{-1}{1-t} y dt + d\nu ; \quad y(0) = 0 \]
together with a third equation of the form

\[ dz = x d\nu - y dw \]

Suppose we wish to evaluate the expected values of \( z(1) \), \( z^2(1) \) and \( z^4(1) \). For the solution of the Brownian bridge equations, as we have as discussed above,

\[ \mathcal{E} x^2 = \mathcal{E} y^2(t) = t(1-t) \]

In view of the fact that \( x(t) \) is Gaussian and zero mean, we see that for \( k \) an even integer

\[ \mathcal{E} x^k(t) = \mathcal{E} y^k(t) = \frac{k!}{(k/2)!} \left( \frac{t(1-t)}{2} \right)^{k/2} \]

Thus

\[ \mathcal{E} x^4(t) = \mathcal{E} y^4(t) = 3t^2(t-1)^2 ; \quad \mathcal{E} x^2(t) y^2(t) = t^2(1-t)^2 \]

and so

\[ \mathcal{E}(x^2(t) + y^2(t))^2 = \mathcal{E}(x^4(t) + 2x^2(t)y^2(t) + y^4(t)) = 8t^2(1-t)^2 \]

Using the Itô rule we have

\[ dz^2 = 2z(x d\nu - y dw) + (x^2 + y^2) dt \]

which implies that

\[ \mathcal{E} z^2(t) = 2 \int_0^t \tau(1-\tau) d\tau = t^2 - 2t^3/3 \]

Now turning to the quartic terms, we have

\[ dz^2(x^2+y^2) = 2z(x d\nu - y dw)(x^2+y^2) + (x^2+y^2)^2 dt + \frac{2}{1-t} (x^2+y^2)^2 dt + 2z^2(x dw + y d\nu) + 2z^2 dt \]

so that

\[ \frac{d}{dt} \mathcal{E} z^2(x^2 + y^2) = -2 \frac{1}{1-t} \mathcal{E} z^2(x^2 + y^2) + 2 \mathcal{E} z^2 \]

\[ dz^4 = 4z^3(x d\nu - y dw) + 6z^2(x^2 + y^2) dt \]

To simplify the notation let \( a = \mathcal{E} z^2(x^2 + y^2) \) so that

\[ \dot{a} = -2 \frac{1}{1-t} a + 8t^2(1-t)^2 + 2t^2 - 4t^3/3 = -2 \frac{1}{1-t} a + 10t^2 - \frac{52}{3} t^3 + 8t^4 \]

Using the variation of constants formula, the solution of this equation is given by

\[ a(t) = \int_0^t \left( \frac{1-t}{1-\tau} \right)^2 \left( 10\tau^2 - \frac{52}{3} \tau^3 + 8\tau^4 \right) d\tau \]

Observe that

\[ \frac{8t^4 - (52/3)t^3 + 10t^2}{t^2 - 2t + 1} = 8t^2 - (4/3)t - (2/3) - \frac{4/3}{(1-t)^2} \]
3.6. Finite Difference Approximations

The numerical treatment of stochastic differential equations requires analysis that goes beyond the usual finite difference approximations that serve as the starting point for the more familiar \( \dot{x} = f(x) \). The source of the difficulties is, of course, related to the unusual differentiation rule in stochastic calculus. Here we discuss the connection between stochastic differential equations and difference equation approximations with a view toward bringing out the limitations on numerical techniques in this domain.

Consider the difference equation

\[
x((t+h)) = x(t) + hf(x(t)) + hg(x(t))n(t)
\]

with \( n(t) \) being a zero mean Gaussian random variable. If we think of this as an approximation to a stochastic differential equation of the Itô type we need to investigate the relationship between the Itô description and the \( f, g \) and \( n \) that enter here. In particular, because the step size \( h \) is often subject to adjustment based on empirical evidence, the role of step size change will be interest. We begin by comparing

\[
x(h) = x(0) + hf(x(0)) + hg(x(0))n(0)
\]

with the value of \( x(h) \) that results if we replace \( h \) by \( h/2 \) and take two steps. In the latter situation we have

\[
x(h/2) = x(0) + (h/2)f(x(0)) + (h/2)g(x(0))n(0)
\]

and, repeating this again

\[
x(h) = x(0) + (h/2)f(x(0)) + (h/2)g(x(0))n(0) + (h/2)f(x(h/2)) + (h/2)g(x(h/2))n(h/2)
\]

If we expand \( f \) and \( g \) in their Taylor series we can express this as

\[
x(h) = x(0) + hf(x(0)) + h/2g(x(0))n(0) + h/2g(x(0))n(h/2) + (h/2)^2g'(x(0))g(x(0))n(0)n(h/2) + \ldots
\]
It is useful to focus on the variance associated with \( x(h) \) in these two cases. In the case of a single step of size \( h \) we have

\[
\mathcal{E} \left( x(h) - x(0) \right) (x(h) - x(0))^T = f^2(x(0)) + h^2 \left( g(x(0)) g^T(x(0)) \right) \mathcal{E} n^2(0)
\]

In the case of two steps of size \( h/2 \) we have

\[
\mathcal{E} \left( x(h) - x(0) \right) (x(h) - x(0))^T = f^2(x(0)) + \frac{h^2}{4} \left( g(x(0)) g^T(x(0)) \right) \mathcal{E} n^2(0) + \frac{n^2(h/2)}{}
\]

Thus, if we wish the variance of the solution to stay the same when we divide the interval in half, we must double the variance of the noise. For this reason it is common to adopt the notation

\[
x(t + h) = x(t) + hf(x(t)) + \sqrt{h} g(x(t)) n(t)
\]

In such a notation the variance of the noise term \( n(t) \) remains constant as \( h \) is changed.

Notice that if \( \psi(\cdot) \) is a twice differentiable function of \( x \), then

\[
\psi \left( x(t + h) \right) = \psi \left( x(t) + hf(x(t + h)) + \sqrt{h} g(x(t + h)) n(t) \right)
\]

so that

\[
\psi \left( x(t + h) \right) = \psi \left( x(t) \right) + h \left( \frac{\partial \psi}{\partial x} , f(x(t)) \right) + \sqrt{h} \left( \frac{\partial \psi}{\partial x} , g(x(t)) \right) n(t)
\]

\[
+ \frac{h}{2} \left( \frac{\partial^2 \psi}{\partial x^2} , g \right) n^2(t) + \text{h.o.t.}
\]

This analysis is consistent with the Itô differentiation rule.

### 3.7 A Digression on Stratonovic Calculus

An important aspect of the Wiener process is the formula for the quadratic variation

\[
\lim_{\Delta_i \to 0} \sum |w(t_{i+1}) - w(t_i)|^2 = |a - b|
\]

where \( \Delta_i = t_{i+1} - t_i \) and the sum is over subdivisions of the interval \([a, b] \). This fact underlies the useful calculation rule, \( dw \cdot dw = dt \). In order to define what is meant by the stochastic differential equation

\[
dx = f(x) dt + g(x) dw
\]

one integrates to get

\[
x(t) - x(0) = \int_0^t f(x(\sigma)) d\sigma + \int_0^t g(x(\sigma)) dw
\]

and transfers the difficulty to that of interpreting the integral. In the figure below we illustrate part of the construction of the (non anticipatory) Itô integral

\[
\int_a^b g(x(t)) dw(t) = \lim_{\Delta_i \to 0} \sum g(x(t_i))(w(t_i + \Delta_i) - w(t_i))
\]
On the other hand, in the (partially anticipatory) Stratonovic calculus one uses a different definition of the integral

\[
\int_a^b g(x(t))dw(t) = \lim_{\Delta_i \to 0} \sum g(x(t_i + \Delta_i/2))(w(t_i + \Delta_i) - w(t_i))
\]

Using the notation and setup of the figure shown below, we explore the relationship between the Itô and Stratonovic integrals.

The Itô integral is based on the interpretation of

\[
I = \int_a^b g(x(t)) \, dw_t
\]

in which \(g(x(t))\) is evaluated at the left end point of each subdivision of \([a, b]\) and \(dw_t\) is treated as occurring "in advance" of any change in \(x(t)\). That is to say, we define the integral as the limit of sums of the type

\[
I = \sum_i g(x(t_i))(w(t_i + \Delta_i) - w(t_i))
\]

On the other hand, we could have used instead

\[
I = \sum_i g(x(t_i + \Delta_i/2)) (w(t_i + \Delta_i) - w(t_i))
\]

evaluating \(g\) in the middle of the subdivision. Unlike the situation one has in Riemann integration where in the limit as \(\Delta_i\) goes to zero such a change makes no difference, in this case one gets a different integral. Since this "central difference" interpretation will be useful later, we will now discuss the precise way in which it differs from the Itô interpretation. These remarks lead to a different, but for our purposes equally expressive, version of the stochastic calculus. It is also widely used. The key observation is that from Taylor series we get

\[
g(x(t + \Delta_i/2)) = g(x(t)) + \frac{\partial g}{\partial x}(x(t)) \Delta_i x/2
\]

If we evaluate \(\Delta_i x\) as

\[
\Delta_i x = f(x + \Delta_i x/2) \Delta_i t + g(x + \Delta_i x/2) \Delta_i w
\]

we see that

\[
g(x(t + \Delta_i x/2) \approx g(x(t)) + (\partial g/\partial x)g(x) \Delta_i w + \text{terms of order } \Delta_i t, (\Delta_i w_i)^2 \text{ and higher}
\]

Recall that when we introduced the Wiener process as the limit of a certain construction involving Poisson counters, we found that

\[
\mathcal{E}(w(t) - w(\tau))^2 = |t - \tau|
\]

and that as a consequence of this it was necessary to treat the differential calculus with additional care. In particular, it happened that \((dw)^2\) is, in a certain sense, first order and
equal to the deterministic differential $dt$. The change of variables and expectation formula associated with the Itô calculus is summarized by

$$
\begin{align*}
\, \mathrm{d}x &= f(x) \, \mathrm{d}t + g(x) \, \mathrm{d}w \\
\, \mathrm{d}\psi &= \left\langle \frac{\partial \psi}{\partial x}, f(x) \, \mathrm{d}t + g(x) \, \mathrm{d}w \right\rangle + \frac{1}{2} g^T(x) \frac{\partial^2 \psi}{\partial x^2} g(x) \, \mathrm{d}t \\
\, \frac{\, \mathrm{d}\mathcal{E}\psi}{\, \mathrm{d}t} &= \left\langle \frac{\partial \psi}{\partial x}, f(x) \right\rangle + \frac{1}{2} g^T(x) \frac{\partial^2 \psi}{\partial x^2} g(x)
\end{align*}
$$

When we switch and represent dynamics by differential equations which are to be interpreted as central differences, these equations change. The change can be best appreciated by noting that a Taylor series expansion yields

$$
g(x(t + \Delta/2)) = g(x(t)) + \frac{1}{2} \frac{\partial g}{\partial x}(x(t)) \Delta + \frac{1}{2} (\Delta/2)^2 \frac{\partial^2 g}{\partial x^2}(x)
$$

But $\Delta x$ is $f(x) \Delta t + g(x) \Delta w$ and since $(\Delta w)(\Delta t)$, but not $(\Delta w)^2$, is of order higher than $\Delta t$, we see that

$$
\int_{\text{Ito}} g(x(t)) \, \mathrm{d}w = \int_{\text{central}} g(x_i) \, \mathrm{d}w - \frac{1}{2}\int (\partial g/\partial x)g \, \mathrm{d}t
$$

Thus we have a different calculus; different from, but equally expressive as, the Itô calculus. Using barred $\mathrm{d}$’s to indicate differentials in the Stratonovic sense, we can summarize via

$$
\begin{align*}
\, \overline{\mathrm{d}}x &= f(x) \, \overline{\mathrm{d}}t + g(x) \, \overline{\mathrm{d}}w \\
\, \overline{\mathrm{d}}\psi &= \left\langle \frac{\partial \psi}{\partial x}, f(x) \, \overline{\mathrm{d}}t + g(x) \, \overline{\mathrm{d}}w \right\rangle \\
\, \frac{\, \overline{\mathrm{d}}\mathcal{E}\psi}{\, \overline{\mathrm{d}}t} &= \mathcal{E} \left( f(x) + \frac{1}{2} \left( \frac{\partial g}{\partial x} \right) g(x) \right)
\end{align*}
$$

If $x$ satisfies the Itô equation

$$
\, \mathrm{d}x = f(x) \, \mathrm{d}t + \sum_{i=1}^{m} g_i(x) \, \mathrm{d}w_i
$$

then it satisfies the Stratonovic equation

$$
\, \overline{\mathrm{d}}x = (f(x) - \frac{1}{2} \sum_{i=1}^{m} \frac{\partial g}{\partial x} g_i) \, \overline{\mathrm{d}}t + \sum_{i=1}^{m} g_i(x) \, \overline{\mathrm{d}}w_i
$$

### 3.8 The Fokker-Planck Equation

Hand in hand with the Itô differentiation formula goes the evolution equation for the probability density. Consider in $\mathbb{R}^n$ the Itô equation

$$
\, \mathrm{d}x = f(x) \, \mathrm{d}t + \sum_{i=1}^{m} g_i(x) \, \mathrm{d}w_i
$$
and let \( \psi \) be a smooth function having compact support. Then
\[
d\psi = \left\langle \frac{\partial \psi}{\partial x}, f(x) dt + \sum_{i}^{m} g(x) dw_{i} \right\rangle + \frac{1}{2} \sum_{i=1}^{m} \left\langle \frac{\partial^{2} \psi}{\partial x^{2}} g_{i}(x), g_{i}(x) \right\rangle dt
\]
If we are interested in \( \frac{d}{dt} E \psi \), we have on one hand
\[
\frac{d}{dt} E \psi = \frac{d}{dt} \int_{\mathbb{R}^{n}} \psi(x) \rho(t,x) dx
\]
On the other hand,
\[
\frac{d}{dt} E \psi = E \left\langle \frac{\partial \psi}{\partial x}, f(x) \right\rangle + \frac{1}{2} \sum_{i=1}^{m} E \left\langle \frac{\partial^{2} \psi}{\partial x^{2}} g_{i}(x), g_{i}(x) \right\rangle
\]
Integrating this by parts we get
\[
\frac{d}{dt} E \psi = \int_{\mathbb{R}^{n}} \left\langle \frac{\partial \psi}{\partial x}, f(x) \right\rangle + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} \left( \frac{\partial}{\partial x_{j}} g_{i}^{j}(x) g_{i}^{k}(x) \right) \rho(t,x) dx
\]
Putting these two formulas for \( \frac{d}{dt} E \psi \) together we have
\[
0 = \int_{\mathbb{R}^{n}} \psi \left( \frac{\partial \rho}{\partial t} + \left\langle \frac{\partial}{\partial x}, \rho(t,x) f(x) \right\rangle - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} \left( \frac{\partial^{2}}{\partial x_{j} \partial x_{k}} g_{i}^{j}(x) g_{i}^{k}(x) \right) \rho(t,x) \right) dx
\]
The only way this can hold true for all \( \psi \) is if
\[
\frac{\partial \rho(t,x)}{\partial t} = -\left\langle \frac{\partial}{\partial x}, \rho(t,x) f(x) \right\rangle + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^{2}}{\partial x_{j} \partial x_{k}} g_{j}(x) g_{k}(x) \rho(t,x)
\]
This is the Fokker-Planck equation; the evolution equation for the probability law.

**Example 1:** The Fokker Planck equation corresponding to the Itô equation
\[
dx = -x dt + dw
\]
CHAPTER 3. WIENER PROCESSES AND DIFFERENTIAL EQUATIONS

is

$$\frac{\partial \rho(t, x)}{\partial t} = \frac{\partial x \rho(t, x)}{\partial x^2} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \rho(t, x)$$

If $\rho(0, x)$ is given then one can verify that

$$\rho(t, x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \sigma(t)}} e^{-\frac{(x-e^{-t} \eta)^2}{2\sigma(t)}} \rho(0, \eta) d\eta$$

with $\sigma(t) = \frac{1}{2}(1 - e^{-2t})$

**Example 2:** Seeking an interesting topic for a term paper, a novice finance student asks the following question. If I can solve for the joint probability distribution for the pair of Itô equations

$$dx = -ax dt + bx dw_1$$

$$dy = -cy dt + fy dw_2$$

with $w_1$ and $w_2$ independent, can I also solve for the joint probability density when

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} -a & 0 \\ 0 & -c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} dt + \begin{bmatrix} -b & h \\ g & -f \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} dw_1 + \begin{bmatrix} -b' & h' \\ g' & -f' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} dw_2$$

Assuming suitable initial conditions, write down a joint density $\rho(t, x, y)$ in the first case using the fact that the logarithm of $x$ and $y$ are Gaussian for all time if they are at $t = 0$. (Do not forget that we are dealing with Itô equations here.) Should the student expect to be successful in the more general situation? Explain your answer carefully, including the role of a term of the form $AB - BA$.

**Solution:** If we have $dx = -ax dt + bx dw$ then we can apply the Itô rule to get the differential of $\psi(x) = \ln x$ as

$$d\psi(x) = -adt + bdw - \frac{1}{2} \frac{b^2 x^2}{x^2} dt = (-a - \frac{1}{2}b^2) dt + bdw$$

This can be solved for a Gaussian density of mean $(c_1 - a - b^2/2)t$ and variance $c_2 + tb^2$. The same remarks apply to the second equation. The matrix system can not be solved in closed form because the linear system is time varying and does not decouple.

**Example 3:** Consider the stochastic equations

$$dx = -z x dt + dw$$

$$dz = -2zN$$

with $z(0) \in \{ -1, 1 \}$ and $N$ a Poisson counter of rate $\lambda$. Instead of writing $\rho(t, x, z)$ we write $\rho(t, x, 1) = \rho_+(t, x)$ and $\rho(t, x, -1) = \rho_-(t, x)$. In this notation one has the following pair of equations for the density

$$\frac{\partial \rho_+(t, x)}{\partial t} = \lambda(\rho_-(t, x) - \rho_+(t, x)) + \frac{\partial}{\partial x} 11 x \rho_+(t, x) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \rho_+(t, x)$$

$$\frac{\partial \rho_-(t, x)}{\partial t} = \lambda(\rho_+(t, x) - \rho_-(t, x)) + \frac{\partial}{\partial x} 9 x \rho_-(t, x) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \rho_-(t, x)$$
Example 4: Consider the equation
\[
\begin{bmatrix}
\frac{dx}{dy}
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} dt + \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} (dN_1 - dN_2)
\]
with \(N_1\) and \(N_2\) being Poisson counters of rate \(\lambda\) and \(w\) and \(\nu\) being standard Wiener processes. Find a differential equation for the mean and the variance of \((x, y)\). Find the Fokker-Planck equation for \(\rho(t, x, y)\). Discuss the influence of the size of \(\lambda\) on the existence of a steady-state solution. (Assume throughout that \(w, \nu, N_1, \) and \(N_2\) are all independent, and that \(\theta\) is constant) Working in terms of matrices, and using \(\Sigma\) to denote the expectation of \(xx^T\), we have
\[
\dot{\Sigma} = -2\Sigma + I + \mathcal{E}((x + \Theta x)(x + \Theta x)^T - xx^T)\lambda + ((x - \Theta x)(x - \Theta x)^T - xx^T)\lambda
\]
This, in turn, simplifies to
\[
\dot{\Sigma} = -2\Sigma + I + 2\Theta \Sigma \Theta^T \lambda
\]
Clearly, we see that if \(|\lambda|\) is larger than 1 this system might be unstable, depending on the eigenvalues of the operator \(L(\cdot) = \Theta(\cdot)\Theta^T\).

In terms of components everything is much less attractive.
\[
d\mathcal{E} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix} \mathcal{E} \begin{bmatrix} x \\ y \end{bmatrix} dt + \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix} \mathcal{E} \begin{bmatrix} x \\ y \end{bmatrix} (\lambda dt - \lambda dt)
\]
and
\[
\mathcal{E} \begin{bmatrix} x \\ y \end{bmatrix} = e^{t \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}} \mathcal{E} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}
\]
\[
\lim_{t \to 0} \mathcal{E} \begin{bmatrix} x \\ y \end{bmatrix} = 0
\]
\[
dx^2 = 2x(-x)dt + 2x dw + dt + [(x + x \cos \theta + y \sin \theta)^2 - x^2]dN_1 + [(x - x \cos \theta - y \sin \theta)^2 - x^2]dN_2
\]
From this we can take expectations to get
\[
\frac{d}{dt} \mathcal{E} x^2 = -2\mathcal{E} x^2 + (\mathcal{E}(x \cos \theta + y \sin \theta)^2)\lambda + 1
\]
\[
\frac{d}{dt} \mathcal{E} x^2 = -2\mathcal{E} x^2 + 2\lambda \sin 2\theta \mathcal{E} xy + \lambda \mathcal{E} x^2 + \lambda \mathcal{E} y^2 + 1
\]
We omit further details.

For the Fokker-Planck equation we have
\[
\frac{\partial \rho(t, x, y)}{\partial t} = 2\left(\frac{\partial \rho(t, x, y)}{\partial x} + \frac{\partial \rho(t, x, y)}{\partial y}\right) + \frac{1}{2} \left(\frac{\partial^2 \rho(t, x, y)}{\partial x^2} + \frac{\partial^2 \rho(t, x, y)}{\partial y^2}\right) + f \lambda (\rho(t, x_-, y_-) - 2\rho(t, x, y) + \rho(t, x_+, y_+)) + \rho(t, x, y)
\]
where
\[ x_\pm = x \mp (x \cos \theta + y \sin \theta) \]
\[ y_\pm = x \mp (y \cos \theta - x \sin \theta) \]
and
\[ f^{-1} = \det \begin{bmatrix} 1 + \cos \theta & \sin \theta \\ -\sin \theta & 1 + \cos \theta \end{bmatrix} \]

**Example 5:** Suppose that \( x \) and \( y \) satisfy the Itô equation
\[
\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} -adt & dw \\ -dw & -adt \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} dN
\]
with \( a = 1/2 \) and
\[ b = -1 \pm \frac{1}{2} \sqrt{4 + \frac{1}{x^2 + y^2}} \]
Assuming that \( x^2(0) + y^2(0) = 1 \), describe the set of points that sample trajectories will visit. Hint: Use the Itô rule to compute \( d(x^2 + y^2) \).

**Solution:** Applying the Itô rule to the function \( \psi(x, y) = x^2 + y^2 \) we get
\[
d\psi(x, y) = 0 dt + 0 dw + [(x + bx)^2 - x^2 + (y + by)^2 - y^2] dN
\]
which evaluates to
\[ d\psi(x, y) = (x^2(2b + b^2) + y^2(2b + b^2)) dN = dN \]
For the given definition of \( b \) this is of the form \( d\psi = dN \). The solutions will only visit points lying on circles centered at the origin and having radii that are positive integers.

### 3.9 Computing Temporal Correlations

We can use the same multivariable method for computing temporal correlations that were discussed in section 2.7. If \( x \) satisfies the Itô equation
\[
dx = -f(x)dt + g(x)dw
\]
then for \( \tau > t \), \( x(t)x(\tau) \) satisfies
\[
d_x x(t)x^T(\tau) = x(t)f^T(x(\tau))d\tau + x(t)g^T(x(\tau))dw, \]
Thus we can take expectation to get
\[
\frac{d}{d\tau} \mathbb{E} x(t)x^T(\tau) = \mathbb{E} x(t)f^T(x(\tau))
\]
This relationship can often be used with a formula for $\mathcal{E}x(t)x^T(t)$ to compute correlations.

**Example:** Consider the Itô equation

$$\begin{align*}
    dx &= (-x + v)dt ; \quad x(0) = 0 \\
    dv &= (-v - x)dt + dw ; \quad v(0) = 0
\end{align*}$$

Compute the steady state value of the covariance matrix

$$
    \Sigma(\tau) = \mathcal{E} \left[ \begin{array}{c}
        x(t) \\
        x(t+\tau)
    \end{array} \right] \left[ \begin{array}{c}
        x(t) \\
        x(t+\tau)
    \end{array} \right]
$$

The steady state value of the covariance matrix

$$
    \Sigma(\tau) = \mathcal{E} \left[ \begin{array}{c}
        x(t) \\
        x(t+\tau)
    \end{array} \right] \left[ \begin{array}{c}
        x(t) \\
        x(t+\tau)
    \end{array} \right]
$$

can be obtained from the steady state solution of the variance equation associated with $(x, v)$,

$$
    0 = \begin{bmatrix}
        -1 & 1 \\
        -1 & -1
    \end{bmatrix} \begin{bmatrix}
        \sigma_{11} & \sigma_{12} \\
        \sigma_{21} & \sigma_{22}
    \end{bmatrix} + \begin{bmatrix}
        \sigma_{11} & \sigma_{12} \\
        \sigma_{21} & \sigma_{22}
    \end{bmatrix} \begin{bmatrix}
        -1 & 1 \\
        1 & -1
    \end{bmatrix} + \begin{bmatrix}
        0 & 0 \\
        0 & 1
    \end{bmatrix}
$$

by solving

$$
    \Sigma(\tau) = \Sigma_\infty e^{A^T \tau}
$$

Note that

$$
    e^{A^T \tau} = \begin{bmatrix}
        e^{-\tau} \cos \tau & e^{-\tau} \sin \tau \\
        -e^{-\tau} \sin \tau & e^{-\tau} \cos \tau
    \end{bmatrix}
$$

The steady state value of $\Sigma$ is

$$
    \Sigma_\infty = \begin{bmatrix}
        \frac{1}{8} & \frac{1}{8} \\
        \frac{1}{8} & \frac{1}{8}
    \end{bmatrix}
$$

Thus in steady state we have

$$
    \mathcal{E}x(t)x^T(t+\tau) = \begin{bmatrix}
        \frac{1}{8} e^{-\tau} (\cos \tau - \sin \tau) & \frac{1}{8} e^{-\tau} (\cos \tau + 3 \sin \tau) \\
        \frac{1}{8} e^{-\tau} (\cos \tau + 3 \sin \tau) & \frac{1}{8} e^{-\tau} (\cos \tau + \sin \tau)
    \end{bmatrix}
$$

### 3.10 Linear Equations

The combination of the Itô rule and the results of 3.5 on computing expectations gives an effective method for computing the statistical properties of solutions of stochastic equations of the form

$$
    dx = Axdt + \sum_{i=1}^{m} B_i xdw_i + \sum_{i=1}^{k} b_i dv_i
$$
where \( w_i \) and \( v_i \) are independent Wiener processes. The special cases

\[
\begin{align*}
    dx &= Ax dt + \sum_{i=1}^{k} b_i dv_i
\end{align*}
\]

which involve only additive white noise terms have been used extensively as models for physical and economic systems. The observation that in terms of an enlarged vector \((1, x^T)^T\) we can write

\[
    d \begin{bmatrix} 1 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & A \\ 0 & 0 & B_i \end{bmatrix} \begin{bmatrix} 1 & x \end{bmatrix} dt + \sum_{i=1}^{m} \begin{bmatrix} 0 & 0 & 0 \\ b_i & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} dw_i + \sum_{i=1}^{m} \begin{bmatrix} 0 & b_i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} dv_i
\]

means that it is never necessary to display the inhomogeneous term explicitly. For this reason we begin by developing the basic properties of

\[
    dx = Ax dt + \sum_{i=1}^{m} B_i x dw_i
\]

**Fact 1:** If \( x \) satisfies a linear sample path equation then

\[
    \frac{d}{dt} \mathcal{E} x(t) = A \mathcal{E} x(t)
\]

**Proof:** Take expectations of both sides of (1). Use the fact that \( \mathcal{E} g(x) dw = 0 \).

**Fact 2:** If \( x \) satisfies a linear sample path equation then for each positive integer \( p \) the set of moments of order \( p \) satisfy a linear homogeneous differential equation.

**Proof:** We introduce the notation \( x^{[p]} \), where \( x \) is an \( n \)-vector and \( p \) is a positive integer, to denote the \((n+p-1)\)-component vector whose entries are the independent monomials homogeneous of degree \( p \). That is,

\[
    x = \begin{bmatrix}
        x_1 \\
        x_2 \\
        \vdots \\
        x_n
    \end{bmatrix}; \quad x^{[p]} = \begin{bmatrix}
        x_1^p \\
        x_1^{p-1} x_2 \\
        \vdots \\
        x_1 x_2^{p-1} \\
        \vdots \\
        x_1 x_2^{p-1} \\
        \vdots \\
        x_n^p
    \end{bmatrix}
\]

If \( \dot{x} = Ax \), then it is easy to see that \( x^{[p]} \) also satisfies a linear equation so that we have, for \( A^{[p]} \) suitably defined,

\[
    \frac{d}{dt} x^{[p]} = A^{[p]} x^{[p]}
\]

We take this as a definition of \( A^{[p]} \). In applying the Itô rule to (3.1) we get

\[
    dx^{[p]} = A^{[p]} x^{[p]} dt + \sum_{i=1}^{m} B_{i[p]} x^{[p]} dw_i + \text{“Itô term”}
\]

the Itô term denoting the contributions due to

\[
    \left< \frac{\partial^2 x^{[p]}}{\partial x_i \partial x_j} g_i, g_j \right> dt
\]


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in using the Itô rule. Because the entries of \( x^{[p]} \) are homogeneous of degree \( p \), their second derivatives are homogeneous of degree \( p - 2 \). Since \( B_i x dW_i \) is linear in \( x \), we see that the Itô term is homogeneous of degree \( p \) in \( (x_1, x_2, \ldots, x_n) \) or, what is the same, linear in \( x^{[p]} \). Thus after taking expectation, we get

\[
\frac{d}{dt} \mathcal{E} x^{[p]} = L_{[p]} \mathcal{E} x^{[p]}
\]

for some \( L_{[p]} \). One can show that \( L_{[p]} \) is given by

\[
L_{[p]} = \left( A - \frac{1}{2} \sum_{i=1}^{m} B_i^2 \right)_{[p]} + \frac{1}{2} \sum_{i=1}^{m} (B_{[i]}^{[p]})^2
\]

Fact 3: If \( x \) satisfies a linear sample path equation then for \( \tau \geq 0 \)

\[
\mathcal{E} x(t)x^T(t + \tau) = \mathcal{E} x(t)x^T(t)e^{AtT}
\]

Proof: It is easy to see that

\[
\frac{d}{d\tau} \mathcal{E} x(t)x^T(t + \tau) = \mathcal{E} x(t)x^T(t + \tau)A^T
\]

which is solved with initial condition \( \mathcal{E} x(t)x^T(t) \).

3.11 Asymptotic Behavior

Under suitable hypothesis the probability density \( \rho(t, x) \) associated with an Itô equation will approach a limit as \( t \) goes to infinity. This means that the process is approaches a “steady state” in the sense that the statistical properties approach constant values. In many cases it is only this steady state which is experimentally observable and a great deal of the study of statistical physics and classical communication theory is based on steady state analysis.

For a one dimensional linear system the situation is as follows. If we have

\[
dx = ax dt + dw
\]

then the 2nd moment equation is

\[
\frac{d}{dt} \mathcal{E}x^2 = 2a \mathcal{E}x^2 + 1
\]

and so

\[
\mathcal{E}x^2(t) = e^{2at} \mathcal{E} \left( x^2(0) + \frac{1}{2a} \right) - \frac{1}{2a}
\]

Thus we see that if \( a \) is positive the variance goes to infinity with increasing \( t \) whereas if \( a \) is negative it goes to \( -1/2a \) with increasing \( t \). In fact, since the response to \( \rho(0, x) = \delta(x - x_0) \) is

\[
\rho(t, x) = \frac{1}{\sqrt{2\pi\sigma(t)}} e^{-a(x-x_0e^{at})^2/2\sigma(t)}
\]
we see that, regardless of the form of the initial distribution, $\rho$ goes to
$$\rho(x) = \sqrt{-a/\pi} e^{ax^2}$$
as $t$ goes to infinity provided that $a$ is negative. In this case the situation may be summarized as follows: If the nonrandom part of the stochastic equation is asymptotically stable then the density approaches a steady state as $t$ goes to infinity.

This same conclusion is valid in many situations. In particular, for linear systems of the form
$$dx = Ax dt + \sum_{i=1}^{m} b_i dw_i$$
we have a differential equation for the second moment $\Sigma(t) = E(x(t)x^T(t))$ taking the form
$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A^T + b_1 b_1^T + \cdots + b_m b_m^T$$
(To see this, apply the Itô rule taking advantage of the fact that the $w_i$ are independent and the fact that the second derivative $\frac{\partial^2 x_n x_m}{\partial x_i \partial x_j}$ takes a simple form) This has a solution which may be expressed as
$$\Sigma(t) = \int_0^t e^{A(t-\sigma)} BB^T e^{A^T(t-\sigma)} d\sigma + e^{At} \Sigma(0) e^{A^T t}$$
where $B = (b_1, b_2, \ldots, b_n)$. If the eigenvalues of $A$ have negative real parts, then $\Sigma(t)$ approaches a limit as $t$ goes to infinity. The density $\rho$ also approaches a limit in this case
$$\lim_{t \to \infty} \rho(t,x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma_\infty}} e^{-\frac{1}{2} x^T \Sigma^{-1}_\infty x}$$
where $\Sigma_\infty$ satisfies
$$A\Sigma_\infty + \Sigma_\infty A^T + BB^T = 0$$
or
$$\Sigma_\infty = \int_0^\infty e^{At} BB^T e^{A^T t} dt$$

**Example 1:** Solve the Fokker-Planck equation for
$$dx_1 = x_2 dt$$
$$dx_2 = -x_1 dt + dw$$
In this case $E x(t) = e^{At} E x(0) = \bar{x}(t)$ and $\Sigma(t)$ is given by
$$\Sigma(t) = \int_0^t \begin{bmatrix} \cos(t-\sigma) & \sin(t-\sigma) \\ -\sin(t-\sigma) & \cos(t-\sigma) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(t-\sigma) & -\sin(t-\sigma) \\ \sin(t-\sigma) & \cos(t-\sigma) \end{bmatrix} d\sigma$$
$$\Sigma_\infty = \int_0^\infty e^{At} BB^T e^{A^T t} dt$$
With these definitions \( \rho(t, x) = \frac{1}{\sqrt{(2\pi)^d \det \Sigma(t)}} e^{-\frac{1}{2} (x - \bar{x}(t))^T \Sigma^{-1}(t)(x - \bar{x}(t))} \)

**Example 2:** Consider the scalar Itô equation
\[
\frac{dx}{dt} = -x dt - \alpha x dw + \beta x dv \quad ; \quad x(0) = 0
\]
where \( w \) and \( v \) are independent Brownian motions. Derive an equation for the \( p \)th moment of \( x \) and give conditions on \( \alpha \) and \( \beta \) for the \( p \)th moment to have a finite limit as \( t \) goes to infinity. Applying the Itô rule with \( \psi(x) = x^p \) and \( p \geq 2 \), we need the first and second derivative of \( \psi(x) \),
\[
\frac{\partial \psi(x)}{\partial x} = px^{p-1}, \quad \frac{\partial^2 \psi(x)}{\partial x^2} = p(p-1)x^{p-2}. \]
Hence, we get
\[
\frac{dx^p}{dt} = px^{p-1}(-x dt - \alpha x dw + \beta x dv) + \frac{1}{2} p(p-1)x^{p-2} \left( g_1(x)^2 + g_2(x)^2 \right) dt
\]
Thus
\[
\frac{1}{2} px^p \left( -2 dt - 2\alpha dw + 2\beta dv + (p-1)\alpha^2 dt + (p-1)\beta^2 dt \right)
\]
\[
= \frac{1}{2} px^p \left( ((p-1)(\alpha^2 + \beta^2) - 2) dt - 2\alpha dw + 2\beta dv \right)
\]
Because \( E dw = E dv = 0 \), the expectation law yields
\[
\frac{d}{dt} E x^p = \frac{1}{2} p ((p-1)(\alpha^2 + \beta^2) - 2) E x^p
\]
The solution is expresses as
\[
E x^p = e^{p ((p-1)(\alpha^2 + \beta^2)/2 - 1) t} E x^p(0)
\]
Consequently we may say that for \( p \geq 2 \), \( E x^p \) has a finite limit as \( t \to \infty \) if \( p ((p-1)(\alpha^2 + \beta^2)/2 - 1) \leq 0 \) or
\[
\alpha^2 + \beta^2 \leq \frac{2}{p-1}
\]
When \( p = 0 \) and \( p = 1 \), the expectation of the moment has always a finite limit.

**Example 3:** Consider the piecewise linear stochastic equation of the Itô type
\[
\frac{dx}{dt} = f(x) dt + \sqrt{a} dw
\]
where
\[
f(x) = \begin{cases} 
-x + 2 & x \geq 1 \\
-x - 2 & x \leq -1 \\
x & |x| < 1 
\end{cases}
\]
We want to find the steady state solution of the corresponding Fokker Planck equation by piecing together, in a continuous way, three Gaussian solutions and normalizing appropriately. It will happen that one of “Gaussians” has a positive quadratic term in the exponent
but this will not cause difficulty. We will also show that as \( a \) approaches zero the steady state density tends, in a weak sense, to the sum of two delta functions.

The solution of the variance equation in the central region \( |x| \leq 1 \) is

\[
\rho = be^{x^2/a}
\]

In the side regions where \( |x| \geq 1 \) we have

\[
\rho = ce^{-(x-2)^2/a}
\]

and

\[
\rho = de^{-(x+2)^2/a}
\]

By symmetry, \( c = d \). Continuity at \( \pm 1 \) gives an overall solution of the form

\[
\rho_{ss}(x) = \begin{cases} 
  ce^{-(x-2)^2/a} & x \geq 1 \\
  ce^{-2/a}e^{x^2/a} & |x| < 1 \\
  ce^{-(x+2)^2/a} & x \leq -1
\end{cases}
\]

We must chose \( c \) so that the area under \( \rho_{ss} \) is one and so

\[
c^{-1} = \int_{-\infty}^{-1} e^{-(x-2)^2/a} dx + \int_{-1}^{1} e^{-2/a}e^{x^2/a} dx + \int_{1}^{\infty} e^{-(x+2)^2/a} dx
\]

### 3.12 Stochastic Approximation

There are various senses in which a sequence of real-valued random variables might be said to converge. It is useful to keep in mind that if \( \{x_1, x_2, \ldots, x_k, \ldots\} \) is a sequence of random variables then there is associated with each element of the sequence a different probability distribution, say \( P_1, P_2, \ldots, P_k, \ldots \). Suppose, that these distributions have densities, \( p_1, p_2, \ldots, p_k, \ldots \). The densities are \( L_1 \) functions mapping the real line into the positive half-line. If the sequence of random variables is to converge then the sequence of densities must in some, as yet to be described sense, “converge” to a delta function. Such an analysis requires a careful choice of topologies. On the other hand, if we are content to consider convergence in the sense that

\[
\lim_{k \to \infty} \int_{\mathbb{R}} (x - x_0)^2 p_k(x) dx = 0
\]

then convergence is much easier to deal with. The literature on stochastic approximation is usually concerned with discrete time models. there is usually a more or less obvious way to pass from a discrete-time description to a continuous-time description. Here we only consider the latter.

Consider the following result on the asymptotic properties of the deterministic equation.

**Lemma:** Let \( a \) and \( b \) be integrable functions with \( b \) nonnegative. The solutions of

\[
\dot{x}(t) = a(t)x(t) + b(t)
\]
3.12. STOCHASTIC APPROXIMATION

go to zero as \( t \) goes to infinity if

\[
\lim_{t \to \infty} \int_0^t a(\sigma) d\sigma \to -\infty
\]

and

\[
\lim_{t \to \infty} \int_0^t b(\sigma) d\sigma < \infty
\]

**Proof:** The solution of this equation is given by

\[
x(t) = e^{\int_0^t a(\tau) d\tau} x(0) + \int_0^t e^{\int_\tau^t a(\sigma) d\sigma} b(\eta) d\eta
\]

The asymptotic properties of \( a \) imply that the first term goes to zero. Because of the convergence of the integral of \( b \), given any \( \epsilon > 0 \) there exists a time \( T \) such that

\[
\int_T^\infty b(\tau) d\tau \leq \epsilon
\]

For \( t \geq T \) we have

\[
x(t) = e^{\int_0^T a(\tau) d\tau} x(0) + \int_0^T e^{\int_\tau^T a(\sigma) d\sigma} b(\eta) d\eta + \int_T^t e^{\int_\tau^t a(\sigma) d\sigma} b(\eta) d\eta
\]

The first two terms on the right go to zero and the third can be made arbitrarily small by choice of \( T \).

Now consider the scalar stochastic equation

\[
dw = a(t) x(t) dt + dw
\]

It has an associated variance equation

\[
\dot{\sigma}(t) = -2a(t)\sigma(t) + a^2(t)
\]

Applying the lemma we see that if

\[
\lim_{t \to \infty} \int_0^t 2a(\sigma) d\sigma \to -\infty
\]

and

\[
\lim_{t \to \infty} \int_0^t a^2(\sigma) d\sigma < \infty
\]

then the variance goes to zero and we can assert that the random variable \( x(t) \) converges to zero as \( t \) goes to infinity in mean square sense.

The main fact here is that \( a \) must go to zero at just the right rate if \( x \) is to go to zero. This same analysis can be used much is different if \( x \) is replaced by \( f(x) \) in the right-hand side of this equation, as long as \( f(\cdot) \) is a first and third quadrant, monotone increasing function.
3.13 Exit Times

Consider the problem of determining the probability that the solution of
\[ dx = f(x)dt + g(x)dw; \ x(0) \in S_1 \]
leaves an open connected set \( S \supset S_1 \) before time \( t \). Such problems arise in the analysis of the life expectancy of a machine, the time to financial ruin, etc. One way to formulate this is to consider a modified process which satisfies the given equation as long as \( x \in S \) and satisfies \( dx = 0 \) once \( x \) reaches the boundary of \( S \). The corresponding Fokker-Planck equation in the set \( S \) is
\[
\frac{\partial \rho(t, x)}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x) \rho(t, x) + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} g_i(x) g_j(x) \rho(t, x)
\]
In terms of a physical picture we can imagine that the boundary of \( S \) absorbs the process and so, insofar as motion inside \( S \) is concerned, the appropriate boundary condition is \( \rho(t, x) = 0 \) for \( x \) on the boundary of \( S \). To answer the original question then, we would need to solve for \( \rho \) subject to this boundary condition. The probability that \( x \) does not leave \( S \) on the interval \([0, t]\) is then just
\[
p = \int_{S} \rho(t, x) dx
\]
with the integral extending over the open set \( S \).

**Example:** Suppose that we have a Gauss-Markov process
\[ dx = -x dt + dw; x(0) = 0 \]
and want to know the probability that \( x(t) \) has not left the interval \([-\pi, \pi]\) over the period \( 0 \leq t \leq 1 \). The Fokker-Planck equation is
\[
\frac{\partial \rho(t, x)}{\partial t} = \frac{\partial}{\partial x} x \rho(t, x) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \rho; \ \rho(t, -1) = \rho(t, 1) = 0
\]
The nature of the boundary conditions suggest that we seek an even solution using an expansion in terms of trigonometric functions.
\[
\rho(t, x) = \sum_{n=0}^{\infty} p_n(t) \cos nx
\]
Standard separation of variable techniques yield an equation for the individual \( p_n \),
\[
p_n = p_n - n^2 p_n - \frac{1}{n} p_n
\]
Because the probability that \( x \) has not left \([-\pi, \pi]\) in the interval \([0, t]\) is just \( \rho_0(t) \) we see that
\[
\text{prob} = e^{-t}
\]
3.14 Exercises Chapter 3

1. Find the Fokker-Planck equation associated with the linear equation

\[ dx = -x \, dt + dw ; \quad x(0) = 0 \]

Verify by calculation that the solution is of the form

\[ \rho(t, x) = \frac{1}{\sqrt{2\pi a(t)}} e^{-(x-b(t))^2/2a(t)} \]

and find \( a \) and \( b \).

2. Suppose that

\[ dx = dw ; \quad \rho(0, x) = \phi(x) \]

Show that the solution to the Fokker-Planck equation is

\[ \rho(t, x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} \phi(y) \, dy \]

Evaluate the integral for \( \rho(0, x) = \frac{1}{2} e^{-|x|} \).

3. Find the Fokker-Planck equation associated with the pair of linear equations

\[ dx = -y \, dt \]
\[ dy = -x \, dt + dw \]

4. If \( x_1 \) and \( x_2 \) satisfy the Itô equations

\[ dx_1 = dw_1 \]
\[ dx_2 = dw_2 \]

then the Fokker Planck equation is just

\[ \frac{\partial \rho(t, x, y)}{\partial t} = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \rho(t, x, y) = \frac{1}{2} \nabla^2 \rho(t, x, y) \]

That is to say, the right-hand side is one-half the Laplacian. If we make the change of variables

\[ \theta = \tan^{-1} \frac{x_2}{x_1}, \]
\[ r = \sqrt{x_1^2 + x_2^2} \]

then find the evolution equation for the probability density \( \hat{\rho}(t, \theta) \). (This is partially done in the notes.) You should find that \( \hat{\rho} \) satisfies the equation

\[ \frac{\partial \hat{\rho}(t, \theta)}{\partial t} = \frac{r}{2} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \hat{\rho}(t, \theta) = \frac{1}{2} \nabla^2 \hat{\rho}(t, \theta) \]

where \( \nabla^2 \) now stands for the Laplacian in radial coordinates. What accounts for the \( r \) and \( 1/r \) terms appearing on the left?

5. In the previous chapter we saw the use of Poisson counters in modeling processes that can only take on values in a certain range or even a certain set. Wiener processes can be used to model processes which take on values in an interval. If we want a process that takes on values on \([0, \infty)\) then we might use the log normal idea, leading to

\[ dx = ax \, dt + bw \, dw ; \quad x(0) > 0 \]
Show that the solution of
\[ dx = a(x - 1)(x - 3)dt + b(x - 1)(x - 3)dw \quad 1 < x(0) < 3 \]
remains within the interval \((1, 3)\). Hint: Suppose that we make a change of variables so as to map the interval \((1, 3)\) to the whole line \((-\infty, \infty)\). For example, the choice \(y = \ln \left( \frac{x - 1}{3 - x} \right)\) is a possibility. Then using the Itô rule we have
\[
\frac{dy}{x - 1} = \frac{dx}{3 - x} \left( a(x - 1)(x - 3)dt + b(x - 1)(x - 3)dw \right) + b^2 ((x - 1)^2 + (3 - x)^2) dt
\]
One can solve \(y = \ln \left( \frac{x - 1}{3 - x} \right)\) for \(x\). In fact \(x = (3e^y - 1)/(1 + e^y)\). Check what has been claimed here and finish off the calculation of an equation for \(y\).

6. Solid evidence confirming that many parts of analysis are self-consistent notwithstanding, it is sometimes worthwhile to verify a particular fact with a direct calculation. On one hand, for
\[ dx = (adt + bdw)x \quad x(0) = \bar{x} \]
we can compute directly the value of \(E_x(t)\). On the other hand, we can change variables \(y = \ln x\) and get an equation for \(y\) of the form
\[
\frac{dy}{a - b^2/2} + dw \quad y(0) = \ln \bar{x}
\]
Because of the simple form of this equation we can write down the density by inspection,
\[
\rho(y) = \frac{1}{\sqrt{2\pi b^2 t}} e^{-\frac{1}{2}(y - \ln \bar{x} - (a - b^2/2)t)^2}
\]
Use the transformation of densities formula to get the density of \(x\); call it \(\rho_x\). Evaluate the integral
\[
E_x = \int_{-\infty}^{\infty} y \rho_x(x) dx
\]
What is involved if you are given a general \(\rho(x(0))\) and not a deterministic initial condition?

7. Define \(f(x)\) as
\[
f(x) = \begin{cases} 
-1 & x > 1 \\
-x & |x| \leq 1 \\
1 & x < -1 
\end{cases}
\]
Show that
\[ dx = f(x)dt + dw \]
Has a steady state probability density of the form
\[
\rho(x) = \begin{cases} 
\alpha e^{-x^2/4} & |x| \leq 1 \\
\beta e^{-2x} & |x| > 1 
\end{cases}
\]

8. Find the differential equation for the probability density \(\rho(t, x)\) associated with the Itô equation
\[ dx = axdt + bx dw \quad x(0) = 1 \]
and solve assuming the initial condition \(x(0) = 1\). Hint: You may want to think about the change of variables \(y = \ln x\).
9. If $x_1$ and $x_2$ satisfy the Itô equations

\[
\begin{align*}
\frac{dx_1}{dt} &= dw_1 \\
\frac{dx_2}{dt} &= dw_2
\end{align*}
\]

and if we make a change of variables $\theta = \tan^{-1}(x_2/x_1)$, $r = ((x_1)^2 + (x_2)^2)^{\frac{1}{2}}$, then find an expression for the probability density $\rho(t, r, \theta)$ assuming that the initial density is rotationally symmetric. If $x_1(0) = x_2(0) = 0$ find the probability that $x_1^2(t) + x_2^2(t) \leq a$.

10. Consider the stochastic differential equation

\[
\begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt}
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{2} dt & dw \\
-dw & -\frac{1}{2} dt
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

Show that $x_1^2(t) + x_2^2(t) = \text{constant}$. Show that if $\theta = \tan^{-1}(x_1/x_2)$, then

\[d\theta = dw_1\]

for some suitably defined Wiener process $w_1$. Show that in terms of $\theta$ the probability density satisfies

\[
\frac{\partial \rho(t, \theta)}{\partial t} = \frac{\alpha}{2} \frac{\partial^2 \rho(t, \theta)}{\partial \theta^2}
\]

for a suitable value of $\alpha$. Find $\alpha$. Solve this equation for $\rho$ on $-\pi \leq \theta \leq \pi$ given that $\theta(0) = 0$.

11. Let $w$ be a standard Wiener process, and let $dx = dw$ with $x(0) = 0$. Evaluate

\[m(t) = \mathbb{E} e^{x^2(t)}\]

12. Consider, again, the polar coordinate representation of two-dimensional Wiener process as developed in example two of section 3.4.

\[
\begin{align*}
\frac{dr}{dt} &= \frac{1}{r} dt + \sin \theta dv + \cos \theta dw \\
\frac{d\theta}{dt} &= \frac{\cos \theta dv - \sin \theta dw}{r} + \frac{\sin \theta \cos \theta}{r^2} dt
\end{align*}
\]

The corresponding Fokker-Planck equation makes apparent the rotational symmetry present.

\[
\frac{\partial \rho(t, r, \theta)}{\partial t} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \rho(t, r, \theta)
\]

The term in parentheses on the right is just the Laplacian expressed in polar coordinates. Show that this equation admits the rotationally symmetric solution

\[\rho(t, r, \theta) = \frac{r}{2\pi t} e^{-r^2/2t}\]

13. Consider the Itô equation

\[
\begin{align*}
dx &= dw_1 \\
dy &= dw_2 \\
dz &= x dw_2 - y dw_1
\end{align*}
\]

Show that

\[
\frac{d}{dt} \mathbb{E} z^2(t) = \mathbb{E} (x^2(t) + y^2(t)) = 2t
\]

and that

\[
\frac{d}{dt} \mathbb{E} z^2(t)(x^2(t) + y^2(t)) = \mathbb{E} (x^2(t) + y^2(t))^2 + \mathbb{E} z^2(t)
\]
14. Consider the Itô equations for \(x, y, z\) in the previous problem. Show that the corresponding Fokker-Planck Equation is

\[
\frac{\partial \rho(x,t)}{\partial t} = \frac{1}{2} \left( \frac{\partial}{\partial x} - y \frac{\partial}{\partial z} \right)^2 \rho(t,x) + \frac{1}{2} \left( \frac{\partial}{\partial y} - x \frac{\partial}{\partial z} \right)^2 \rho(t,x)
\]

Show that for each real value of \(\lambda\) the expression \(\rho(t, x, y, z) = e^{-2\lambda t} e^{-\lambda(x^2+y^2)} \cos 2\lambda z\) satisfies this equation. It does not have a constant integral with respect to the variables \(x, y, z\). Discuss the possibility of forming a weighted combination of solutions of this type to obtain a solution which remains normalized with growing values of \(t\).

15. Consider the Itô equations

\[
\begin{align*}
    dx &= x dt + (xz/\sqrt{x^2+y^2}) dw + zy \sqrt{x^2+y^2} dt \\
    dy &= -y dt - (zy/\sqrt{x^2+y^2}) dw - zx \sqrt{x^2+y^2} dt \\
    dz &= -z \sqrt{x^2+y^2} dt
\end{align*}
\]

Show that as \(t\) goes to infinity

\[
\mathcal{E} x^2 \to 0 \\
\mathcal{E} y^2 \to 0
\]

and

\(z(t) \to 0\)

16. In the treatment of noise in linear electrical circuits one models a resistor-inductor circuit (following Nyquist and Johnson) via an Itô equation \(L \, di = -Ri \, dt + \sqrt{kRT} \, dw\) where \(T\) is the temperature with respect to an absolute scale and \(k\) is a suitable constant (Boltzman’s constant). Determine the expected value of the energy stored in the inductor in steady state.

17. Consider the Wiener process \(x\)

\[
dx = dw \quad ; \quad x(0) = 0
\]

Let \(t\) be a positive number and suppose that it is known that \(x(t) = 0\). For \(0 \leq \tau \leq t\) find the probability density for \(x(\tau)\) conditioned on the fact that \(x(t) = 0\). What is the probability density for

\[
n = \int_0^t x(\sigma) d\sigma
\]

conditioned on the fact that \(x(t) = 0\)?

18. Suppose that \(x_1\) and \(x_2\) satisfy the differential equations

\[
dx_1 = x_2 \, dt
\]
\[ dx_2 = (-x_1 - x_2) dt + dw \]

Suppose that the process has reached steady state. Show that the probability that \( x(t)x(t+\tau) \) is positive is

\[ p(\tau) = \frac{1}{\pi} \tan^{-1} \alpha \]

with \( \alpha \) being derived from the autocorrelation function.

19. Given that

\[ \frac{dx}{dy} = \frac{y}{\alpha x dt + \beta y dt + d\omega} \]

for what values of \( \alpha \) and \( \beta \) will \( \rho(t, x, y) \) have a limiting value as \( t \) goes to infinity? Suppose that \( \beta = -3 \) and \( \alpha = -2 \), find

\[ \lim_{t \to \infty} E_y(t)y(t + \tau) \]

20. Compute the probability that the solution of

\[ dx_1 = dw_1 \quad ; \quad x_1(0) = 0 \]
\[ dx_2 = dw_2 \quad ; \quad x_2(0) = 0 \]

lies in the disk \( \{x_1, x_2 | x_1^2 + x_2^2 \leq 1 \} \) at \( t = 1 \). Evaluate the expected value of

\[ m(t) = e^{x_1(t)x_2(t)} \]

21. Consider the modified diffusion equation

\[ \frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial^2 \rho}{\partial x^2} - \rho - x \frac{\partial \rho}{\partial x} \]

Find an Itô equation of the form

\[ dx = f(x) dt + g(x) dw \]

such that the above diffusion equation describes the probability density. Solve for the density given \( x(0) = 0 \).

22. Consider the equations

\[ dx = x dt + dw - dt \quad ; \quad x(0) = (0) \]
\[ dy = xy dt \quad ; \quad y(0) = 1 \]

Find the probability density \( \rho(t, x, y) \)?

23. Evaluate \( \mathcal{E} x^2(t) \) for \( x \) satisfying

\[ \frac{dx}{dx} = -2z dN \]
\[ \frac{dz}{dx} = (\alpha + z) x dt + dw \]

Here \( N \) is a Poisson counter of rate \( \lambda \) and \( w \) is a standard Wiener process.

24. Consider a system of the form

\[ dx = (-2z x - x) dt + dw \]
\[ dz = -2z dN \quad ; \quad z(0) \in \{+1 \} \]

with \( N \) a Poisson Counter of rate \( \lambda \). The \( x \) equation is “unstable” with growth \( e^t \) when \( z = -1 \) but is stable with decay \( e^{-3t} \) when \( z = +1 \). Of course \( z = +1 \) and \( z = -1 \) are equally likely and using the methods of section 3.5 we see that

\[ \frac{\partial}{\partial x} \mathcal{E} x^2 = -4\mathcal{E} z x^2 - 2\mathcal{E} z x^2 + 1 \]
\[ \frac{\partial}{\partial z} \mathcal{E} x^2 = -4\mathcal{E} x^2 - 2\mathcal{E} x^2 - 2\lambda \mathcal{E} x^2 + \mathcal{E} z \]

Show that the solutions of these equations approach a constant regardless of the initial conditions.
25. Consider the pair of equations
\[
\begin{align*}
    dx_1 &= -2x_1 dN ; \quad x_1(0) \in \{-1, 1\} \\
    dx_2 &= -10x_2 dt + x_1 dt + dw 
\end{align*}
\]
with \(N\) being a Poisson counter of rate \(\lambda\) and \(w\) being a Weiner process. Compute
\[
\phi(t, \tau) = \mathcal{E} x_2(t) x_2(t + \tau) \quad ; \quad \tau \geq 0
\]
Does there exist a limiting value for \(\mathcal{E} x_2^2(t)\) as \(t\) goes to infinity?

26. Compute the autocorrelation function for the steady state of \(x(t)\) where
\[
\begin{align*}
    dx(t) &= -x(t) dt + z(t) x(t) dt + dw \\
    dz &= -2z dN ; \quad z(0) = 1 
\end{align*}
\]
with \(N\) being a Poisson counter of rate \(\lambda\).

27. Consider the pair of stochastic differential equations
\[
\begin{align*}
    dx &= z x dt + dw \\
    dz &= -2z dN \quad ; \quad z(0) \in \{-1, +1\}
\end{align*}
\]
where \(N\) is a Poisson counter of rate \(\lambda\). The Ito rule gives
\[
d\psi(x, z) = \frac{\partial \psi}{\partial x} z x dt + [\psi(x, -z) - \psi(x, z)] dN + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} dz^2 + \frac{\partial \psi}{\partial x} dw
\]
Assume that the initial density of \(x\) exists and that a density exists for all positive time, say \(\rho: \mathbb{R}^+ \times \mathbb{R} \times \{1, -1\} \to \mathbb{R}\).
Writing \(\rho(t, x, 1)\) as \(\rho_+(t, x)\) and \(\rho(t, x, -1)\) as \(\rho_-(t, x)\) show that
\[
\frac{\partial \rho_+(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2 \rho_+(t, x)}{\partial x^2} + \frac{\partial \rho_+(t, x)}{\partial x} \left( \frac{1}{2} \frac{\partial^2 \rho_-(t, x)}{\partial x^2} + \frac{\partial \rho_-(t, x)}{\partial x} \right) + \lambda \left( \rho_+(t, x) - \rho_-(t, x) \right) + \lambda \left( \rho_-(t, x) - \rho_+(t, x) \right)
\]

28. Consider the pair of Itô equations
\[
\begin{align*}
    dx &= z(t) x dt + dw \quad ; \quad x(0) \in \mathbb{R}^1 \\
    dz &= -2z(t) dN \quad ; \quad z(0) \in \{\pm 1\}
\end{align*}
\]
where \(w\) is a standard Wiener process and \(N\) is a Poisson counter of rate \(\lambda\). Find the Fokker-Planck equation for this system and express it in terms of two coupled diffusion equations. Does there exist a limiting value as \(t \to \infty\) for \(\mathcal{E} x^2(t)\)?

29. (Heat baths and equipartition of energy) Let \(x\) take on values in \(\mathbb{R}^n\), let \(S\) be a real skew-symmetric matrix and let \(\epsilon\) be a positive number. Consider the Itô equation
\[
dx = (S - \epsilon G G^T) x dt + \sqrt{\epsilon} G dw
\]
Assume that \((G, SG, \ldots, S^{n-1}G)\) has rank \(n\). In steady state the variance satisfies
\[
\Sigma(S - \epsilon G G^T)^T + (S - \epsilon G G^T) \Sigma = -\epsilon G G^T
\]
Show that \(\Sigma = (1/2) I\) is the unique solution of this equation for all \(\epsilon > 0\). Now consider
\[
\begin{bmatrix}
    dx \\
    dz
\end{bmatrix} = \begin{bmatrix}
    S - GG^T & B \\
    -B^T & \Omega
\end{bmatrix} \begin{bmatrix}
    x \\
    z
\end{bmatrix} dt + \begin{bmatrix}
    G dw \\
    0
\end{bmatrix}
\]
Evaluate the steady state variance assuming \(\Omega = -\Omega^T\) and rank \((B, \Omega B, \ldots, \Omega^k B) = k\), where \(k = \text{dim } x + \text{dim } z\).
30. If $dx = dw$ show that
\[ dx^n = nx^{n-1}dw + n(n-1)x^{n-2}dt \]
so that
\[ \frac{d}{dt}x^n = n(n-1)x^{n-2} \]

31. Show that if $dx = -x/(1-t) + dw$ then on the interval $[0,1]$
\[ E_x^4 = 3t(t-1) \]

32. If $dx = -x/(1-t) + dw$ and $dy = -y/(1-t) + dv$ and we let
\[ dz = xdv - ydw \]
Then
\[ E_z^2 = t^2 - 2t^3/3 \]
NOTES AND REFERENCES

1. [1] Gaussian densities/distributions are also called normal densities/distributions. Most applied mathematical treatments of partial differential equations do not solve the heat equations in quite enough generality for our purposes, but the basic ideas are widely discussed. For example, see [8]

2. [2–3] This kind of limiting approach starting from Poisson process was worked out by Paul Levy around 1940. A more recent and highly readable account can be found in McKean [9]. Alternative points of view are explored in Wong [10]

3. Notice that we can keep the same point of view as was used in Chapter 2.

4. The Fokker-Planck equation plays a fundamental role in many physical problems. It is also the point of departure for the study of the very important conditional density equations to be taken up in Chapter 6.


7. A classic text on stationary processes, including material on the expected number of zero-crossings is Cramer [13].

8. The exercises involving coupled diffusion equations are roughly analogous to problems in quantum mechanics involving spin. See, for example,


Chapter 4

Pseudorandom Processes

4.1 Pseudorandom Number Generators

A deterministic process, such as the calculation of the decimal expansion of \( \pi \), may produce a sequence of digits that appears to be random if one checks the relative frequency of occurrence of a particular digit or pair of digits even though there may be sophisticated tests powerful enough to reveal the deterministic origin of the sequence. Do there exist "universal" tests for randomness and what tests would one use to attempt to use to discover if a sequence is truly random? Are all points in the sample space represented with the right frequency? Are there correlations between successive points? Any finite list of tests could be extended by considering more complex tests. The sequence of digits of \( \pi \) would fail certain highly nonlinear tests for randomness but in many applications these tests would be irrelevant.

The generation by computer of "random" numbers and sequences of random numbers is obviously quite important for simulation and even for some types of computation. Most high level computer languages make available random number generators that produce an empirical distribution appearing to be consistent with the uniform distribution on \([0,1]\). Of course the real situation is a bit different. The random numbers produced by computer are all rational and, far from being independent. Successive elements in a sequence are related by a deterministic rule.

Consider the equation

\[ x(k + 1) = (kx(k) + b) \mod 1 \]

If \( k \) is large, say about \( 10^5 \), then multiplication by \( k \) can be thought of as shifting the decimal point of \( x \) five places to the right. The addition of \( b \) further rearranges matters and reducing modulo one serves to make everything to the left of the decimal point irrelevant. Thus successive terms in the sequence, while deterministically related, are related in such a highly nonlinear way that insofar as linear correlation is concerned they may appear to be independent, depending on the exact values of \( x(0), k \) and \( b \).
CHAPTER 4. PSEUDORANDOM PROCESSES

4.2 Uniform versus Gaussian Distributions

According to the central limit theorem, under rather weak hypothesis, the sum of \( n \) independent, identically distributed zero mean, variance \( \sigma \), random variables has a density which when suitably scaled tends to a Gaussian with increasing \( n \). That is, the density for the random variable \( y_N \)

\[
y_N = \sum_{i=1}^{N} \frac{x_i}{\sigma \sqrt{n}}
\]
tends to a Gaussian of unity variance as \( N \) goes to \( \infty \). Expressed in terms of convolution, we have for a unity variance distribution

\[
g(y_2) = f(x) * f(x)
\]
as the density of

\[
y_2 = (x_1 + x_2)
\]
Thus the density for \( y_n \), in the limit as \( n \) goes to infinity, is

\[
\lim_{n \to \infty} \prod_{i=1}^{n} f\left(\frac{x}{\sqrt{n}}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}
\]

One can investigate how this works using the density \( f(x) = 1 \) for \( |x| \leq \frac{1}{2} \) and zero otherwise. (In this case the variance is \( \frac{1}{12} \).) The number of terms required to get a good approximation depends on the circumstances. Naturally the probability that \( (x_1 + x_2 + \cdots + x_n) \) exceeds \( n/2 \) is zero. Thus this remains of limited validity even for \( n \) large. However, near the origin, say \( |x| \leq 3 \) the match is already quite good for \( n = 20 \).

4.3 The Results of Jacobi and Bohl, Sierpinski and Weyl

In this section we sketch two results on deterministic processes.

**Theorem 1:** Let \( x \) be a scalar, \( 0 \leq x < 1 \), and \( \omega \) a real number. If \( f(x) = (x + \omega) \mod 1 \) then the orbits of \( x(k+1) = f(x(k)) \) are dense in \([0,1]\), if and only if \( \omega \) is irrational.

**Proof:** If \( \omega = \frac{2}{p} \), then \( x(p) = (x(0) + \frac{2}{p}) \mod (1) = x(0) \) and so \( x(p) = x(0) \). Thus all motions are periodic. If \( w \) is irrational then \( f^n(x) \neq f^m(x) \) for \( m \) and \( n \) distinct integers; if not

\[
x + n\omega = x + m\omega
\]
which contradicts the fact that \( (n - m)\omega \) is not an integer. Thus each orbit contains an infinity of points in \([0,1]\). There must be at least one limit point. That is, given \( 1/2 > \epsilon > 0 \) there exists \( m \neq n \) such that \( |f^n(x) - f^m(x)| \leq \epsilon \). This means

\[
|f^{(n-m)}(x)| \leq \epsilon
\]
4.4. OTHER DIFFERENCE EQUATIONS

because \( |f(x) - f(y)| = |x - y| \). (Recall that 0 and 1 are to be thought of as being the same point.)

Let \( n - m = q \). If \( f^q(x) - x \) is positive, then \( f^{2q}(x) - x \) is also. We observe that the sequence \( x, f^q(x), f^{2q}(x), \ldots \) satisfies

\[
|f^{rq}(x) - f^{sq}(x)| > \epsilon
\]

for \( r \) and \( s \) integers between 0 and \( 1/\epsilon \). The same is true of \( f^q(x) - x \) negative. But \( \epsilon \) is an arbitrary positive number so the sequence \( f^k(x) \) is dense in \([0, 1]\).

Vector Version: Suppose \( x(k+1) = x(k) + \omega \) with \( x(k) \in \mathbb{R}^n \). In this case \( x(k) \) winds densely if \( k \cdot \omega \in \mathbb{Z} \) and \( k \in \mathbb{Z}^n \) implies \( k = 0 \).

The following theorem is often credited to Bohl, Sierpinski and Weyl. (See [4].) It shows that the process described in Theorem 1 generates a uniform distribution if \( \omega \) is irrational. It is formulated in terms of a rotation on the circle \( S^1 \).

**Theorem 2:** If \( f : S^1 \to S^1 \) is a rotation of the circle through an angle \( \omega \), if \( \omega \) is incommensurate with \( 2\pi \), and if \( \psi \) is a Riemann integrable function, then

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \psi(f^k(x)) = \int_{S^1} \psi(x) \, dx
\]

with the understanding that \( f^k \) denotes the composition of \( f \) with itself \( k \) times.

**Proof:** (This is only a partial proof.) For the sake of making the steps in the proof more transparent, we transform the problem to the unit circle. Write \( z = e^{2\pi i x} \) and \( \theta = e^{2\pi i \omega} \). We study the difference equation \( z(k+1) = e^{2\pi i \omega} z(k) \). Now

\[
\frac{1}{p} \sum_{k=0}^{p-1} (e^{2\pi i k} z(0))^k = \frac{1}{p} \sum_{k=0}^{p} \theta^k z(0)^k
\]

\[
= \begin{cases} 
1 & p = 0 \\
\frac{1}{p} \sum_{i=1}^{k} a_i z^i & p \neq 0
\end{cases}
\]

If \( \psi(z) = \sum_{i=1}^{k} a_i z^i \) then we just add up the above. This gives the result when \( \psi \) is a polynomial in \( e^{2\pi i x_0} \). We omit the demonstration that any Riemann integrable function can be approximated by such polynomials with sufficient precision so as to complete the proof.

### 4.4 Other Difference Equations

There are completely deterministic procedures for determining the decimal expansion of \( \pi \). Yet if we considered a discrete time stochastic process \( x(t) \) taking on values in the set \{0,1,\ldots,9\} with uniform distribution and independent trials, we would have great difficulty in designing a test which would distinguish the above expansion from a sample function.
of the stochastic process. This is but one of many procedures for generating what we
might call pseudorandom processes. Recently it has become a very popular game to invent
deterministic schemes for generating processes whose behavior looks stochastic. This may
be done using difference equations or differential equations. Because rather less is known
about the differential equation case we begin with a brief discussion of difference equations.

Perhaps the best known pseudorandom difference equation is the so-called “logistic”
equation
\[ x(i + 1) = 1 - k(x(i) - \frac{1}{2})^2 \]
considered on the interval \(0 \leq x \leq 1\) and for \(0 \leq k \leq 1\). In this range of values we may
distinguish between several types of behavior according to

(a) \(0 < k < \frac{1}{4}\)
(b) \(\frac{1}{4} < k \sim .91\)
(c) \(\sim .91 < k < 1\)

In the first case all solutions approach an equilibrium point. In the second region there are
many periodic solutions – more as \(k\) gets larger. In the last case the motion appears to be
quite chaotic indeed.

If \(f: [0, 1] \rightarrow [0, 1]\) is such that for each value of \(y\) there is a finite number of values \(x\)
such that \(f(x) = y\), and if \(f\) is differentiable with a derivative that is nonzero except for a
finite number of points, then we can consider the equation
\[
\rho(x) = \sum_{\text{inverse images}} \frac{1}{|f'(x)|} \rho(f^{-1}(x))
\]
This is the functional equation that a steady state density for \(x\) would need to satisfy if it
were to exist.

**Example 1:** Let \(f(x) = 4x - 4x^2\). In this case \(f(f(...f(x)...))\) is a polynomial of
degree \(2^n\) where \(n\) is the number of compositions. Clearly this has at most \(2^n\) distinct
periodic solutions of period \(n\) because a polynomial of degree \(2^n\) has, at most, \(2^n\) real roots.
It is easy to see that if all the roots are real and distinct then there are \(2^n\) distinct periodic
solutions of period \(n\).

The functional equations that the steady state invariant density would need to satisfy can be found as follows. From
\[ y = 4x - 4x^2 \]
we see that
\[ (y - 1) = -(2x - 1)^2 \]
or
\[ x = \frac{1}{2}(1 \pm \sqrt{1-y}) \]
thus
\[ \rho(y) = \frac{1}{\sqrt{1-y}} (\rho(\frac{1}{2} + \frac{1}{2}\sqrt{1-y}) + \rho(\frac{1}{2} - \frac{1}{2}\sqrt{1-y})) \]
Example 2: Let $f$ be defined by

$$f(x) = \begin{cases} 2x : & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x : & \frac{1}{2} \leq x \leq 1 \end{cases}$$

then the difference equation

$$x(k + 1) = f(x(k))$$

gives rise to the invariant measure equation

$$\rho(x) = \frac{1}{2} \rho(x/2) + \frac{1}{2} \rho((1 - x)/2)$$

Clearly this admits the solution $\rho(x) \equiv 1$. Value $x(0) = a$ gives rise to a periodic solution of period $n$ if

$$f(f(\ldots f(a)\ldots)) = a$$

If $n = 2$ and $a < 1/2$ this is

$$2 - 4a = a$$

4.5 Differential Equations

We have seen that simple difference equations of the form $x(k + 1) = f(x(k))$ can have solutions that are highly unstable but yet are not unbounded. Differential equations can exhibit the same type of behavior but unless time enters explicitly they must be of dimension 3 or more.

Consider the third order equation

$$x^{(3)} + x^{(2)} + 1.25x^{(1)} - 1.8\sin(x) = 0$$

Figure 4.3 shows a trajectory in $(x, \dot{x})$-space. It is confined to a relatively small region of the space but yet it does not appear to be periodic.
4.6 Exercises 4

1. Any given a number in $[0, 1)$ has a binary expansion $x = .a_1a_2a_3 \cdots$. Of course $a_1$ is one if $x \geq 1/2$, $a_2$ is one if $x - a_1 \geq 1/4$, etc. Describe the process of generating the $a_i$ as a pair of scalar difference equations by introducing $b_i = x - a_1, a_2, \ldots, a_{i-1}$ then finding $g_1$ and $g_2$ so that

\begin{align*}
a_{i+1} &= g_1(a_i, b_i) \\
b_{i+1} &= g_2(a_i, b_i)
\end{align*}

4.7 Notes and References

Chapter 5

System Concepts

This chapter marks the transition from pure stochastic processes to the subject of stochastic control. We assemble here a few control and system theoretic tools that will be used in the later chapters. We are interested in modeling systems that have control terms (inputs), stochastic terms, and explicitly observable terms (outputs). One of the basic problems we will discuss is that of finding a stochastic differential equation whose solutions have certain specified statistical properties.

5.1 Modeling

The previous chapters are devoted to explaining the ideas involved in stochastic differential equations and various approaches to learn more about their properties. Looking ahead, there will be more focus on stochastic control, and this means the models we will deal with will include terms that can be manipulated. This introductory section contains some simplified models of stochastic processes that include controls in the problem description.

Example 1: Capital growth with random interest rates Consider the model

\[ dx = (adt + bw)dx + u(t)dt \]

where the variables have the following interpretation

1. \( x(t) \) is the present value of the capital
2. \( a \) is the average interest rate
3. \( w \) is a standard Wiener process and \( b \) is a constant which scales \( w \) so that the stochastic part of the interest rate has an appropriate variance, relative to \( a \).
4. \( u \) is the rate at which new capital is added.

**Example 2: Two instrument capital growth** As a more detailed version of the first example, consider a situation in which one has two different possibilities for investment. Although the model we write down is symmetric in \( x \) and \( y \) we may suppose that \( x \) represents a risky investment and \( y \) a less risky one. If new capital available for invest is represented by \( v(t) \), and if we allow transfer in the amount \( u_2 \) from one to the other

\[
\begin{align*}
    dx &= (a_1 dt + b_1 dw)x + u_1 dt + u_2 dt \\
    dy &= (a_2 dt + b_2 dw)x + (v - u_1)dt - u_2(t)dt
\end{align*}
\]

Consistent with this we would have \( a_1 > a_2 \) and \( b_1 > b_2 \). The variables \( u_1 \) and \( u_2 \) have the interpretation of added capital and transferred capital, respectively. Consider the problem of choosing the controls so as to maximize growth or minimize risk, etc.

**Example 3: Flood control** Consider a sequence of dams on controlling water flow in a river basin. For illustrative purposes we consider a situation in which there are just three dams and consider the rainfall to be aggregated into three random variables.

\[
\begin{align*}
    dx_1 &= -u_1 dt + z_1 dt \quad dz_1 = -2dN_1 + 1 \\
    dx_2 &= -u_2 dt + z_2 dt + u_1 dt \quad dz_2 = -2dN_2 + 1 \\
    dx_3 &= -u_3 dt + z_3 dt + u_2 dt \quad dz_3 = -2dN_3 + 1
\end{align*}
\]

where the variables have the following interpretation

1. \( x_i(t) \) is the height of the water in the \( i \)th dam
2. \( z_i \) is the binary valued state of the rainfall—raining or not raining
3. \( u_i \) is the rate at which water is allowed to escape from the \( i \)th dam.

A possible objective is to select the \( u_i \) so that the probability over a certain interval of the event \( x_3(t) > h_3 \) is as small as possible.

**Example 4: Communication Network Control** Let \( hz \) be the incoming stream of traffic. We model it as a random process \( z \) times a constant \( h \) with \( dz = (1 - 2z) dN \), with \( z \) taking on the values 0 and 1. We assume that we have a buffer of a certain capacity which at time \( t \) has \( v(t) \) bytes in it. This gives

\[
\begin{align*}
    dz &= (1 - 2z) dN \\
    dv &= -u dt + hz
\end{align*}
\]

where \( u \) represents the rate at which messages are allowed to leave the buffer. Of course there are limitations on the variables. We need \( v \leq c \) where \( c \) is the capacity of the buffer.
and we need \( u = 0 \) if \( v = 0 \). The goal of the control policy might be to minimize the variance of the output. That is, introduce a time period \( T \) and select \( u \) so as to minimize

\[
\mathcal{E} \int_0^T (u - \mathcal{E}u)^2 \, dt
\]

If the messages leave the buffer only to enter a second one, the model can be expanded. Again, we have both control variables and stochastic variables entering into the description of the problem.

### 5.2 Deterministic Linear Systems

A deterministic, continuous time, differential equation based, input-output model takes the form

\[
\dot{x} = f(x, u, t); \quad y = h(x, t)
\]

A stochastic, differential equation based, input-output model might take the form

\[
dx = f(x, u, t)dt + \sum_{i=1}^{k} g_i(x, u)dw_i + \sum_{j=1}^{l} \tilde{g}_i(x, u)N_i \quad ; \quad dy = h(x)dt + i(x)d\nu
\]

Mathematical models of the form

\[
\dot{x} = Ax + Bu; \quad y = Cx
\]

with \( x, u, \) and \( y \) vector valued are called linear systems. If \( A, B, \) and \( C \) are constant then it is said to be a time invariant linear system. If \( \Phi(t, \sigma) \) satisfies the equations

\[
\frac{d}{dt} \Phi(t, \sigma) = A(t)\Phi(t, \sigma) : \Phi(\sigma, \sigma) = I
\]

then \( \Phi \) is said to be a fundamental solution of \( \dot{x} = Ax \). If \( A \) is constant then \( \Phi(t, \sigma) \) is just \( e^{A(t-\sigma)} = I + A(t-\sigma) + A^2(t-\sigma)^2/2 + \cdots \). The so-called variation of constants formula gives \( y \) in terms of \( x(\sigma) \) and \( u \)

\[
y(t) = C(t)\Phi(t, \sigma)x(\sigma) + \int_\sigma^t C(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau
\]

The function \( T \) defined by \( T(t, \sigma) = C(t)\Phi(t, \sigma)B(\sigma) \) is sometimes called the weighting pattern. If \( A, B, \) and \( C \) are constant then it takes the form

\[
T(t, \sigma) = Ce^{A(t-\sigma)}B
\]

with exponential as defined as above.

A linear system is said to be controllable if for any given value of \( x(\sigma) \) and any given \( t > \sigma \) there is a control \( u(\cdot) \) defined on \([\sigma, t] \) such that \( u(\cdot) \) drives the system to the state zero at time \( t \). The matrix

\[
W(\sigma, t) = \int_\sigma^t \Phi(t, \tau)B(\tau)B^T(\tau)\Phi^T(t, \tau)d\tau
\]
is called the **controllability gramian** for the interval $[\sigma, t]$. Notice that $W(\sigma, t)$ is automatically nonnegative definite. An application of the control $u(t) = B^T(t)\Phi^T(t, \sigma)W^{-1}(t, \sigma)x(\sigma)$ shows that if $W^{-1}(t, \sigma)$ exists then the system is controllable. If it is positive definite then the system is controllable. If $A$ and $B$ are constant then $W(\sigma, t)$ is positive definite if and only if the matrix $(B, AB, \ldots, A^{n-1}B)$ (commas denote column partition) has rank equal to $n = \dim x$.

The problem of observability relates to the possibility of determining the value of $x(t)$ from a knowledge of $y(\cdot) = cx(\cdot)$ over some interval $[t_0, t_1]$. The answer to this type of question can also be reduced to the investigation of the definiteness of a symmetric matrix. Consider the expression

$$
\Phi^T(t_1, t_0)c^T(t_1)c(t_0) + \int_{t_0}^{t_1} \Phi^T(t, t_0)\Phi(c(t_0)dtx_0 + \int_{t_0}^{t_1} \Phi^T(t, \sigma)u(\sigma)d\sigma dt
$$

the matrix $M(t_1, t_0) = \int_{t_0}^{t_1} \Phi^T(t, t_0)\Phi(c(t)ds$ is called the **observability gramian**.

If $A, B,$ and $C$ are constant matrices then one can associate to the system

$$
\dot{x} = Ax + Bu; y = Cx
$$
a matrix of rational functions

$$
G(s) = C(Is - A)^{-1}B
$$

called the **transfer function**. It has a number of properties that make it useful in doing systems analysis. In the first place, if we recall that if the real parts of the eigenvalues of $A$ are all less than $\sigma$ then $x(t)e^{-\sigma t}$ approaches zero as $t$ goes to infinity and we may define the the Laplace transform for $Re s > \sigma$ by the integral

$$
L(x(\cdot))(s) = \int_0^{\infty} e^{-st}x(t)dt
$$

It is not too hard to see that the Laplace transform of the matrix exponential $e^{At}$ is given by

$$
\int_0^{\infty} e^{-st}e^{At}dt = (Is - A)^{-1}
$$

Thus we see that the transfer function is the Laplace transform of the weighting pattern.

It is of importance to observe that if the real parts of the eigenvalues of $A$ are negative then not only do all the unforced solutions of $\dot{x} = Ax + Bu$ go to zero as $t$ goes to infinity, but in addition, the solution corresponding to $u = u_0\cos \omega t$ approaches the form $x(t) = x_1 \cos \omega t + x_2 \sin \omega t$ and we can compute vectors $x_1$ and $x_2$ from $u_0$ and the transfer function. In fact, if we write

$$
C(Ii\omega - A)^{-1}B = ReG(i\omega) + iImG(i\omega)
$$

then

$$
x_1 = Im(i\omega)u_0; x_0 = ReG(i\omega)u_0
$$
5.2. DETERMINISTIC LINEAR SYSTEMS

This justifies the use of the term “frequency response”.

When a linear model describes certain types of physical situations additional symmetries and structure maybe present. For example, if
\[ \dot{x} = Ax + Bu ; \quad y = Cx \]
describes a passive electrical network with \( u \) being a vector of currents associated with a collection of terminal pairs and \( y \) being the corresponding voltages, then \( \langle u, y \rangle \) is the rate of flow of energy into the system. If the stored electrical energy is given by \( x^T Q x \) then of course
\[
\frac{d}{dt} x^T Q x = x^T (QA + A^T Q)x + x^T QBu + u^T B^T Qx
\]
thus we must have \( B = C^T \) and \( QA + A^T Q \leq 0 \).

A linear stationary system that is controllable and observable and symmetric in the sense that \( CA_iB \) for \( i = 1, 2, \ldots \) is a symmetric matrix is said to be passive if there exists a symmetric positive definite matrix \( \Sigma \) such that
\[
\frac{d}{dt} x^T(t) \Sigma x(t) \leq y^T(t) u(t)
\]

Example: Consider the linear Itô equation
\[
\begin{align*}
\dot{x} &= y \, dt \\
\dot{y} &= z \, dt + a \, dw \\
\dot{z} &= -x \, dt - 2y \, dt - z \, dt + dw
\end{align*}
\]
Display the equation for the steady state variance as a function of the parameter \( a \). Is there any value of \( a \) such that the steady state variance is not positive definite? You can answer this by solving for the variance if you like but there are other ways to get the answer.

Solution: Writing this in vector matrix notation we have
\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & -2 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} dt +
\begin{bmatrix}
0 \\
a \\
1
\end{bmatrix}
dw
\]
The steady state variance equation is
\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & -2 & -1
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}
+ \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & -1 \\
1 & 0 & -2 \\
0 & 1 & -1
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 \\
0 & a^2 & a \\
0 & a & 1
\end{bmatrix}
= 0
\]
It can be solved one component at a time. From the first row, \( \sigma_{12} = 0, \sigma_{22} = -\sigma_{13} \) and \( \sigma_{23} = \sigma_{11} + \sigma_{13} \). Turning to the second row, we see that \( \sigma_{23} = -a^2/2 \) and that \( \sigma_{33} - \sigma_{23} - 2\sigma_{22} = -a \). Thus \( \sigma_{11} + \sigma_{13} = -a^2/2 \) and \( \sigma_{33} - 2\sigma_{22} = -a - a^2/2 \). Finally, the last equation is \( \sigma_{13} + 2\sigma_{23} + \sigma_{33} = 1/2 \). Thus, in addition to \( \sigma_{12} = 0 \) and \( \sigma_{23} = -a^2/2 \) we have
\[
\sigma_{11} + \sigma_{13} = -a^2/2
\]
\[ \sigma_{22} = -\sigma_{13} \]
\[ \sigma_{33} - \sigma_{22} = -a^2 \]
\[ \sigma_{13} + \sigma_{33} = a^2 \]

This leads to \( \sigma_{33} - 2\sigma_{22} = a + a^2 \), and \( \sigma_{22} + \sigma_{33} = 1/2 - a^2 \) which implies that \( \sigma_{33} = 1/4 + a/2 \) and \( \sigma_{22} = 1/4 - a/2 - a^2 \), and

Thus
\[
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}
= \begin{bmatrix}
1/2 + a + a^2 & 0 & -1/2 - a - 3a^2/2 \\
0 & 1/2 + a + 3a^2/2 & -a^2/2 \\
-1/2 - a - 3a^2/2 & -a^2/2 & 1/2 + a + 5a^2/2
\end{bmatrix}
\]
The determinant vanishes, signaling a lack of positive definiteness, if \( 1 - a - a^2 = 0 \).

An easier way to do the problem is to observe that the eigenvalues of \( A \) have negative real parts and thus there is a steady state solution. The steady state solution will be positive definite if the system is controllable. The controllability matrix is
\[
[b, Ab, A^2b] = \begin{bmatrix}
0 & a & 1 \\
a & 1 & -2a - 1 \\
1 & -2a - 1 & a - 1
\end{bmatrix}
\]
and its determinant is \( -a^3 - 3a^2 - 2a - 1 \). The covariance matrix fails to be positive definite when the system fails to be controllable.

### 5.3 Covariance and the Power Spectrum

We now turn to a more general situation which illustrates the much broader scope of the previous somewhat special reasoning.

We begin with the analysis of stationary stochastic processes also called stationary time series. Here the Fourier transform plays a role and a little complex variable notation is required. We will use \( s = \sigma + i\omega \) and \( \bar{s} = \sigma - i\omega \). According to one definition the Fourier transform is
\[
F_1(i\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt
\]
Another definition puts a constant in front
\[
F_2(i\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt
\]
The inversion formulae are then
\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} F_1(i\omega) d\omega
\]
5.3. COVARIANCE AND THE POWER SPECTRUM

\[ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} F_2(i\omega) d\omega \]

These integrals are well defined if \( f \) is square integrable (in Riemann sense or Lebesgue sense) on \((-\infty, \infty)\). Recall that if the function \( f \) is square integrable and periodic on \([-\pi, \pi]\) then we have a Fourier series representation

\[ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} ; \quad c_n = \int_{-\pi}^{\pi} f(t)e^{int} dt \]

It is useful to recall that on the interval \([-\pi, \pi]\) we have

\[ \int_{-\pi}^{\pi} e^{int} e^{-int} dt = 2\pi ; \quad \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} e^{int} \frac{1}{\sqrt{2\pi}} e^{-int} dt = 1 \]

Thus it is the functions \((1/\sqrt{2\pi})e^{int}\) that are orthonormal on \([-\pi, \pi]\).

\[ \int_{-\pi}^{\pi} f^2(t) dt = 2\pi \sum_{n=-\infty}^{\infty} c_n \bar{c}_n = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2 \]

This is often called Parseval’s relation. For the Fourier transform there is an similar result. If \( f \) is square integrable then

\[ \int_{-\infty}^{\infty} f^2(t) dt = \int_{-i\infty}^{i\infty} F_2(i\omega) F_2(-i\omega) d\omega \]

Now if we have a stochastic process such as one generated by

\[ dx = Ax dt + b dw ; \quad y = cx \]

then in steady state we have \( \mathcal{E} x(t) = 0 \) and

\[ \mathcal{E} x x^T = \Sigma ; \quad A\Sigma + \Sigma^T A^T = -bb^T \]

Thus when steady state has been achieved, the \( y(t) \) is not a square integrable function. In fact,

\[ \mathcal{E} \int_{-T}^{T} y^2(t) dt = 2T c \Sigma c^T \]

However, there is a limiting situation which captures measurable quantities. Observe that

\[ \mathcal{E} \frac{1}{2T} \int_{-T}^{T} y^2(t) dt = c \Sigma c^T \]

so that the time-average value of \( y^2(t) \) is \( c \Sigma c^T \) and this can be determined experimentally, and identified with the average power in the signal \( y \). Using the Fourier transform we can write this as

\[ \mathcal{E} \frac{1}{2T} \int_{-T}^{T} y^2(t) dt = \frac{1}{2T} \int_{-\infty}^{\infty} F_T(i\omega) F_T(-i\omega) d\omega \]
This makes it clear that when using Fourier theory in connection with stationary stochastic processes it is not the standard formulae that will play a role but rather expressions such as

\[ F_3(i\omega) = \lim_{T \to \infty} \frac{1}{2T} \mathcal{E} \int_{-T}^{T} e^{-i\omega t} y(t) dt \]

Wiener chose to call this kind of modification of Fourier theory, generalized harmonic analysis.

Now consider the idea of a band pass filter. If \( f \) is an \( L_2 \) function having a Fourier transform \( F_2 \) then we can recover \( f \) from \( F_2 \) as

\[ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} F_2(i\omega) d\omega \]

now consider the possibility of restricting the integral as in

\[ f_{\omega_0}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega_0} e^{i\omega t} F_2(i\omega) d\omega \]

This captures only the part of the original \( f \) that is representable by frequencies between \(-\omega\) and \( \omega \). Thus we see that for \( \omega_1 > \omega_2 \)

\[ f_{\omega_1}(t) - f_{\omega_2}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\omega_2}^{-\omega_1} e^{i\omega t} F_2(i\omega) d\omega + \frac{1}{\sqrt{2\pi}} \int_{\omega_1}^{\omega_2} e^{i\omega t} F_2(i\omega) d\omega \]

The average “power” in a stochastic signal that lies in the frequency range between \( \omega_1 \) and \( \omega_2 \) can thus be expressed as a difference. Now in the context of generalized harmonic analysis we replace the deterministic functions by suitable expected values. Taking the limit

\[ \Phi(\omega) \overset{\text{def}}{=} \lim_{\delta \to 0} \frac{f_{\omega + \delta} - f_{\omega}}{\delta} \]

where \( \Phi(\cdot) \) is called the power spectral density.

A real valued stochastic process is said to be wide-sense stationary if the statistical properties \( \mathcal{E} y(t) \) and \( \mathcal{E} y(t)y(\tau) \) are, respectively, independent of \( t \) and dependent only on \( |t - \tau| \). A stationary process is said to be ergodic if all statistical properties can be computed as time averages. For example, if \( y \) is ergodic, then

\[ \mathcal{E} y(t)y(t + \tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} y(t)y(t + \tau) dt \]

with \( y \), as it appears on the right, being any particular sample path.

In applications one is often interested in processes of this type and usually measures the power spectrum rather than the autocorrelation function. For a sine wave, \( y(t) = a \sin \omega t \) the power is concentrated at \( \omega \) and the power density (power per unit bandwidth) at that point is a delta function of strength \( a^2 \). In general, the power spectrum, defined for zero mean, wide-sense stationary processes by

\[ \Phi(\omega) = \lim_{T \to \infty} \frac{1}{2T} \left| \int_{-T}^{T} e^{-i\omega t} y(t) dt \right|^2 \]
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is a measure of the “power” at $\omega$ in the process $y$.

For stationary, ergodic processes, the power spectrum and the autocorrelation function are related by the well known Wiener-Khinchin theorem which states that the power spectrum is simply the Fourier transform of the autocorrelation function. We sketch how this goes. By definition

$$\Phi(\omega) = \lim_{T \to \infty} \frac{1}{2T} \left| \int_{-T}^{T} e^{-i\omega t} \phi(t) dt \right|^2$$

which we can write as

$$\Phi(\omega) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} e^{-i\omega t} y(t) e^{i\omega \tau} y(\tau) dt d\tau$$

Let $\sigma = t - \tau$ and use the ergodic property to get

$$\Phi(\omega) = \lim_{T \to \infty} \int_{-T}^{T} e^{-i\omega \sigma} E(y(t) y(t - \sigma)) d\sigma$$

which is the desired result.

Because the power spectrum of a real valued stochastic process is clearly nonnegative at each frequency, we see that there must be some nontrivial conditions on a function $\phi$ in order for it to be an autocorrelation function. These are best brought out by means of the Parseval relation which states that for complex valued functions which are square integrable on the interval $(-\infty, \infty)$ and related by

$$\Phi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \phi(t) dt$$

we have

$$\int_{-\infty}^{\infty} |\phi(t)| dt = \int_{-\infty}^{\infty} |\Phi(\omega)| d\omega$$

**Example:** Let $y(t) = e^{-|t|}$ defined on $(-\infty, \infty)$. Its Fourier transform is

$$\hat{y}(\omega) = \frac{1}{\sqrt{2\pi}} \left( \int_{0}^{\infty} e^{-i\omega t} e^{-t} dt + \int_{-\infty}^{0} e^{-i\omega t} e^{t} dt \right) = \frac{1}{\sqrt{2\pi}} \frac{1}{1 + \omega^2}$$

If $\Phi(\omega)$ is nonnegative on the real axis, then of course for any $u(\cdot)$

$$0 \leq \int_{-\infty}^{\infty} \Phi(\omega) |\hat{u}(\omega)|^2 d\omega$$

Since multiplication goes over into convolution after Fourier transformation, we have

$$0 \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(t - \tau) u(t) u(\tau) dtd\tau$$
Functions $\phi$ which satisfy this inequality are said to be positive definite (in the sense of Bochner). The autocorrelation function of a stationary process is positive definite.

One can gain some appreciation for what it means for a function to be positive definite by letting $u$ be a zero except for a string of narrow pulses centered at $t_1, t_2, \ldots, t_n$. In this case define

$$
\Phi(u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(t-\tau)u(t)u(\tau) \, dt \, d\tau
$$

Then the integral is approximated by

$$
\left[ \alpha_1, \alpha_2, \ldots, \alpha_n \right] \begin{bmatrix} 
\phi(t_1-t_1) & \phi(t_1-t_2) & \cdots & \phi(t_1-t_n) \\
\phi(t_2-t_1) & \phi(t_2-t_2) & \cdots & \phi(t_2-t_n) \\
\cdots & \cdots & \cdots & \cdots \\
\phi(t_n-t_1) & \phi(t_n-t_2) & \cdots & \phi(t_n-t_n) 
\end{bmatrix} \begin{bmatrix} \alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n 
\end{bmatrix}
$$

where $\alpha_i$ is the area under the $i^{th}$ pulse. Clearly we require that the symmetric matrix on the right should be positive definite.

5.4 The Zero-Crossings Problem

We consider a problem that comes up frequently in applications. We wish to evaluate the expected number of times a gauss-markov process will cross a given level in a given period of time. For motivation observe that if the random variable is capital and if your capital goes negative then one faces the possibility of “ruin” or, if the variable is a voltage on the terminal of a threshold detector some instruments are designed to give a click if the signal crosses some threshold, etc.

In section 1.5 we showed that if $(x, y)$ are Gaussian random variables with identical variance and a two-by-two covariance matrix $\Sigma$ then the probability of $xy < 0$ is

$$
p_{neg} = \frac{2}{\pi} \tan^{-1} \sqrt{\frac{\sigma_{22} - \sigma_{12}}{\sigma_{11} + \sigma_{12}}}
$$

Thus we see that the probability that $x(t)$ and $x(t+\tau)$ have opposite signs is given in terms of properties of the correlation matrix. Now of course if $x(t)$ and $\dot{x}(t)$ have the same sign then $x(t)$ is not reversing its sign.

We see that the covariance matrix for a stochastic process $x(t), x(t+\tau)$ looks like

$$
\Sigma = \mathbb{E} \begin{bmatrix} x(t)x(t) & x(t)x(t+\tau) \\
x(t)x(t+\tau) & x(t+\tau)x(t+\tau) 
\end{bmatrix}
$$

If we want to find out something about zero crossings we ask about the probability that $x(t)$ and $x(t+\tau)$ have different signs. The density of zero crossings is independent of time.
5.5. STOCHASTIC REALIZATION PROBLEM

for a stationary process so we can focus on a small period of time. We start by making the assumption that the auto correlation function is differentiable at $\tau = 0$. In this case we have

$$\mathbb{E} x(t)x(t+\tau) = \mathbb{E} x^2(t) + \frac{1}{2} \phi''(0)\tau^2 + ...$$

Thus if we let the variance of $x$ be $a$ and insert the appropriate values into the above formula for $p_{neg}$ we get

$$p_{neg} = \frac{2}{\pi} \tan^{-1} \sqrt{\frac{-\phi''(0)\tau^2/2}{2a - \phi''(0)\tau^2/2}}$$

For values of $\tau$ that are negligible in comparison with $\phi(0)$ this simplifies to

$$p_{neg} = \frac{2}{\pi} \sqrt{\frac{-\phi''(0)\tau^2}{4\phi(0)}}$$

Thus for small values of $\tau$ the probability that $x(t)$ and $x(t+\tau)$ have the same sign is calculated as above and the per unit time density of the zero crossings is

$$d = \frac{1}{\pi} \sqrt{\frac{-\phi''(0)}{\phi(0)}}$$

5.5 Stochastic Realization Problem

The basic problem in stochastic realization is that of finding a dynamical model, say of the form

$$dx = f(x)dt + \sum g_i(x)dw_i + \sum \tilde{g}_i(x)dN_i$$

and a map $h(\cdot)$ such that $y(t) = h(x(t))$ has certain, preassigned, statistical properties. Stated in this way, the stochastic realization question is a very general one and one for which a general answer is not likely to be too useful. In control theory and communication theory one is often interested in specifying just the mean and covariance; most commonly

$$\mathbb{E} y(t) = 0$$
$$\mathbb{E} y(t)y(t+\tau) = \phi(\tau)$$

and then asking for a linear model of the form

$$dx = Axdt + Bdw$$
$$y = cx$$

such that $y$ has the desired statistical properties. We will see that this particular version of the stochastic realization problem has a solution which is essentially unique.

Before discussing the mathematical treatment of the problem of stochastic realizations we give a few examples.
Blackbody Realization: The most striking example of a stochastic realization problem comes from physics and concerns the so-called “blackbody radiation problem”. In the late 1890’s experimenters in Berlin were measuring the power spectra of the radiant energy coming from the sun. The existing theories as developed by Wien and Rayleigh-Jeans did not agree with each other and did not explain the experimental results in a uniform way. Letting $\omega$ denote the frequency of the electromagnetic radiation, Max Planck noticed that for appropriate values of $\alpha$ and $\beta$ the function

$$\Phi(\omega) = \frac{\beta|\omega|^3}{e^{\alpha|\omega|} - 1}$$

provided an excellent fit to the observed data, and more importantly, suggested his quantum hypothesis in order to explain this function. Thus it is fair to say that insofar as Planck’s contributions are concerned, the “old quantum mechanics” grew out of an attempt to solve a particular stochastic realization problem in which the power spectrum of the “signal” was given and an dynamical system explanation was sought.

Speech Recognition and Synthesis: No two people pronounce a given word the same way. This does not cause much difficulty for humans but it is the source of difficulty for engineers building artificial speech recognition systems. It means that there is not a single acoustical pressure wave that corresponds to the word “dizziness” but rather, a class of signals. Conversely, it is not appropriate to model “dizziness” by a single signal but rather it should be thought of as a family of possible realizations. Consider a finite state process of the form

$$dx = \sum_{i=1}^{r} \phi_i(x) dN_i$$

$$y(t) = \sum_{i=1}^{r} \phi_i(x) b_i(t)$$

where the $b_i(t)$ are fixed deterministic or stochastic processes. According to this point of view spoken English corresponds to a set of stochastic models.

Figure 5.1: A waveform of the author saying the word “spectral” (left) and “factor” (right).

Turbulence: Other problems of this type which are not yet solved include an explanation of the power spectra of certain fluid mechanics problems. In this area the nature of the underlying model is still very much up in the air. In the 1940’s Kolmogorov applied physical principles to reason that the power spectrum associated with turbulence should, for large value of $w$ behave like $\pi(\omega) = k\omega^{-5/2}$. The status of this theory is still uncertain.
5.6 Linear Stationary Realizations

As we have seen in section 2.5, the covariance associated with
\[ dx = Ax dt + \sum_{i=1}^{m} b_i dw_i \quad ; \quad x(t) \in \mathbb{R}^n \]
is given by
\[ \mathcal{E}x(t)x^T(t+\tau) = \Sigma(t)e^{A^T\tau} \quad ; \quad \tau \geq 0 \]
where \( \Sigma \) satisfies
\[ \dot{\Sigma} = A\Sigma + \Sigma A^T + \sum_{i=1}^{m} b_i b_i^T \]

This means that in the limit as \( t \) goes to infinity, we obtain an autocorrelation function which depends on \( \tau \) only provided that the eigenvalues of \( A \) have negative real parts. Under this hypothesis we have a unique solution \( \Sigma \) to
\[ A\Sigma + \Sigma A^T + \sum_{i=1}^{m} b_i b_i^T = 0 \]
Moreover, this solution is nonnegative definite and can be expressed as
\[ \Sigma = \sum_{i=1}^{m} \int_{0}^{\infty} e^{At}b_i b_i^T e^{A^Tt} dt \]
It is not too hard to see that \( \Sigma \) is actually positive definite if the system
\[ \dot{x} = Ax + \sum_{i=1}^{m} b_i u_i \]
is controllable. That is to say, \( \Sigma \) is positive definite if the set \( \{ A^i b_j \} \) spans \( \mathbb{R}^n \).

In any case, assuming only that \( \mathcal{R}e \lambda(A) < 0 \) we have for \( \tau > 0 \)
\[ \lim_{t \to -\infty} \mathcal{E}x(t)x(t+\tau) = \left( \sum_{i=1}^{m} \int_{0}^{\infty} e^{At}b_i b_i^T e^{A^Tt} dt \right) e^{A^T\tau} \]
If \( y = cx \) is any linear functional on \( x \), then
\[ \lim_{t \to -\infty} \mathcal{E} [y(t)y^T(t+\tau)] = c\Sigma e^{A^T\tau} c^T \]
where \( \Sigma \) is as above.
5.7 Spectral Factorization

Given a linear stochastic equation of the form

$$dx = Ax dt + B dw; \quad x(t) \in \mathbb{R}^n$$

with $(A, B)$ a controllable pair ($(B, AB, \ldots, A^{n-1}B)$ has rank $n$), we know that if the eigenvalues of $A$ have negative real parts, then

$$\lim_{t \to \infty} \mathcal{E} x(t) x^T(t + \tau) = \Sigma e^{A\tau}; \quad \tau \geq 0$$

$$\lim_{t \to \infty} \mathcal{E} x(t) x^T(t + \tau) = e^{-A\tau} \Sigma; \quad \tau \leq 0$$

Using the definition of the Fourier transform we obtain

$$\Phi(\omega) = \int_{-\infty}^{\infty} e^{-i\omega \tau} \phi(\tau) d\tau$$

$$= \int_{0}^{\infty} e^{-i\omega \tau} \Sigma e^{A\tau} d\tau + \int_{-\infty}^{0} e^{-i\omega \tau} e^{-A\tau} \Sigma d\tau$$

$$= -\Sigma (-i\omega I + A^T)^{-1} - (i\omega I - A)^{-1} \Sigma$$

We can use the fact that $\Sigma$ satisfies $\Sigma A^T + A \Sigma + BB^T = 0$ to reexpress this. Adding and subtracting $i\omega \Sigma$ we get

$$(A - i\omega) \Sigma + \Sigma (Ii\omega + A^T) = BB^T$$

Pre and post multiplication by $(A - i\omega)^{-1}$ and $(A^T + i\omega)^{-1}$ to get

$$\Sigma (Ii\omega + A^T)^{-1} + (A - Ii\omega)^{-1} \Sigma = (Ii\omega - A)^{-1} BB^T (Ii\omega - A^T)^{-1}$$

Thus

$$\Phi(\omega) = (-Ii\omega - A)^{-1} BB^T (Ii\omega - A^T)^{-1}$$

This shows that the Fourier transform of the autocorrelation function is hermitian and nonnegative. Looking at $y = cx$ we get for the Fourier transform of $\mathcal{E}cx(t)cx(t + \tau) = \phi(\tau)$

$$\mathcal{F}(\phi(\cdot)) = c(-Ii\omega - A)^{-1} bb^T (Ii\omega - A^T)^{-1} c^T$$

The solution of the present version of the stochastic realization problem is based on this identity and the following lemma. (*The spectral factorization lemma.*)

**Lemma:** Given an even function $\psi(s) = \psi(-s)$ which is real, rational, and nonnegative on $s = i\omega$ for all $\omega$, there exists $r(s)$, real, rational, and having no poles in the half plane $\Re s > 0$ such that

$$\psi(s) = r(s)r(-s)$$

Moreover, $r(s)$ is analytic wherever $\psi$ is.

**Proof:** See Finite Dimensional Linear Systems.
5.7. SPECTRAL FACTORIZATION

Suppose we are given a bounded, continuous real valued function \( \phi(\cdot) = \phi(-\cdot) \) which for \( t \geq 0 \) is of the form

\[
\phi(t) = \sum \alpha_{ij} t^i e^{\lambda_j t}; \quad (\lambda_j \text{ may be complex})
\]

positive definite in the sense of Bochner. Its Fourier transform \( \Phi \) is nonnegative and hence can be expressed as

\[
\Phi(\omega) = r(i\omega)r(-i\omega)
\]

with \( r(s) \) a real rational function with its poles in \( \mathbb{R} \) \( \text{Re} s > 0 \). (Poles on \( \mathbb{R} \) \( \text{Re} s = 0 \) are ruled out by the fact that they lead to infinite total power in the power spectrum.) In view of these remarks we can express the inverse Fourier transform as

\[
(F^{-1}r)(t) = ce^{At}b
\]

Tracing backwards the above formulae yield the identity

\[
\lim_{t \to \infty} E y(t) y(t + \tau) = \phi(\tau)
\]

for

\[
\begin{align*}
dx &= Axdt + bd\omega \\
y &= cx
\end{align*}
\]

Thus to solve the problem of realizing a given stationary covariance \( \phi(\cdot) \) with a Wiener process model it is enough to

(i) Transform \( \phi(\cdot) \)
(ii) Factor the transform using spectral factorization
(iii) Express \( r(s) \) as \( c(Is - A)^{-1}b \)
(iv) Construct

\[
\begin{align*}
dx &= Axdt + bd\omega \\
dy &= \begin{bmatrix} \sqrt{10} \\
\sqrt{2}
\end{bmatrix} \begin{bmatrix} x_1 \\
x_2
\end{bmatrix}
\]

Example: Let the power spectrum be

\[
\Phi(w) = \frac{1}{1+w^2} + \frac{1}{10+2w^2} = \frac{11-2w^2}{(1+w^2)(3+w^2)} \cdot \sqrt{2} \frac{\sqrt{5} - iw}{1-iw})
\]

A corresponding realization of the covariance is provided by

\[
\begin{bmatrix}
 dx_1 \\
dx_2
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\
-3 & -4
\end{bmatrix} \begin{bmatrix} x_1 \\
x_2
\end{bmatrix} dt + \begin{bmatrix} 0 \\
1
\end{bmatrix} dw; \quad dy = [\sqrt{10}, \sqrt{2}] \begin{bmatrix} x_1 \\
x_2
\end{bmatrix} dt
5.8 The Gauss-Markov Heat Bath

This section describes an application of the above ideas to a basic problem in statistical mechanics. The stochastic realization problems which appears here is distinguished from those just considered by a need to realize jointly a certain “coupling behavior” (e.g., an impedance) together with a power spectrum. This idea is illustrated very nicely by the Nyquist-Johnson model for a resistor. In fact the model proposed by them, when used with the theory of linear systems, leads to a rather nice packaging of a certain circle of ideas in statistical mechanics. The advantage of the system theory formalization is that it allows one to discuss the interaction between systems in a very precise way. We will illustrate this with a discussion of heat baths. In the process we will need to clarify the following ideas.

1. Linear passive systems and stochastic dynamical systems
2. The fluctuation dissipation equality and temperature.
3. The heat bath as a stochastic dynamical system.

By a linear, finite dimensional, Gaussian system (FDLGS) we understand a pair

\[ dx = Ax \, dt + Bu \, dt + G \, dw \]

\[ y = Cx + Du \]

Here \( dw \) is a vector valued Wiener process and \( x, u, y \) are all vector valued. This system is said to be minimal if \((A, B)\) is a controllable pair and \((A, C)\) is an observable pair.

We refer to

\[ G(s) = C(I - A)^{-1}B + D \]

as the transfer function associated with the system and we call

\[ \phi(s) = (D + C(-I - A)^{-1}G)^T(D + C(I - A)^{-1}G) \]

the power spectrum of the system. We say that the system is externally reciprocal if \( G(s) = G^T(s) \). The system is said to be deterministically passive if the eigenvalues of \( A \) have real parts that are negative and if \( G(i\omega) + G^T(-i\omega) \geq 0 \). A deterministically passive system of this form is said to be monotemperaturic if there exists a proportionality factor \( \beta \geq 0 \) such that the power spectrum of \( y \) and the frequency response are related by

\[ (C(I - A)^{-1}G + D)^T(C(I - A)^{-1}G + D) = \beta[C(-I + A)^{-1}B^T + D^T + C(I - A)^{-1}B + D] \]  

\( (*) \)

In the language of thermodynamics, this equation expresses a frequency independent fluctuation-dissipation proportionality. In the language of system theory it expresses a proportionality between the power spectrum and the parahermetian part of the transfer function. If equality \( (*) \) holds we call \( \beta \) the temperature of the system. Several justifications...
for this definition can be given. One is that (*) expresses a property of electrical networks constructed from linear constant inductors, capacitors and resistors in Nyquist-Johnson form and all of the same temperature. It is best thought of as the fluctuation-dissipation equality in the context of linear systems excited by white noise.

One of the beautiful facts about the linear theory of equilibrium thermodynamics is the equipartition of energy theorem which states that the expected value of the energy of each mode of a system in equilibrium is the same. For example, a balloon in still air can be expected to have as much kinetic energy as an $O_2$ molecule. This idea finds its expression here in terms of the interconnection of lossless systems with monotemperaturic systems.

We will say that a linear stochastic system has the equipartition property if there exists a positive number $\beta$ such that whenever it is connected to a conservative system with impedance $Z(s)$ the resulting system has a unique invariant measure and for this measure

$$\mathcal{E}_{yy} = \beta G_0$$
$$\mathcal{E}_{uu} = \beta Z_0$$

regardless of the impedance $Z(s)$ of the conservative system.

The explanation of this definition is that if one chooses normal coordinates for the lossless system then it appears as

$$\dot{x} = \Omega x + Bu; \quad y = B^T x$$

Equipartition means that $\mathcal{E}_{xx} = \beta I$ for some $\beta$ and thus $\mathcal{E}_{yy} = \beta B^T B$. However $Z(s) = B^T Bs^{-1} + B^T ABs^{-2} + \ldots$. Similar but more involved calculations apply to $\mathcal{E}_{uu}$. The following theorem is easily verified.

**Theorem 1:** A linear stochastic system has the equipartition property if and only if it is monotemperaturic.

Monotemperaturic linear stochastic systems have the following canonical form, relating to the Darlington normal form for passive electrical networks [5].

**Theorem 2:** If $S$ is a minimal, externally reciprocal, monotemperaturic system then we can make a linear change of coordinates on $x$ such that it takes the form ($\Omega = -\Omega^T$)

$$dx = (\Omega - \frac{1}{2\beta}GG^T)xdt + Budt + Gdw$$
$$dy = B\dot{u}dt + D\dot{u}dt + \sqrt{2\beta} d\dot{u}$$

Conversely, any system of this form is monotemperaturic.

**Proof:** That systems of this form are monotemperaturic is just a calculation.

Consider the deterministic system

$$\dot{x} = (A - GG^T)x + Bu_1 + Gu_2; \quad y_1 = Bu; y_2 = G^T x$$

Adopt the notation

$$[\dot{y}_1] = [G_{11}(s) \quad G_{12}(s)]\quad [\dot{u}_1]$$
$$[\dot{y}_2] = [G_{21}(s) \quad G_{22}(s)]\quad [\dot{u}_2]$$
The characterization of monotemperaturic implies that this system is conservative and hence it can be realized with \( A - \frac{1}{2} GG^T \) skew symmetric.

The classical equipartition of energy theorem is quite elegantly expressed by saying that a linear stochastic differential equation of the form

\[
    dx = (S - BB^T)x dt + Cudt + Bdw; \quad y = Cx
\]

\( (S = -S^T) \) has the property that its steady state variance is \( \frac{1}{2}I \) and if we interconnect it with a lossless system

\[
    z = \Omega x + Gu; \quad v = G^T z : \quad \Omega = \Omega^T
\]

then the random process \((x, z)\) has variance \( \frac{1}{2}I \) as well.

We can use these ideas to explain the Rayleigh-Jeans law in the following way. If we have a one parameter family of monotemperaturic systems of the form

\[
    dx = (S - \epsilon^2 BB^T)x dt + \epsilon Bdw; \quad y = cx; \quad S = -S^T
\]

Then in steady state

\[
    \mathcal{E} y(t)y(t + \tau) = \frac{1}{2}ce^{A^T\tau}c^T
\]

where

\[
    A_{\epsilon} = S - \epsilon^2 BB^T
\]

There is, of course, a limiting value for \( A_{\epsilon} \) as \( \epsilon \) goes to zero and the resulting power spectrum is pure line spectrum. The lines in the line spectrum correspond to the eigenvalues of \( S \). Thus the number of eigenvalues of \( S \) in an interval \( \omega_0 < \omega < \omega_0 + \delta \) determine the amount of power in the interval provided we select the linear functional \( c \) in such a way as to weight all modes equally.

One feature of monotemperaturic systems is that they form an “interconnectable set” in that if we interconnect two systems which satisfy this form of the fluctuation-dissipation equality then the resulting system does also. Within the realm of linear Gaussian systems there is a generalization of this idea which is significant, namely the fact that if \( \text{Re} \ g(i\omega) = \chi(\omega)\psi(i\omega) \) for a system \( S_1 \) and if the same is true for a system \( S_2 \) then it remains true for the interconnection of \( S_1 \) and \( S_2 \). Let us denote by \( P(\chi) \) the set of passive systems with this property.

**Theorem 3:** Suppose that

\[
    dx = (Ax + bu)dt + Bdw; \quad y = cx
\]

is given with \((A, b, c)\) a minimal triple and \((A, B)\) a controllable pair. Suppose further that the eigenvalues of \( A \) have negative real parts and that \( \text{Re} \ g(i\omega) \geq 0 \) for all \( \omega \). If \( q \) and \( p \) satisfy

\[
    \begin{bmatrix}
    \dot{q} \\
    \dot{p}
    \end{bmatrix} =
    \begin{bmatrix}
    0 & \omega \\
    -\omega & 0
    \end{bmatrix}
    \begin{bmatrix}
    q \\
    p
    \end{bmatrix} + \begin{bmatrix}
    \epsilon \\
    0
    \end{bmatrix} u^T; \quad y^T = [\epsilon, 0] \begin{bmatrix}
    q \\
    p
    \end{bmatrix}
\]
We define an interconnected system by
\[
\begin{bmatrix}
 dx \\
 dq \\
 dp \\
\end{bmatrix} =
\begin{bmatrix}
 A & b & 0 \\
 -c & 0 & \omega \\
 0 & -\omega & 0 \\
\end{bmatrix}
\begin{bmatrix}
x \\
q \\
p \\
\end{bmatrix}
+ \begin{bmatrix}
 Bd\omega \\
 0 \\
 0 \\
\end{bmatrix}
\]

If \( G(s) = c(I - A)^{-1}b \) and if \( AQ + QA^T = BB^T \) with \( \phi(s) = c(I - A)^{-1}Q(-I + A^T)c^T \), then
\[
\Sigma_0 = \begin{bmatrix}
 Q & 0 \\
 0 & \alpha I \\
\end{bmatrix}
\]
with \( \alpha = \frac{\psi(i\omega)}{\mathcal{R}eG(i\omega)} \),
is the weak coupling limit of the steady state variance limiting value as \( \epsilon \) goes to zero.

**Proof:** The proof of this theorem involves solving for the limiting value of the solution of a system equation of the form \( (A + \epsilon B)x = c \) with \( A \) singular. In the first place, for \( \mathcal{R}e \, G(i\omega) > 0 \) we see that the eigenvalues of
\[
\bar{A} = \begin{bmatrix}
 A & \epsilon b & Q \\
 -c & 0 & \omega \\
 0 & -\omega & 0 \\
\end{bmatrix}
\]
have negative real parts for \( \epsilon \neq 0 \). Thus for \( \epsilon \neq 0 \) there is a unique positive definite variance \( \Sigma(\infty) \) which is the solution of
\[
\bar{A}(\epsilon)\Sigma(\infty) + \Sigma(\infty)\bar{A}^T = \begin{bmatrix}
 BB^T & 0 \\
 0 & 0 \\
\end{bmatrix} = M
\]
To solve this we make use of the representation
\[
\Sigma(\infty) = \int_0^\infty e^{\bar{A}(\epsilon)t}Me^{\bar{A}(\epsilon)^t}dt
\]
transformed by Parseval’s theorem to
\[
\Sigma(\infty) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} (Is - \bar{A}(\epsilon)^{-1}M(I - \bar{A}(\epsilon))^{-1})ds
\]
this formula then readily yields the desired result.

We may then think of any system
\[
dx = Ax dt + Budt + d\omega; \quad y = cx
\]
with \( c(I - A)^{-1}b \) positive real as defining a **heat bath** in the sense that if we couple it loosely an oscillator and allow it to come to equilibrium the oscillator will possess a certain expected energy, independent of the details of the coupling or the oscillator.

Now it is well known in a three dimensional cavity the number of eigenfrequencies for the wave equation between \( \omega_0 \) and \( \omega_0 + \delta \) is approximately \( \delta \omega_0^2 \) for \( \omega_0 \) not too small. Thus if the coupling of each oscillator to \( y \) (as represented by \( c \)) is of the same strength, then the power spectral density is proportional to \( \omega^2 \) as indicated above.
In the classical theory one takes $\chi(\omega)$ to be independent of the frequency and identifies its value, via physical reasoning, with $1/2$ the temperature times Boltzmann’s constant. In a quantum theory, in order to be consistent with the black body radiation curve, it would be necessary to take $\chi(\omega)$ to be a member of the one-parameter-family

$$\chi(\omega) = \frac{h\omega/T}{e^{\hbar \omega/kT} - 1}$$

We make some additional remarks here. In the first place, for an oscillator with weak coupling to a heat bath we find that the steady state value of the expected value of the energy depends on the natural frequency $\omega$ of the oscillator and is simply $\chi(\omega)$. If we connect with weak coupling a lossless first order system – a capacitance or an inductance – we get $\chi(0)$ or $\chi(\infty)$ respectively.

### 5.9 Reducibility

Let $A$ be the infinitesimal generator of a finite state, continuous time Markov process whose state space is a subset of the real line. If $\{x_1, x_2, \ldots, x_n\}$ are the values taken on by this process, then

$$y(t) = x(t)$$

is a real valued stochastic process and its autocorrelation function

$$\mathcal{E}y(t)y(t + \tau) = \phi(t, \tau)$$

can be computed. In particular we can ask if there exists a limit

$$\lim_{t \to \infty} \mathcal{E}y(t)y(t + \tau)$$

As might be expected, the existence and uniqueness of such a limit is dependent on the properties of $A$.

One says that a finite dimensional generator $A$ is *irreducible* if there is no permutation matrix $P$ such that

$$PAP^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix}$$

with $A_{11}$ and $A_{22}$ both square. It is clear that an infinitesimal generator $A$ is always singular. If, in fact, there are two distinct probability vectors $p_1$ and $p_2$ such that $Ap_1 = Ap_2 = 0$, then of course

$$A(\alpha p_1 + (1 - \alpha)p_2) = 0$$

for some value of $\alpha$ it will happen that $\alpha p_1 + (1 - \alpha)p_2$ has nonnegative entries with some entries zero. By a permutation we can arrange these to be at the top. Thus

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ x_1 \\ \vdots \\ x_t \end{bmatrix} = 0$$
Since $x_l$ has positive entries and since $A$ is nonnegative off the diagonal, we see that the upper right-hand block of $A$ must be zero and that $A$ must be reducible. Thus we have shown the following.

**Lemma:** If there is not a unique steady state probability distribution for $\dot{p} = Ap$, then $A$ is reducible.

### 5.10 Covariance Generation with Finite State Processes

We now turn our attention to the question of generating a covariance using a finite state continuous time Markov process. This represents a considerable departure from the usual Gauss-Markov theory discussed in the previous section. The main question here is this. Given a stationary covariance $\phi(\tau)$ under what circumstances does there exist a finite state continuous time Markov process $x(\cdot)$ taking on values in a finite set $X$ and a function $f : X \to \mathbb{R}$ (the real numbers) such that $y(t) = f[x(t)]$ has a specified mean and covariance?

Because we have shown that any finite state continuous time Markov process is equivalent to one which can be expressed as

$$dx(t) = \sum_{i=1}^{N} B_i x(t) dN_i(t) ; \quad x(t) \in \mathbb{R}^n$$

with $N_i(t)$ a standard Poisson counting process with rate $\lambda_i$, this work is a natural complement to the Gauss-Markov covariance generation problem.

We find it convenient to associate with each state of an $n$ state Markov process a point in $\mathbb{R}^n$. This lets us visualize the process as jumping between points in a vector space and allows us to use certain familiar formulas from linear system theory. Let $e_i$ be the $i$th standard basic element in $\mathbb{R}^n$

$$e_i = [0, 0, \ldots, 0, 1, 0, \ldots, 0]^T$$

and let $x$ be a process which takes on values in the set $\{e_1, e_2, \ldots, e_n\}$. If $p_i(t)$ is the probability that $x(t) = e_i$, then our assumptions imply that there exists a constant matrix $A$ such that

$$\begin{bmatrix}
\dot{p}_1(t) \\
\dot{p}_2(t) \\
\vdots \\
\dot{p}_n(t)
\end{bmatrix} = 
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
p_1(t) \\
p_2(t) \\
\vdots \\
p_n(t)
\end{bmatrix}$$

Notice that because of the way we have embedded the states in $\mathbb{R}^n$ we have

$$p(t) = E x(t)$$

As we discussed in chapter 2, the entries of $A$ satisfy the conditions
(i) \( a_{ij} \) \( \geq 0 \) if \( i \neq j \)
(ii) \( \sum_{i=1}^{n} a_{ij} = 0 \)

Such matrices are called infinitesimally stochastic. Of course Peron-Frobenius theory implies that \( A \) has a nontrivial null space. If we ask that the null space be one dimensional, then we are assured that there is a unique steady state probability distribution. For these reasons we will assume

(iii) the kernel of \( A \) is one dimensional.

Such processes are called irreducible.

As an immediate consequence of the definitions we see that

\[ \mathcal{E}x(t) = e^{At}x(0) \]

Because the \( i \)th and \( j \)th components of \( x \) are never simultaneously nonzero and because the components take on only the values zero and one

\[ \Sigma(t) = \mathcal{E}x(t)x^T(t) = \begin{bmatrix} p_1(t) & 0 & \ldots & 0 \\ 0 & p_2(t) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_n(t) \end{bmatrix} \]

Moreover an elementary application of Bayes’ rule yields

\[ \mathcal{E}x(t)x^T(\tau) = \Sigma(t)e^{A^T(\tau-t)} ; \quad \tau \geq t \]

The equations (1) and (2) together with one more remark will yield a restatement of our question in linear algebraic form. The additional remark is this. Any map of the state space of the Markov process into \( \mathbb{R} \) is of the form \( c^T x \) for some \( c \) in \( \mathbb{R}^n \). Simply let the \( i \)th component of \( c \) be \( f(e_i) \) where \( f : X \rightarrow \mathbb{R} \). Brining these remarks together we have the following theorem.

**Theorem 1**: A stationary covariance \( \phi(t - \tau) = \hat{\phi}(t, \tau) \) is realizable by a zero mean process which is a real valued function of a finite state, continuous time irreducible Markov process if and only if

\[ \phi(t) = c^T \Sigma e^{At} ; \quad t > 0 \]

for some pair \((A, c)\) where \( A \) is an \( n \) by \( n \) matrix satisfying (i), (ii), (iii); \( c \) is an \( n \) vector such that \( c^T x = 0 \) for all \( x \) in the kernel of \( A \), and

\[ \Sigma = \begin{bmatrix} p_1 & 0 & \ldots & 0 \\ 0 & p_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_n \end{bmatrix} \]
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with \((p_1, p_2, \ldots, p_n)\) being the probability vector in the kernel of \(A\).

The determination of what functions can be expressed in the form required by equation (4.3) is made difficult by the requirement that \(A\) be infinitesimally stochastic. To get around the awkwardness of this constraint we focus attention on a special class of matrices. By a circulant matrix we understand a square matrix of the form

\[
M = \begin{bmatrix}
m_0 & m_1 & m_2 & \cdots & m_{n-1} \\
m_{n-1} & m_0 & m_1 & \cdots & m_{n-2} \\
& \ddots & \ddots & \ddots & \\
m_2 & m_3 & m_4 & \cdots & m_1 \\
m_1 & m_2 & m_3 & \cdots & m_0
\end{bmatrix}
\]

Associated with each such \(M\) there is a polynomial \(\hat{m}(z) = m_0 + m_1 z + \cdots + m_{n-1} z^{n-1}\).

The eigenvalues of a circulant matrix are simply the values

\[
\lambda_k = \hat{m}(e^{ik\theta}) ; \quad \theta = \frac{2\pi}{n} ; \quad k = 0, 1, \ldots, n - 1
\]

Thus \(M\) meets condition (i), (ii), (iii) if and only if

(i') the coefficients of \(\hat{m}(z)\) are all nonnegative except for the constant term.

(ii') \(\hat{m}(1) = 0\)

(iii') \(\hat{m}(z)\) does not vanish for \(z\) an \(n\)th root of unity unequal to one.

Under these assumptions one sees easily that the solution of

\[
\dot{p} = Mp
\]

for \(p(0)\) a probability vector, tends to

\[
p_\infty = \frac{1}{n} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}
\]

Thus using such an \(M\) we see that the matrix \(\Sigma\) in equation (4.3) becomes \(\frac{1}{n} I\) and we have

\[
\tilde{\phi}(t) = \frac{1}{n} e^{t M} c ; \quad t > 0
\]

The condition that the mean should vanish, \(c^T p_\infty = 0\), is also easily interpretable. In fact if

\[
c^T = (c_0, c_1, \ldots, c_{n-1})
\]

we introduce

\[
\tilde{c}(z) = c_0 + c_1 z + \cdots + c_{n-1} z^{n-1}
\]
In this notation \( c^T p_\infty = 0 \) becomes
\[
\hat{c}(1) = 0
\]

We can also express \( \tilde{\phi} \) succinctly in terms of \( m(z) \) and \( c(z) \) (see [2]).
\[
\tilde{\phi}(t) = \sum_{z:z^n=1} \hat{c}(z)\hat{c}(z^{-1})e^{\hat{m}(z)t} \quad ; \quad t \geq 0
\]
The following lemma gives a somewhat more satisfactory form of this.

**Lemma 1**: The set of finite state continuous time realizable covariances include those expressible as
\[
\tilde{\phi}(t) = \sum_{k=1}^{n-1} r_k e^{\hat{m}(e^{2\pi ik/n})t} \quad ; \quad t \geq 0
\]
with \( m \) satisfying (i'), (ii') and (iii') and the \( r_k \) real and nonnegative. In particular
\[
\psi(t) = re^{-\sigma t} \cos \omega t \quad ; \quad t \geq 0
\]
is realizable if \( r \) and \( \sigma \) are real and positive and \( \omega \) is real.

**Proof**: Of course \( \hat{c}(z)\hat{c}(z^{-1}) \) is, for \( z \) on the unit circle, real and nonnegative. Since \( \hat{c}(1) = 0 \), we must have \( \hat{c}(z)\hat{c}(z^{-1}) \) vanishing at \( z = 1 \) but otherwise we may pick the coefficients so that \( \hat{c}(z) \) has arbitrary complex values at the roots of unity consistent with \( c(z) = c(\overline{z}) \). Adding up the contribution from \( \rho = e^{2\pi ik/n} \) we get
\[
\tilde{\phi}_k(t) = 2|\hat{c}(\rho)|^2 e^{Re\hat{m}(\rho)t} \cos Im\hat{m}(\rho)t
\]
But since \( \hat{c}(\rho) \) is arbitrary, we see that \( |\hat{c}(\rho)|^2 \) can be any real nonnegative number. The general form given in the lemma then follows.

To show that the specific \( \psi \) given in the lemma is expressible in this way we make a particular choice of \( n \) and \( \hat{m}(z) \). Let
\[
\hat{m}(z) = a(1 - \alpha z - (1 - \alpha)z^2) \quad ; \quad a, \alpha > 0
\]
and let \( \sigma > 0 \) and \( \omega \) be given. At \( z = e^{i2\pi/n} = \cos(2\pi/n) + i\sin(2\pi/n) \) the ratio of the real to the imaginary parts of \( \hat{m} \) is
\[
\gamma = \frac{1 - \alpha \cos(2\pi/n) - (1 - \alpha)\cos(4\pi/n)}{1 - \alpha \sin(2\pi/n) - (1 - \alpha)\sin(4\pi/n)}
\]
Inspection of this equation shows that for any negative \( \gamma \) we can choose an integer \( n \) large enough so as to have a solution for \( \alpha \). (As \( \gamma \) approaches zero, \( n(\gamma) \) goes to infinity.) Thus we can, with this choice of \( m(z) \) adjust the magnitude and argument of \( m(e^{2\pi i/n}) \) as needed to get the function \( \psi(\cdot) \) of the lemma. Of course we pick \( c \) in such a way as to vanish on all \( n^{th} \) roots of unity except the two which enter in this discussion.

Lemma 1 makes it clear how to realize nonnegative linear combinations of the basic terms labeled there \( \psi \); take the direct sum of realizations of the type constructed in its
proof. However these realizations
(a) will not, in general, satisfy the irreducibility condition, and
(b) give no suggestions as to how to realize covariances such as \( \phi(t) = 10e^{-|t|} - e^{-5|t|} \) which are differences of positive definite functions but still positive definite.

We now establish the results necessary to get around these difficulties.

If \( \phi(t) \) is a continuous, even, positive definite function, then according to the well known representation theorem of Bochner it can be expressed as
\[
\phi(t) = \int_{[0,\infty)} \cos \omega t d\mu
\]
for some nonnegative measure \( \mu \). Of course if \( \mu \) is absolutely continuous with respect to Lebesgue measure, then we can write
\[
\phi(t) = \int_0^\infty \cos(\omega t) \Phi(\omega) d\omega \quad ; \quad \Phi(\omega) \geq 0
\]
displaying the power spectrum explicitly.

However, if we assume that \( \phi \) is not only positive definite but in addition it is strictly positive definite in the sense that
\[
\phi' (t) = \phi(t)e^{2\epsilon |t|}
\]
is for, some \( \epsilon > 0 \), also square integrable and positive definite, then we can express \( \phi \) as
\[
\phi(t) = \int_0^\infty e^{-\epsilon |t|} \cos(\omega t) \Phi(\omega) d\omega
\]
with \( \phi \) analytic. (This follows from Payley-Wiener theory; the Fourier transform of \( \phi(t)e^{-\epsilon |t|} \) analytic in a strip of width \( 2\epsilon \) centered on the \( \omega \)-axis and \( \Phi \) is its Fourier transform.) Let \( \{t_i\}_{i=1}^r \) be any finite set of real numbers. We can approximate simultaneously the integrals \( \phi(t_i) \) by Riemann sums and thus obtain
\[
\omega(t) = \sum_{i=1}^m e^{-\epsilon t_i} \cos \omega_i t \phi(\omega_i) s_i + \epsilon(t)
\]
with \( |\epsilon(t_i)| \) less than any preassigned positive number. Both \( \psi \) and the approximation go to zero as \( |t| \to \infty \); in view of the continuity of \( \phi \) we see that it can be uniformly approximated by a linear combination of \( \psi \)-like terms with positive coefficients. The following theorem summarizes.

**Theorem 2**: Any continuous, strictly positive definite function is the uniform limit of a sequence \( \phi_n \) of the form
\[
\phi_n(t) = \sum_{k=1}^{m(n)} \alpha_k e^{-\epsilon t} \cos \omega_k(t) \quad ; \quad \alpha_k > 0
\]
We now address the second problem mentioned at the start of this section.

**Lemma 2:** Stationary covariances of the form appearing in theorem 2 can be realized by a pair \((A, c)\) satisfying conditions (i), (ii) and (iii).

**Proof:** Let \((A_k, c_k)\) be a realization of \(\alpha_k e^{-\epsilon t/2} \cos \omega_k t\) of the form given in the proof of lemma 1; i.e., of the circulant form. (Note we have \(\epsilon/2\) as the decay factor.) Then for

\[
A = \begin{bmatrix}
A_1 & 0 & \ldots & 0 \\
0 & A_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & A_m
\end{bmatrix}; \quad c = \begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_m
\end{bmatrix}
\]

it follows that

\[
c^T e^{At} c = \sum_{k=1}^m \alpha_k e^{-\epsilon t/2} \cos \omega_k t
\]

Now subtract from \(A\) the infinitesimally stochastic matrix

\[
F = \frac{\epsilon}{2} I - \frac{\epsilon}{2n} \begin{bmatrix}
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1
\end{bmatrix}
\]

where \(n\) is the sum of the dimensions of the \(A_i\). Clearly \(A - F\) is infinitesimally stochastic and irreducible. Notice that

\[
c^T e^{(A-I/2)t} c = \sum_{k=1}^m \alpha_k e^{-\epsilon t} \cos \omega_k t
\]

(This time the decay factor agrees with theorem 2.) Finally, because of the null spaces of each of the \(A_i\) we see that

\[
A \begin{bmatrix}
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1
\end{bmatrix} = 0 ;
\]

and

\[
c^T \begin{bmatrix}
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1
\end{bmatrix} = 0
\]

Together these imply \(c^T e^{(A-F)t} c = c^T e^{(A-I/2)t} c\) and so \((A - F, c)\) meets all requirements.

**Example:** Consider a stationary stochastic process with power spectrum

\[
\phi(\omega) = \frac{1}{\omega^2 + 1} + \frac{2}{\omega^2 + 4}
\]
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Find a linear system of the form
\[ dx = Ax \, dt + B \, dw \; ; \; y = c^T x \]
such that in steady state \( y \) has this power spectrum. Do there exist values for the real numbers \( a, b \) and the rate of a Poisson counter \( N \) such that when steady state is reached the system
\[ dz = -2z \, dN \; ; \; z(0) \in \{-1, 1\} \]
\[ \dot{x} = ax + bz \]
has this same power spectrum? For the first part we have the calculation
\[
\frac{1}{\omega^2 + 1} + \frac{2}{\omega^2 + 4} = \frac{3\omega^2 + 6}{(\omega^2 + 1)(\omega^2 + 4)} = \frac{(\sqrt{6} + \sqrt{3}i\omega)(\sqrt{6} - \sqrt{3}i\omega)}{(\omega^2 + 1)(\omega^2 + 4)}
\]
We can the use
\[
\begin{bmatrix}
  dx_1 \\
  dx_2
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 \\
  -2 & -3
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} \, dt +
\begin{bmatrix}
  0 \\
  dw
\end{bmatrix}
\]
with \( y = \sqrt{3}x_1 + \sqrt{6}x_2 \).

For the second part, we begin by writing
\[
\begin{bmatrix}
  dz \\
  dx
\end{bmatrix} =
\begin{bmatrix}
  -2dN & 0 \\
  bdt & adt
\end{bmatrix}
\begin{bmatrix}
  z \\
  x
\end{bmatrix} \, dt
\]
Noticing that
\[ \mathcal{E}z = e^{-2\lambda} \mathcal{E}z(0) \]
and that some routine calculations show
\[ \mathcal{E}z^2 = 1 \]
\[ \lim_{t \to \infty} \mathcal{E}xz = \frac{b}{2\lambda - a} \]
\[ \lim_{t \to \infty} \mathcal{E}x^2 = \frac{b^2}{2a^2 - 4a\lambda} \]
From the denominator we see that we need \( \{a, \lambda\} = \{-1, -2\} \). Notice that
\[
= \lim_{t \to \infty} \mathcal{E}
\begin{bmatrix}
  z(t)z(t+\tau) & z(t)x(t+\tau) \\
  z(t)x(t+\tau) & x(t)x(t+\tau)
\end{bmatrix}
\begin{bmatrix}
  1 & \frac{b}{2\lambda - a} \\
  \frac{b}{2a^2 - 4a\lambda} & \frac{b}{2a^2 - 4a\lambda}
\end{bmatrix}
\exp\left[-2\lambda \, 0\right]
\]
One way to see that it is impossible to realize the given autocorrelation function is to notice that the process labeled \( z \) drives the process labeled \( x \) and there is no feedback from \( x \) to \( z \). The power spectrum of the \( z \) process is the Fourier transform of \( e^{-2\lambda|t|} \) which is \( 4\lambda/(\omega^2 + 4\lambda^2) \). The Power spectrum of the \( x \) process is obtained by multiplying this by \( g(i\omega)g(-i\omega) \) where \( g \) the transfer function of the \( x \) system \( g(s) = b/(s - a) \). This product has a constant numerator, no zeros, whereas the desired power spectrum is not of this form.
5.11 Exercises Chapter 5

1. Under what circumstances is the process

\[ y(t) = (\sin t) e^T x + (\cos t) d^T x \]

second order (wide sense) stationary if

\[ dx = Ax dt + B dw \]

and the \( x \) process is in steady state?

2. It has been suggested that there is a role in financial analysis for two state variable investment models of the form

\[
\begin{bmatrix}
    dx_1 \\
    dx_2
\end{bmatrix} = \begin{bmatrix}
    a_1 dt + b_1 dw \\
    0 \\
    a_2 dt + b_2 dw
\end{bmatrix} \begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} + \begin{bmatrix}
    u_1 - v \\
    r - u_1 + v
\end{bmatrix} dt
\]

In such models randomness in the interest rate results in noise terms that multiply the state variables. i) Use the Itô calculus to compute the variance associated with the scalar equation

\[ dx = (adt + b dw)x + u(t) dt \quad x(0) = 1 \]

where \( u \) is deterministic. ii) Show that one can reduce the problem of computing the mean and variance of the solution of

\[
\begin{bmatrix}
    dx_1 \\
    dx_2
\end{bmatrix} = \begin{bmatrix}
    a_1 dt + b_1 dw \\
    g dt \\
    a_2 dt + b_2 dw
\end{bmatrix} \begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix}
\]

to the solution of a system of linear differential equations. Write down an equation for the evolution of the expected value of \( x_1^2, x_1 x_2, x_2^2 \).

3. Describe all the 2-dimensional Wiener process realizations of the power spectrum

\[ \Phi(\omega) = \frac{1}{1 + \omega^2} + \frac{4}{1 + 4\omega^2} \]

4. Find a realization for a stationary, zero-mean process with autocorrelation function equal to \( 10e^{-3|\tau|} \).

5. Find a realization for the process \( y \) whose autocorrelation function is \( \phi(\tau) = e^{-2|\tau|} + 5e^{-3|\tau|} \).

In fact, find two distinct realizations, one involving Wiener processes and one involving Poisson processes.

6. Find a finite state continuous time jump process realization of the power spectrum

\[ \Phi(\omega) = \frac{1}{\omega^4 + \omega^2 + 1} \]

7. Find a constant matrix \( A \) and vectors \( b \) and \( c \), such that for

\[ dx = Ax dt + b dw \quad y = cx \]

the power spectrum of \( y \) is given by

\[ \phi(w) = \frac{1 + w^2}{(1 - 7w^2)^2 + 1} \]

Describe the sense in which your answer is unique.
8. Consider an intensity matrix for a four-state continuous time jump process
\[
A = \begin{bmatrix} -a & 0 & d & d \\ 0 & -a & e & e \\ b & b & -f & 0 \\ c & c & 0 & -f \end{bmatrix}
\]
If \( p_i \) is the probability that \( x = x_i \) and if \((x_1, x_2, x_3, x_4) = (m, n, m, n)\) then compute the expectation of \( x(t)x(t+\tau) \). Hint: Show that the eigenvalues of the matrix \( A \) satisfy
\[
(s + a)(s + f) - af = 0
\]

9. Let \( c \) be an \( n \) dimensional row vector whose entries are chosen from \( \{0, 1\} \). Let \( A \) be an infinitesimal generator of a continuous time jump process and let \( p(0) \) be a probability vector. Consider \( \Psi(t) = ce^{At}p(0) \)

Show that if \( A \) is a circulant matrix corresponding to \( -1 + z \) then
\[
\Psi(t) = \sum_{k=0}^{n} cke^{-t}e^{2\pi ikt}
\]

10. Find the values of \( a, b, \lambda_1 \) and \( \lambda_2 \) such that the stochastic differential equations driven by Poisson counters
\[
\begin{align*}
&dx = -2xdN_1 \quad ; \quad x(0) \in \{-1, 1\} \quad ; \quad E[N_1(t)] = \lambda_1 t \\
&dy = -2ydN_2 \quad ; \quad y(0) \in \{-1, 1\} \quad ; \quad E[N_2(t)] = \lambda_2 t
\end{align*}
\]

with
\[
z(t) = ax(t) + by(t)
\]
generate an autocorrelation function for \( z \) which is, in steady state,
\[
E[z(t)z(t+\tau)] = 13e^{-|\tau|} + 2e^{-2|\tau|}
\]

11. Let \( x(t) \) take on values in the set of unit vectors in \( e_1, e_2, \ldots, e_n \in \mathbb{R}^n \). The continuous time jump process evolves according to
\[
dx = \sum E_{ij}xdN_{ij}
\]
There is an associated output \( y = c_kx(t) \) with \( c_k \) being random variable taking on values in a finite set \( C \). More specifically, assume that any time \( x \) changes its value, \( c \) is selected from \( C \), drawn according to a probability law that assigns the probability \( p_k \) to the event \( c = c_k \). Find the expected value of \( y(t) \) and the autocorrelation function for \( y \).

12. The web address http://www.measuringworth.com/DJA/result.php contains the closing level of the Dow Jones Industrial averages on a day by day basis over the last 10 years. Find the average value of this number for the year January 1, 2006 to Dec 31, 2006 and build a stochastic model for the difference between the average value and the daily value for that year. A suggestion would be to break the data up into 20 day periods and to determine an empirical autocorrelation function over a 20 day period. Fit this with a sum of exponentials and use the data to suggest a model of the form
\[
dx = axdt + bw \quad ; \quad y = cx
\]
Also determine the empirical power spectrum. Criticize your model.
13. Find a realization for the process $y$ whose autocorrelation function is $\phi(\tau) = e^{-2|\tau|} + 5e^{-3|\tau|}$.
   In fact, find two distinct realizations, one involving Wiener processes and one involving Poisson processes.

14. This question is about hidden Markov models. Let $x(t)$ take on values in the set of unit vectors in $e_1, e_2, ..., e_n \in \mathbb{R}^n$. The continuous time jump process evolves according to

   $$dx = \sum E_{ij} xdN_{ij}$$

   There is an associated output $y = c_k x(t)$ with $c_k$ being random variable taking on values in a finite set $C$. More specifically, assume that any time $x$ changes its value, $c$ is selected from $C$, drawn according to a probability law that assigns the probability $p_k$ to the event $c = c_k$. Find the expected value of $y(t)$ and the autocorrelation function for $y$. 
5.12 Notes and References

1. A standard reference here is:


Chapter 6

Estimation Theory

Given a noisy observation of a stochastic process and given some a priori statistical characterization of the process, how can one extract the best estimate of the process from its noisy version? This type of problem is ubiquitous in science and engineering. In this chapter we recast this question as a problem in calculating conditional densities and develop some of the basic results on sequential estimation.

6.1 Preliminaries

We begin with the following basic and insightful situation. Let $x$ be a Markov process which evolves in discrete time, taking on values in a finite set $X = \{x_1, x_2, \ldots, x_N\}$. Let $p_i(k)$ be the probability that $x(k) = x_i$ and suppose that $p(k+1) = Fp(k)$. Let $y$ be a second discrete time stochastic process taking on values in a set $Y$. We postulate that $y(k)$ depends on $x(k)$ through a probabilistic law expressed in terms of a conditional probability

$$p(y(k) = y| x(k) = x_j) = c(y, x_j)$$

If we are given the values $y(0), y(1), \ldots, y(k)$, what can we infer about $x(k)$? This problem, modified in various ways, is to be investigated.

Bayes’ rule facilitates the calculations we need. Given $y(t) = y_i$, the probability that $x(t) = x_j$ is given by

$$p(x = x_j|y = y_i) = \frac{1}{p(y = y_j)} p(y = y_i|x = x_j) \cdot p(x = x_j)$$

If we consider the conditional probability at time $t$ to be a vector whose entries are the conditional probabilities of $x_1, x_2, \ldots, x_n$, respectively, then Bayes’ rule tells us

$$p(x|y) = \frac{1}{\alpha} p(y|x) \cdot p(x)$$
where the dot indicates element by element multiplication of the two vectors and $\alpha$ is the normalizing scalar which shows up in Bayes' rule. The power of this approach comes from the fact that this can be used repeatedly together with the probability law for the evolution of $x$ to solve a general class of problems.

There is one particular way in which $x_i$ conditions $y$ that occurs frequently in applications. Suppose that the values $x(k)$ assumes are the real numbers, $\{x_1, x_2, \ldots, x_n\} \in \mathbb{R}$, and that $y(k)$ is related to $x(k)$ by the equation $y(k) = x(k) + n(k)$ where $n(k)$ is a real valued random variable whose density is $\xi(n)$. Let $C(y)$ denote the diagonal matrix

$$C(y) = \begin{bmatrix}
\xi(y - x_1) & 0 & \cdots & 0 \\
0 & \xi(y - x_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \xi(y - x_n)
\end{bmatrix}$$

Using this and the evolution equation $p(k+1) = Fp(k)$ we get

$$p(x(k+1)) = \frac{1}{\alpha}p(x|y(k) = y_i)p(x(k)) = \frac{1}{\alpha}C(y(k))Fp(x(k))$$

Summarizing what this shows, we see that for

$$p(k + 1) = Fp(k) : y(k) = x(k) + n(k)$$

with the elements of random sequence $\{n(k)\}_{k=0}^{\infty}$ being independent for distinct values of $k$ but distributed according to the same density, $\xi$, the conditional density propagates according to

$$p(x|y) = \frac{1}{p(y)}p(y|x)p(x) = \frac{1}{p(y)}\xi(y - x) \cdot p(x)$$

with $p$ being a vector whose $i^{th}$ component is the probability of $x = x_i$. This can be written in vector notation as

$$\begin{pmatrix} p(x_1|y) \\
p(x_2|y) \\
\vdots \\
p(x_n|y) \end{pmatrix} = \frac{1}{p(y)} \begin{bmatrix}
\xi(y - x_1) & 0 & \cdots & 0 \\
0 & \xi(y - x_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \xi(y - x_n)
\end{bmatrix} \begin{pmatrix} p(x_1) \\
p(x_2) \\
\vdots \\
p(x_n) \end{pmatrix}$$

If we insert $p(x(k))$ in place of $p(x)$ and then use $p(x(k)) = Fp(x(k-1))$, we see that a propagation by $F$ followed by an observation gives

$$p(x|y(i) : 0 \leq i \leq k) = \frac{1}{p(y)}C(y)Fp(x(i)|y(i) : 0 \leq i \leq k - 1)$$

with $C(y)$ being the diagonal matrix appearing in the previous equation. This is a key result. In discussing it we adopt the abbreviated notation

$$\hat{p}(k + 1) = \frac{1}{p(y)}C(y(k))F\hat{p}(k)$$
6.2. A CONTINUOUS TIME FORMULATION

Of course this equation is nonlinear in $\tilde{p}$ because of the normalization factor but, as we will see shortly, this nonlinearity can be circumvented to a large extent.

**Example:** If $x$ takes on just two values, $\pm 1$, then $\tilde{p}$ is a two dimensional vector. Let the noise term $n$ be a Gaussian random variable with zero mean and variance $\sigma$. The conditional density equation then takes the form

$$
\begin{bmatrix}
\tilde{p}_1(k+1) \\
\tilde{p}_2(k+1)
\end{bmatrix} = r
\begin{bmatrix}
e^{-(y(k)-1)^2/2\sigma} & 0 \\
0 & e^{-(y(k)+1)^2/2\sigma}
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{12} \\
\tilde{a}_{21} & \tilde{a}_{22}
\end{bmatrix}
\begin{bmatrix}
\tilde{p}_1(k) \\
\tilde{p}_2(k)
\end{bmatrix}
$$

with $r$ being the scalar which normalizes the sum of the entries of the vector on the right-hand side to one.

6.2 A Continuous Time Formulation

We now recast this problem in continuous time form. Thinking through what one means by a continuous time additive noise measurement is the most significant new element. Let $x$ be a continuous time Markov process taking on values in a finite set of real numbers $X$ and suppose that $\tilde{p}_i$ is the probability that $x = x_i$. If we allow ourselves more and more unbiased independent measurements of a quantity, we would, by the law of large numbers, reduce the error variance to zero. More precisely, if we have measurements $y_i = x + n_i$ for $i = 1, 2, \ldots, m$ with the $n_i$ being independent, zero mean, Gaussian random variables, each of variance $\sigma$, then

$$
\mathcal{E} \frac{1}{m} \sum_{i=1}^{m} y_i = x
$$

and

$$
\mathcal{E} \left( \frac{1}{m} \sum_{i=1}^{m} y_i - x \right)^2 = \frac{1}{m^2} \sum_{i=1}^{m} (n_i)^2 = \frac{\sigma}{m^2}
$$

On the other hand, if when we double the number of measurements we also multiply the variance by 2, then

$$
\mathcal{E} \left( \frac{1}{m} \sum_{i=1}^{m^2} y_i - x \right)^2 = \frac{1}{m^2} \sum_{i=1}^{2m} (n_i)^2 = \frac{\sigma}{m^2}
$$

so we do not change the variance of the measurement. The fixed variance model, according to which, we simply make more measurements with the same variance, is therefore unsuitable as a way to model a noisy, continuous time, observation process whereas the model outlined previously seems more reasonable as a starting point for an analysis in which the frequency of the measurements is allowed to go to infinity.

Consider a stochastic process $x$ and the transition from a known conditional probability at time $t$, $p(t)$ to a conditional probability at time $t + h$. The transition must take into account both the passage of $h$ units of time and the effect of the observations made between time $t$ and time $t + h$. These observations are of the form $y(\tau) = x(\tau) + n(\tau)$ with $n(\tau)$
being Gaussian with zero mean, and a variance that must go to infinity with the sampling frequency. Previously we have noted that if the observation occurs at the right-hand end point then

\[
\begin{bmatrix}
\hat{p}_1(t+h) \\
\hat{p}_2(t+h) \\
\vdots \\
\hat{p}_n(t+h)
\end{bmatrix}
= rC F
\begin{bmatrix}
\hat{p}_1(t) \\
\hat{p}_2(t) \\
\vdots \\
\hat{p}_n(t)
\end{bmatrix}
\]

where

\[
C =
\begin{bmatrix}
e^{-\frac{(y(t+h)-x_1)^2}{2\sigma}} & 0 & \cdots & 0 \\
0 & e^{-\frac{(y(t+h)-x_2)^2}{2\sigma}} & \cdots & 0 \\
0 & 0 & \cdots & e^{-\frac{(y(t+h)-x_n)^2}{2\sigma}}
\end{bmatrix}
\]

On the other hand, if they the observations are considered to occur at the midpoint

Before we can consider taking a limit in which \( h \) goes to zero we need to return to the question of scaling the variance \( \sigma \). As suggested above, we replace \( \sigma \) by \( \sigma_0/h \) with \( \sigma_0 \) being regarded as fixed. This incorporates the doubling of the variance along with the doubling of the sampling frequency. With this model, we have to first order in \( h \)

\[
C \approx
\begin{bmatrix}
1 - \frac{h(y(t+h)-x_1)^2}{2\sigma_0} & 0 & \cdots & 0 \\
0 & 1 - \frac{h(y(t+h)-x_2)^2}{2\sigma_0} & \cdots & 0 \\
0 & 0 & \cdots & 1 - \frac{h(y(t+h)-x_n)^2}{2\sigma_0}
\end{bmatrix}
\]

This approximation, along with the replacement of \( F \) by \( F \approx I + hA \) gives

\[
p(t + h) - p(t) = (C(y)F - I)p(t) \approx h(Ap(t))
\]

To proceed, we would like to divide by \( h \) and let \( h \) go to zero but this can not be done as the equation stands. The difficulty is caused by the \( n^2 \) term in the expression

\[
\frac{h}{2\sigma_0} y^2(k) = \frac{h}{2\sigma_0} (x(k) + n(k))^2 = \frac{h}{2\sigma_0} (x^2(t) + 2x(t)n(t) + n^2(t))
\]
6.2. A CONTINUOUS TIME FORMULATION

Notice that $\mathcal{E}n^2 = 1/h$ and thus even with the $h/2\sigma_0$ term in front it does not go to zero with $h$. To get around this we first separate out the common additive term on the diagonal of $C(y)$, i.e., $hy^2(t+h)/2\sigma_0$. Let $y^2(t+h) = \beta(t+h)$ and then write

$$T \approx (1 + \beta)I + \begin{bmatrix} \frac{h(y(t+h)x_1-x_1^2/2)}{\sigma_0} & 0 & \cdots & 0 \\ 0 & \frac{h(y(t+h)x_2-x_2^2/2)}{\sigma_0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{(h(y(t+h)x_n-x_n^2/2)}{\sigma_0} \end{bmatrix}$$

Now observe the following general fact about systems of linear differential equations. If $\dot{z} = (\alpha(t)I + A(t))z$ and $x(t) = A(t)x$ then for $z(0) = x(0)$ the solutions are related by $z(t) = e^{\int_0^t \alpha(\sigma)d\sigma}x(t)$. That is, the addition of a time varying multiple of the identity to $A(t)$ simply re-scales the solution but does not alter the ratios of the components of the solution. Thus the addition of the $\beta I p$ term to the right-hand side simply re-scales the solution. However, because we need to re-scale anyway by virtue of the Bayes’ rule, we can merge these two re-scalings and propagate an unnormalized conditional density using

$$p(t+h) - p(t) = h(A - \frac{1}{2\sigma_0}D^2 + \frac{y}{\sigma_0}D)p(t)$$

where $D$ is

$$D = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix}$$

Dividing both sides by $h$ and taking the limit as $h$ goes to zero then gives

$$d\tilde{p} = (A - \frac{1}{2}D^2)dt + dy \frac{1}{\sigma_0}Dp$$

In most of what follows we scale the noise variance so that $\sigma_0 = 1$ in which case this equation takes the form

$$d\tilde{p} = (A - \frac{1}{2}D^2)dt + Dpd$$

**Example:** Consider the estimation of the values of a random telegraph wave generated by a Poisson counter of rate $\lambda$.

$$dx = -2xdN \quad ; \quad x(0) = +1$$

$$dy = x dt + d\nu$$

The conditional density equation in unnormalized form is

$$\begin{bmatrix} dp_1 \\ dp_2 \end{bmatrix} = \begin{bmatrix} -\lambda dt - 1/2dt + dy & \lambda dt \\ \lambda dt & -\lambda dt - 1/2dt - dy \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$
6.3 More General State Spaces

Now consider the model
\[ \begin{align*}
\frac{dx}{dt} &= f(x)dt + g(x)dw \\
\frac{dy}{dt} &= h(x)dt + d\nu
\end{align*} \]

What is the corresponding conditional density equation? Without becoming too involved in the details, we note that in the previous section it was seen that the unnormalized conditional density equation involves two operators; \( A \), which governs the evolution of the probability of the unobserved process, and \( D \), which is a diagonal operator which multiplies the probability associated with \( x \) by the value that the observation takes on at \( x \).

Reasoning via this analogy we take \( A \) to be the Fokker-Planck operator
\[ A = -\frac{\partial}{\partial x} f(x) + \frac{1}{2} \sum \frac{\partial}{\partial x_i} g_i(x) g_j(x) \]
and let \( D \) be
\[ D = \text{multiplication by } h(x) \]

Putting these ideas together we see that if we denote the Fokker-Planck operator by \( L \) then
\[ \frac{\partial \rho(t, x)}{\partial t} = (L - \frac{1}{2} h^2(x)) \rho + \frac{dy}{dt} h(x) \rho \]
is the central difference (Stratonovici) version of the conditional density equation for a diffusion process.

**Example**: Consider the simplest problem
\[ \begin{align*}
\frac{dx}{dt} &= dw \\
\frac{dy}{dt} &= x dt + d\nu
\end{align*} \]

The conditional density equation in unnormalized form is
\[ \frac{\partial \rho}{\partial t} = \left( \frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{1}{2} x^2 \right) \rho(t, x) + \frac{dy}{dt} \rho(t, x) \]

If we assume a solution of the form
\[ \rho(t, x) = e^{a(t)x^2 + b(t)x + c(t)} \]
we can derive equations for \( a, b, \) and \( c \). This goes as follows. First of all,
\[ \frac{\partial}{\partial t} e^{a(t)x^2 + b(t)x + c(t)} = \left( \dot{a}(t)x^2 + \dot{b}(t)x + \dot{c}(t) \right) e^{a(t)x^2 + b(t)x + c(t)} \]

Likewise,
\[ \left( \frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{1}{2} x^2 \right) e^{a(t)x^2 + b(t)x + c(t)} = \frac{1}{2} (2\dot{a}(t) + (2a(t)x + b(t))^2 - x^2) e^{a(t)x^2 + b(t)x + c(t)} \]
6.3. MORE GENERAL STATE SPACES

Now in order for the conditional density equation to be satisfied we need the the coefficients of $x^2$, $x^1$ and $x^0$ be equated. This gives three equations

$$\dot{a} = 2a^2 - \frac{1}{2}$$
$$\dot{b} = 2ab + \frac{dy}{dt}$$
$$\dot{c} = a + \frac{1}{2}b^2$$

Now if we rewrite our assumed form of the solution of the conditional density equation in the standard Gaussian form then we have

$$e^{a(t)x^2+b(t)x+c(t)} = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\hat{x})^2}{2\sigma}}$$

Thus

$$2\sigma = -1/a$$
$$\hat{x}/\sigma = b$$
$$-\hat{x}^2/2\sigma - \frac{1}{2}\ln(2\pi\sigma) = c$$

From these equations we see that

$$\dot{\sigma} = -\sigma^2 + 1$$
$$\frac{d\hat{x}}{dt} = -\sigma \hat{x} + \sigma \frac{dy}{dt}$$

which are, respectively, the variance and mean equations of the Kalman-Bucy filter.

This analysis can be carried out for the general linear system

$$dx = Adt + Bdw; \quad dy = Cdt + d\nu$$

In light of what was just discovered, it is reasonable to assume from the start a solution of the form

$$\rho(t,x) = \frac{1}{\sqrt{(2\pi)^n\det\Sigma}} e^{-\frac{1}{2}(x-\hat{x})^T\Sigma^{-1}(x-\hat{x})}$$

Now the conditional density equation equation takes the form

$$\frac{\partial \rho(t,x)}{\partial t} = -\left( \frac{\partial}{\partial x}, Ax \rho \right) + \frac{1}{2} \left( \frac{\partial}{\partial x} \right)^T BB^T \left( \frac{\partial}{\partial x} \right) \rho - \frac{1}{2} \langle Cx, Cx \rangle \rho + \left( \frac{dy}{dt}, Cx \right) \rho$$

Substituting the above expression for $\rho$, this leads to the Kalman-Bucy filter equations

$$\dot{\Sigma} = A\Sigma + \Sigma A^T + BB^T - \Sigma C^T C \Sigma$$

and

$$\frac{d\hat{x}}{dt} = (A - \Sigma C^T C)\hat{x} + \Sigma C^T \frac{dy}{dt}$$

This is not the way these equations were presented in the original Kalman-Bucy paper. In fact, the general form of the conditional density equation had not yet appeared.
6.4 An Exponential Representation

There exist representations of solutions of differential equations that will let us understand clearly the true complexity of the unnormalized conditional density equation based on the structure of a certain Lie algebra of operators. This idea has its origin in differential geometry as was brought out explicitly in the papers of Wei-Norman. An earlier paper by Chen covers similar ground in less explicit form. The basic ideas could probably be traced back at least to Lie and Cartan.

To begin with, consider the finite dimensional linear equation:

$$\dot{x} = (uA + vB)x$$

with $u$ and $v$ functions from $\mathbb{R}^1$ to $\mathbb{R}^1$ and $A$ and $B$ constant $n \times n$ matrices. Naively one might expect to find that the fundamental solution $\Phi(\cdot)$ is

$$\Phi(t) = e^{\int_0^t u(\sigma)d\sigma}A + (\int_0^t v(\sigma)d\sigma)B = I + \left(\int_0^t u(\sigma)d\sigma\right)A + \left(\int_0^t v(\sigma)d\sigma\right)B + \frac{1}{2} \left(\left(\int_0^t u(\sigma)d\sigma\right)A + \left(\int_0^t v(\sigma)d\sigma\right)B\right)^2 + \ldots$$

However,

$$\Phi(t) = (uA + vB) + \frac{1}{2}(uA + vB) \cdot \int_0^t (uA + vB)d\sigma + \frac{1}{2} \left(\int_0^t (uA + vB)d\sigma\right)(uA + vB)d\tau + \ldots$$

and in general it is not possible to factor out $(uA + vB)$ from this expression because $A$ and $B$ do not necessarily commute. Thus the above expression for $\Phi$ does not work. However, we can use the matrix identity

$$e^{-A}Be^A = (1 + A + \frac{1}{2!}A^2 + \ldots)B(1 - A + \frac{1}{2!}A^2 \ldots)$$

$$= B + AB - BA + \frac{1}{2}(A^2 B - 2ABA + BA^2) + \ldots$$

$$= B + [A, B] + \frac{1}{2}[A, [A, B]] + \ldots$$

This is sometimes called the Baker-Cambell-Hausdorff formula. Introduce the notation

$$ad^k_A B = [A, [A, \ldots [A, B] \ldots]]; \quad k \geq 1$$

$$ad^0_A B = B$$

For each choice of $A$, $ad^k_A(B)$ is a linear operator acting on $B$. It is common to express the above relationship as

$$\exp ad_AB = ad^0_A B + ad^1_A B + \frac{1}{2!}ad^2_A B + \ldots$$

$$= B + [A, B] + \frac{1}{2}[A, [A, B]] + \ldots$$

and write

$$e^A Be^{-A} = \exp ad_AB$$
6.4. AN EXPONENTIAL REPRESENTATION

Wei and Norman investigated the differential equation

\[ \dot{x} = \left( \sum_{i=1}^{k} u_i B_i \right)x \]

by looking for a solution \( \Phi(t)x_0 \) which can be represented as a product of exponentials

\[ \Phi(t)x_0 = e^{g_1 A_1} e^{g_2 A_2} \cdots e^{g_m A_m} x_0 \]

in which \( g_1, g_2, \ldots, g_m \) are real valued functions of time and the \( A_i \) are somehow generated by the \( B_i \). Without being specific about the latter process, we may differentiate to get

\[ \frac{d}{dt} \left( e^{g_1 A_1} e^{g_2 A_2} \cdots e^{g_m A_m} \right) = \dot{g}_1 A_1 e^{g_1 A_1} \cdots + e^{g_m A_m} + \dot{g}_1 A_2 e^{g_2 A_2} \cdots e^{g_m A_m} + \cdots \]

Inserting exponentials and their inverses we can transform this into an expression in which all terms have a common factor \( e^{g_1 A_1} \cdots e^{g_m A_m} \) on the right. That is,

\[ \frac{d}{dt} \left( e^{g_1 A_1} e^{g_2 A_2} \cdots e^{g_m A_m} \right) = \left( \dot{g}_1 A_1 + e^{g_1 A_1} \dot{g}_2 A_2 e^{-g_1 A_1} + \cdots \right) \left( e^{g_1 A_1} \cdots e^{g_m A_m} \right) \]

Applying the Baker-Campbell-Hausdorff-formula this can be written as:

\[ \dot{g}_1 A_1 + \dot{g}_2 (A_2 + g_1 [A_1, A_2] + \frac{1}{2} g_1^2 [A_1, [A_1, A_2]] + \cdots) + \cdots + \dot{g}_m \psi \]

with \( \psi \) being an expression containing the matrices \( A_1, A_2, \ldots, A_k \) and various products. This results in a set of differential equations for the \( g_i \) provided that the \( A_i \) are chosen so that any \( B_i \) is a linear combination of the \( A_i \), any \([B_i, B_j]\) is a linear combination of the \( A_i \), any \([B_i, [B_j, B_k]]\) is a linear combination of the \( A_i \), etc. The smallest set of \( A \)'s with this property forms a basis for a matrix Lie algebra and will be referred to as the Lie algebra generated by the \( A_i \).

If the \( B_i \) are, themselves, linearly independent as elements of the vector space consisting of all real matrices of a certain size, then we may take \( A_1 = B_1, A_2 = B_2 \) etc for the first \( k \) elements of set \( A_i \). Matters being so, and assuming that the \( A_i \) are a basis for the Lie algebra generated by the \( B_i \), we have

\[ [A_i, A_j] = \sum_k \gamma_{ijk} A_k \]

with certain coefficients \( \gamma_{ijk} \). These are the so-called structure constants of the Lie algebra. If \( \Phi(t)x_0 \) satisfies the differential equation we must have

\[ \dot{g}_1 A_1 + \dot{g}_2 (A_2 + g_1 [A_1, A_2] + \frac{1}{2} g_1^2 [A_1, [A_1, A_2]] + \cdots) + \cdots + \dot{g}_m (\cdots) = u_1 A_1 + \cdots u_m A_m \]
and because the $A_i$ are independent as vectors in $R^{n \times n}$ we get, on equating coefficients a set of equations of the form

$$
\dot{g}_1 = f_1(g_1, \ldots, g_m, \ldots) + u_1 \\
\dot{g}_2 = f_2(g_1, \ldots, g_m, \ldots) + u_2 \\
\vdots \\
\dot{g}_m = f_m(g_1, \ldots, g_m, \ldots) + u_m
$$

We will refer to these as the Wei-Norman equations. They depend on the choice of a particular ordering of the exponential factors. Because $\Phi(0) = I$ we have initial conditions $g_1(0) = g_2(0) = \ldots = g_m(0)$. An analysis shows that the Wei-Norman equations can always be solved on some interval $|t| \leq \epsilon$ however in most cases the solution cannot be continued for all time. A significant point is that the functions $f_1, \ldots, f_m$ only depend on the structure constants $\gamma_{ijk}$. That is, regardless of the representation of the Lie algebra we get the same Wei-Norman equations. We have here a situation such that by solving one set of nonlinear differential equations we simultaneously solve a whole family of linear evolution equations.

**Example:**

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = 
\begin{bmatrix}
a(t) & c(t) \\
0 & b(t)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = 
\begin{bmatrix}
a(t) & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} + 
\begin{bmatrix}
c(t) & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
$$

Let $\mathcal{L}$ be the Lie algebra generated by

$$
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} \text{ and } \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
$$

We choose an ordered basis for $\mathcal{L}$:

$$A_1 = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, A_2 = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} \text{ and } A_3 = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
$$

The differential equations can now be written

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = (A_1 \eta_1 + A_2 \eta_2 + A_3 \eta_3)
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
$$

with $\eta_1 = a - b, \eta_2 = c, \eta_3 = b$. If we look for a fundamental solution having the form

$$
\Phi(t) = e^{A_1 \eta_1} e^{A_2 \eta_2} e^{I \eta_3}
$$

Then

$$
\dot{\Phi} = (A_1 \dot{\eta}_1 + e^{A_1 \eta_1} A_2 \dot{\eta}_2 e^{-A_1 \eta_1} + \dot{\eta}_3 I) e^{A_1 \eta_1} e^{A_2 \eta_2} e^{I \eta_3}
$$

Because we have a basis with $A_3 = I$ the expression for $\dot{\Phi}$ contains the term $\dot{\eta}_3 \Phi$ instead of a term

$$
e^{A_1 \eta_1} e^{A_2 \eta_2} \dot{\eta}_3 A_3 e^{-g_2 A_2} e^{-g_1 A_1}
$$

that is

$$
\dot{\Phi} = (A_1 \dot{\eta}_1 + \dot{\eta}_2 (A_2 + \eta_1 [A_1, A_2] + \frac{1}{2} \dot{\eta}_1 [A_1, [A_1, A_2]] + \ldots) + \dot{\eta}_3 I) e^{A_1 \eta_1} e^{A_2 \eta_2} e^{I \eta_3}
$$
Now \([A_1, A_2] = A_2\) so that
\[
\dot{\Phi} = (A_1 \dot{g}_1 + \dot{g}_2 (A_2 + g_1 A_2 + \frac{1}{2} g_1^2 A_2 + \ldots) + \dot{g}_3 I) e^{g_1 A_1} e^{g_2 A_2} e^{g_3 I}
\]
\[
= (A_1 \dot{g}_1 + \dot{g}_2 e^{g_1} A_2 + \dot{g}_3 I) e^{g_1 A_1} e^{g_2 A_2} e^{g_3 I}
\]
So
\[
A_1 \dot{g}_1 + \dot{g}_2 e^{g_1} A_2 + \dot{g}_3 I = \eta_1 A + \eta_2 A_2 + \eta_3 I
\]
The Wei-Norman equations become
\[
\begin{align*}
\dot{g}_1 &= \eta_1 = a - b \\
\dot{g}_2 &= e^{-g_1} \eta_2 = ce^{-g_1} \\
\dot{g}_3 &= \eta_3 = b
\end{align*}
\]
The differential equations can be solved directly. By solving them we do not only find
a fundamental solution of the particular set of equations but also a fundamental solution
associated with any family of operators that commute according to the same commutation
relations.

The application of this theory to the study of the conditional density equation is straight-
forward. Given an equation of the form
\[
\dot{p} = (A - \frac{1}{2} D^2) p + \frac{dy}{dt} D p
\]
it implies that the Lie algebra generated by the matrices \(A - \frac{1}{2} D^2\) and \(D\) is key in deteriiming
how complicated it will be to solve the equation.

### 6.5 Extrapolation and Smoothing

Although we have focused on the problem of producing conditional probabilities at a time
 corresponding to the end point of the interval over which observations are available, it often
 happens that an estimate is wanted for some future event or for some event in the past.
When it is possible to delay making an estimate of a process until further data has been
received, it is called a smoothing problem. When it is necessary to make an estimate of a
future event it is called an extrapolation problem. In this section we formulate problems of
this type mathematically and derive the appropriate evolution equation for the conditional
density.

The extrapolation problem presents almost no difficulty. Suppose we have a continuous
time, finite state process \(x(t)\) taking on the values \(x_1, x_2, ..., x_n\) let \(p_i\) be the probability
that the value of \(x\) is \(x_i\) and assume that the probability law is \(\dot{p} = Ap\). Introduce \(c = [x_1, x_2, ..., x_n]\)
so that for a given value of \(p(0)\) we have for positive \(t\),
\[
\mathcal{E} x(t) = ce^{At} p(0)
\]
This provides a straightforward solution of the extrapolation problem.
In its basic form, the smoothing problem is that of determining the conditional probability for \( x(t_s) \), given \( y(\cdot) \) over \([t_0, t_f]\) with \( t_0 < t_s < t_f \). This problem is significantly more complicated than the extrapolation problem. To solve it we construct a conditional probability equation for the joint probability of events of the form \( x(t) = x_i \) and \( x(0) = x_j \). If \( x(t) \) takes on values \( \{x_1, x_2, ..., x_n\} \) then this new space of events contains \( n^2 \) elements and the conditional probability will involve \( n^2 \) probabilities. Our notation will be as follows.

Let \( p_{ij}(t) \) denote the probability that \( x(t) = x_i \) and \( x(0) = x_j \). Thus the marginals

\[
a_i = \sum_{j=1}^{n} p_{ij} ; \quad b_j = \sum_{i=1}^{n} p_{ij}
\]

are the probabilities of \( x_i(t) \) and \( x_j(0) \), respectively. Let \( P(t) \) denote an \( n \) by \( n \) matrix whose entries are \( p_{ij} \). As we have observed, the marginal probability for \( x(t) \) is obtained by taking row sums of \( P \) and the marginal probability of \( x(0) \) is obtained by taking column sums of \( P \). On the interval \([0, t_f]\) we have, in the absence of observations,

\[
\dot{P} = AP ; \quad P(0) = \text{diag}(p_1, p_2, ..., p_n)
\]

The form of \( P(0) \) captures the idea that at \( t = 0 \), \( x(t) \) and \( x(0) \) are the same and so the off-diagonal terms of \( P \) are zero. Now, if we observe \( dy = cx_1 \, dt + d\nu \) on the interval \([0, T]\) then an un-normalized version of \( P \) will evolve following the basic conditional density equation we have been using. The “drift” part is given above and reflects the fact that \( x(0) \) is a constant. There is no observation of \( x(0) \) but there is an observation of \( x(t) \). This results in the conditional probability equation

\[
dP = (A - \frac{1}{2}D^2)P \, dt + dy \, dP
\]

where

\[
D = \text{diag}(x_1, x_2, ..., x_n)
\]

Of course to get the conditional probability of \( x(0) \) we need to normalize \( P \) by dividing by the sum of its entries and marginalize by summing over the rows.

**Example:** Consider the \( \{1, -1\} \)-valued process generated by \( dx = -2xdN \) with \( N \) being a Poisson counter of rate 3. Assume that we observe \( dy = xdt + d\nu \) on the interval \([0, 1]\) and we want to find the probability that \( x(0) = 1 \). In this case the above \( P \) has the interpretation

\[
P = \begin{bmatrix}
p(x(t) = 1, x(0) = 1) & p(x(t) = 1, x(0) = -1) \\
p(x(t) = -1, x(0) = 1) & p(x(t) = -1, x(0) = -1)
\end{bmatrix}
\]

and satisfies

\[
d \begin{bmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{bmatrix} = \left( \begin{bmatrix}
-3 & 3 \\
3 & -3
\end{bmatrix} \right) dt - \frac{1}{2} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} dt + dy \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

with the initial condition

\[
P(0) = \begin{bmatrix}
1/2 & 0 \\
0 & 1/2
\end{bmatrix}
\]
As we have said, the various entries in $P$ have the interpretation of joint probabilities. Thus the probability $f$, that $x(0) = 1$ is given by the value at time one of

$$f = \frac{p_{11} + p_{21}}{p_{11} + p_{12} + p_{21} + p_{22}}$$

In section 6.8 we will return to these questions in the context of linear systems.

### 6.6 The Orthogonality Principle

In the study of inner products on function spaces one often uses the definition

$$\langle f(\cdot), g(\cdot) \rangle = \int f(x)g(x)d\mu$$

In those cases where $d\mu$ is a weighted version of Borel measure one often writes

$$\langle f(\cdot), g(\cdot) \rangle = \int f(x)g(x)w(x)dx$$

Thus far we have concentrated on the problem of propagating the conditional density and have postponed all questions of what to do with the conditional density once we have it. In some cases the goal of the analysis is to estimate the value of some function of the state, say $h(x)$. Naturally

$$\mathcal{E}h(x) = \int h(x)\rho(t,x)dx$$

where $\rho$ is the conditional density. The following fact is basic.

**Theorem:** The estimate $e$ that minimizes the expected value of the square of the error between $h(x)$ and its estimate, is the conditional expectation,

$$e = \int h(x)\rho(t,x)dx$$

**Proof:** In order to prove this we make the following observations. If we observe $y$ and if $z$ is a casual functional of $y$ and if $e$ is $y$-measurable then so is $e + \epsilon z$ for all $\epsilon$. If $e$ minimizes $\mathcal{E}(e - h(x))^2$ then we see that

$$\mathcal{E}(e + \epsilon z - h(x))^2 \geq \mathcal{E}(e - h(x))^2$$

From an analysis of this inequality in a neighborhood of zero we see that

$$\int z(e - h(x))\rho(t,x) = 0$$

That is to say, the error is orthogonal to any past measurable function of the observations.
6.7 Linear Estimation Theory

In some parts of the literature on stochastic differential equations one sees a notation in which a Stratonovic equation is divided by \( dt \) and \( dw/dt \) is written as \( \dot{w} \). This allows for simpler typography and we will use it in this section.

Consider a stochastic process \( y \) which is generated from a unity variance Wiener process \( w \) according to the equations

\[
\dot{x}(t) = A(t)x(t) + B(t)\dot{w}(t) \quad ; \quad \dot{y}(t) = C(t)x(t)
\]

Now suppose that we observe not \( \dot{y}(t) \) itself but \( \dot{y}(t) + \dot{v}(t) \) where \( v \) is a zero mean, unity variance, Wiener process. The question arises as to how, if at all, we can remove the effects of the noise \( v \).

**Lemma:** A stochastic equation of the form

\[
\dot{z}(t) = F(t)z(t) + H(t)(\dot{y}(t) + \dot{v}(t))
\]

has a solution \( z \) which has the same expectation as \( x \) if and only if \( \mathcal{E}z(0) = \mathcal{E}x(0) \) and the matrices \( F \) and \( H \) are chosen so that the equation takes the form

\[
\dot{z}(t) = A(t)z(t) - G(t)[C(t)(x(t) - z(t)) + \dot{v}(t)]
\]

**Proof:** This is a direct application of the rule for computing expectations.

If we seek an estimate which can be generated by a linear stochastic equation in the form described in the lemma, and if we want an unbiased estimate we must constrain the equations for \( z \) in this way. Now from this point on there remains only the question of picking the best choice of \( G \). Notice that if we introduce \( e = x - z \), then

\[
\dot{e}(t) = A(t)e(t) + G(t)[C(t)e(t) + \dot{v}(t)] + B(t)\dot{w}(t)
\]

Writing down the variance equation (c.f. section 2.7) we have

\[
\dot{\Sigma}_{ee}(t) = A(t)\Sigma_{ee}(t) + \Sigma_{ee}(t)A^T(t) + G(t)C(t)\Sigma_{ee} + \Sigma_{ee}(t)C^T(t)G^T(t) + G(t)G^T(t) + B(t)B^T(t)
\]

Now compare this with the solution of the equation

\[
\dot{S}(t) = A(t)S(t) + S(t)A^T(t) - S(t)C^T(t)C(t)S(t) + B(t)B^T(t)
\]

A little manipulation gives

\[
\dot{\Sigma}_{ee}(t) - \dot{S}(t) = \frac{(A(t) + G(t)C(t))(\Sigma_{ee}(t) - S(t)) + (\Sigma_{ee}(t) - S(t))(A(t) + G(t)C(t))^T + G(t)G^T(t) + G(t)C(t)S(t) + S(t)C^T(t)G^T(t) + S(t)C^T(t)C(t)S(t)}{2}
\]

If we set \( \Sigma_{ee}(t_0) = S(t_0) \), then we have

\[
\Sigma_{ee}(t) - S(t) = \frac{(A(t) + G(t)C(t))(\Sigma_{ee}(t) - S(t)) + (\Sigma_{ee}(t) - S(t))(A(t) + G(t)C(t))^T + G(t)G^T(t) + G(t)C(t)S(t) + S(t)C^T(t)G^T(t) + S(t)C^T(t)C(t)S(t)}{2}
\]
This shows that $\Sigma_{ee}(t) - S(t)$ is nonnegative definite and is zero if and only if $G(t) = -S(t)C^T(t)$. Thus we see that the minimum variance estimate of $x$ is generated by

$$d\hat{x} = A\hat{dx} + \Sigma_{ee}(t)C^T(dy - C\hat{dx})$$

**Theorem:** If $z$ satisfies the equations

$$\dot{z}(t) = A(t)z(t) + S(t)C^T(t)(y(t) - C(t)z(t) + \dot{v}(t))$$

with $S(t)$ satisfying

$$\dot{S}(t) = A(t)S(t) + S(t)A^T(t) - S(t)C^T(t)C(t)S(t) + B(t)B^T(t), \quad S(t_0) = \Sigma_{ee}(t_0)$$

then $z$ is the minimum variance estimate of $x$.

There is one additional property of the optimal solution which we need later in the discussion of the separation theorem.

**Theorem:** The optimum error variance and the optimal estimate variance satisfy

$$\Sigma_{xx}(t) = \Sigma_{ee}(t) + \Sigma_{\hat{e}\hat{x}}(t)$$

or equivalently

$$\Sigma_{xx}(t) = \Sigma_{ee}(t) + \Sigma_{\hat{e}\hat{x}}(t)$$

where $\Sigma_{xx} = \Sigma_{xx}^T$, $\Sigma_{\hat{e}\hat{x}} = \Sigma_{\hat{e}\hat{x}}^T$.

To see that the second condition is equivalent to the first observe that

$$x(t)x^T(t) = E(\hat{x}(t) + e(t))(\hat{x}(t) + e(t))^T = E\hat{x}(t)\hat{x}(t) + EEe(t)e^T(t) + E\hat{x}(t)e^T(t) + E(\hat{x}(t)e^T + e(t)\hat{x}(t))$$

and that the third term is zero if and only if $EEe^T$ is zero. To see that $E\hat{x}e^T = 0$ we observe that

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = A(T) \begin{bmatrix} \Sigma_{ee}(t)C^T(t)C(t) \\ 0 \end{bmatrix} + \begin{bmatrix} \Sigma_{ee}(t)C^T(t) \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{c}(t) \\ e(t) \end{bmatrix}$$

Now the resulting variance equation is

$$\frac{d}{dt}E\begin{bmatrix} \dot{\hat{e}}^T \\ \dot{e}^T \end{bmatrix} = \begin{bmatrix} A(t) & \Sigma_{ee}(t)C^T(t)C(t) \\ 0 & A(t) - \Sigma_{ee}(t)C^T(t)C(t) \end{bmatrix} + \begin{bmatrix} \Sigma_{ee}(t)C^T(t)C(t) \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{e}(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} \dot{c}(t) \\ c(t) \end{bmatrix}$$

$$+ E\begin{bmatrix} \dot{\hat{e}}^T \\ \dot{e}^T \end{bmatrix} \begin{bmatrix} A(t) & \Sigma_{ee}(t)C^T(t)C(t) \\ 0 & A(t) - \Sigma_{ee}(t)C^T(t)C(t) \end{bmatrix} + \begin{bmatrix} \Sigma_{ee}(t)C^T(t)C(t) \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{c}(t) \\ c(t) \end{bmatrix}$$

$$+ \begin{bmatrix} \Sigma_{ee}(t)C^T(t)C(t) \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{e}(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} \dot{c}(t) \\ c(t) \end{bmatrix}$$

$$+ \begin{bmatrix} \dot{e}(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} \dot{c}(t) \\ c(t) \end{bmatrix}$$
A short calculation will verify that the solution of this is given by
\[
\begin{bmatrix}
H(t) & 0 \\
0 & \Sigma_{ee}(t)
\end{bmatrix}
\]
where \(H(t)\) is \(E\hat{x}(t)\hat{x}^T(t)\). This calculation depends on verifying that we may take
\[
H(t) = E \hat{x}(t)\hat{x}^T(t) - Ee(t)e^T(t)
\]
where, of course, \(Eee^T = \Sigma_{ee}\) and
\[
\frac{d}{dt} \Sigma_{ee} = \Lambda \Sigma_{ee} + \Sigma_{ee} \Lambda^T - \Sigma_{ee} C^T C \Sigma_{ee} + BB^T
\]

**Example:** Consider the two different models for signal and observation given below. Evaluate the variance of the steady-state estimation error in each case.

\[
dx = -2xdt + dw; \quad dy = xdt + d\nu
\]
\[
dx = -3xdt + 1.5dw; \quad dy = xdt + d\nu
\]
In which case is the estimation error larger and what is a qualitative explanation in terms of the power spectrum? To answer this, begin with a calculation of the covariance equation associated with the first model
\[
\dot{\sigma}_1 = -4\sigma_1 + 1 - \sigma_1^2
\]
and for the second model
\[
\dot{\sigma}_2 = -6\sigma_2 + 2.25 - \sigma_2^2
\]
We can calculate the steady state by setting the derivative equal to zero.
\[
\sigma_1 = \frac{-4 + \sqrt{5}}{2} = \sqrt{5} - 2
\]
\[
\sigma_2 = \frac{-6 + \sqrt{45}}{2} = \frac{3}{2}(\sqrt{5} - 2)
\]
Comparing these we have
\[
\sigma_2 - \sigma_1 = \frac{1}{2}(\sqrt{5} - 2)
\]
Because both systems 1 and 2 are stable there is a steady state for the unobserved process. The steady state variance of the first system when unobserved is 1/4; that of the second system is 2.25/6 which is larger. The signal to noise ratio of the observation process is the same in both cases so we may expect that the estimation error will be larger in the second case.

**Example:** Consider the problem of generating the conditional density for \(x\) when \(x\) satisfies
\[
dx = x_2 dt + dw
\]
and the observation equation is
\[ dy = x_1 dt + ax_2 dt + d\nu \]

Assuming Gaussian initial data, display the differential equation for the error variance. Will the solution fail to have a steady state value for any choice of \( a \)?

**Solution:** The error covariance equation is
\[
\frac{d}{dt} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 0 & -3 \\ 1 & -1 \end{bmatrix} - \\
- \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 & a \\ a & a^2 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}
\]

The \( A \) matrix has an eigenvalue with a positive real part. Thus if the system fails to be observable for any value of \( a \), the error covariance will go to infinity if the unstable mode is unobservable. Otherwise it will have a (finite!) steady state value. The system fails to be observable when
\[
[c; cA] = \begin{bmatrix} 1 & a \\ 3a & 1-3a \end{bmatrix}
\]
is singular and this happens when \( 1 - a - 3a^2 = 0 \) One of these roots results in a value of \( a \) for which the unobservable mode is stable and in this case the variance has a limit. The other root results in the unstable mode being unobservable and in this case no limit exists.

### 6.8 Linear Smoothing

As discussed in section 6.5, it is sometimes possible to have observations up until time \( t + d \) before it is necessary to estimate \( x(t) \). Such problems are called smoothing problems.

Consider the linear system
\[
\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 dt \\ x_2 dt \end{bmatrix} + \begin{bmatrix} bdw \\ 0 \end{bmatrix} ; \ dy = cx_1 dt + d\nu
\]
With $E_1(0) = E_2(0) = 0$ and the initial variance being

$$\Sigma(0) = \begin{bmatrix} P_0 & P_0 \\ P_0 & P_0 \end{bmatrix}$$

The estimation equations are

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 dt \\ x_2 dt \end{bmatrix} + \begin{bmatrix} P_{11} c^T (dy - cx_1 dt) \\ P_{21} c^T (dy - cx_1 dt) \end{bmatrix}$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

and

$$\dot{P} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} A^T & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} bb^T & 0 \\ 0 & 0 \end{bmatrix} -$$

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} c^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Because of the special form of these equations the error variance equation can be decoupled as

$$\dot{P}_{11} = A^T P_{11} + P_{11} A^T + bb^T - P_{11} c^T c P_{11}$$

$$\dot{P}_{21} = P_{21} A^T - P_{21} c^T c P_{11} ; \quad \dot{P}_{22} = -P_{12} c^T c P_{12}$$

The result is that we recover the usual model for $\bar{x}_1$ and an auxiliary equation for $\bar{x}_2$ as well

$$\frac{d}{dt} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} P_{11} \\ P_{21} \end{bmatrix} (c^T (dy - cx_1 dt))$$

Focusing on the $\bar{x}_2$ equation,

$$\frac{d}{dt} \bar{x}_2 = P_{12} c^T (dy - cx_1)$$

we observe that its solution can be written as an integral

$$\bar{x}_2(0) = \bar{x}_1(0) + \int_0^T P_{12} c^T (dy - cx_1 dt)$$

making explicit the way in which the conditional mean of $\bar{x}_2$ depends on the later values of the difference between the estimate of $cx$ and the measurement.

To round out the picture, we note that working directly from the original model, we see that the conditional density $\rho(t, x_1, x_2)$ satisfies

$$\frac{\partial \rho}{\partial t} = -\langle \frac{\partial}{\partial x_1}, A x_1 \rangle \rho - \frac{1}{2} (cx_1)^2 \rho + \frac{1}{2} \left( \frac{\partial}{\partial x_1} \right)^T bb^T \frac{\partial}{\partial x_1} \rho$$

**Example 1:** Consider

$$dx = -x dt + dw ; \quad dy = x dt + d\nu$$
Suppose that \( x \) is distributed with a Gaussian density with \( \mathcal{E}(x(0)) = 1 \) and \( \mathcal{E}(x(0) - 1)^2 = 2 \). The Kalman-Bucy filter will generate the conditional mean and the variance. It takes the form

\[
d\hat{x}(t) = -\hat{x}(t) + p(t)(dy(t) - \hat{x}(t)dt) \; ; \; \hat{x}(0) = 1
\]

with an error variance equation

\[
\dot{p} = -2p + 1 - p^2 \; ; \; p(0) = 2
\]

We use these equations in a forward sweep to generate the causal conditional expectation of \( x(t) \). We then compute the smoothed estimate of \( x(0) \) via

\[
\bar{x}(0) = 1 + \int_0^T qe^T(dy - c\bar{x}(\sigma)d\sigma)
\]

together with the \( p_{21} \) element of the covariance matrix displayed above,

\[
\dot{q} = -q - qp \; ; \; q(0) = 2
\]

**Example 2:** Consider the problem of estimating \( x \) given \( y \) for the scalar system

\[
dx = -xdt + dw \; ; \; dy = xdt + d\nu
\]

Find the error variance for the estimate when this system reaches steady state and determine by how much \( y \) how much does the error variance decreases if we allow a one time unit delay for smoothing before producing an estimate. By previous results we see that the steady state error variance equation is

\[
0 = -2\sigma + 1 - \sigma^2
\]

It has has the solution

\[
\sigma = -1 + \sqrt{2}
\]

If we now set up the augmented variance equation

\[
\frac{d}{dt} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} - \begin{bmatrix} p_{11}^2 & p_{11}p_{12} \\ p_{11}p_{21} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]

with a view toward solving the estimation equations

\[
\frac{d}{dt} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} + \begin{bmatrix} \dot{y} - p_{11}x \\ \dot{y} - p_{12}x \end{bmatrix}
\]

The solution of the variance equation starting from the steady state value

\[
P = \begin{bmatrix} \sqrt{2} - 1 & \sqrt{2} - 1 \\ \sqrt{2} - 1 & \sqrt{2} - 1 \end{bmatrix}
\]

yields

\[
\dot{p}_{12} = -p_{12} - p_{21}(\sqrt{2} - 1)
\]

so

\[
p_{12}(t) = e^{-\sqrt{2}t}(\sqrt{2} - 1)
\]

and

\[
p_{22}(1) = (\sqrt{2} - 1) - \left((\sqrt{2} - 1)\right)^2 \int_0^1 e^{-2\sqrt{2}t}dt
\]
6.9 Identification of Linear Systems

Control theory techniques generally require the a priori knowledge of an accurate model for the dynamical system which is to be controlled. In some situations this is hypothesis is invalid and it is necessary to develop algorithms to identify the system before, or while, control is being exercised. This is a major problem. In this section we address it in the special case of linear systems with constant coefficients. This special case is a good example of applying the tools we have developed.

One approach to system identification is based on the conditional density equation. Although there are very important exceptions, such as the Kalman-Bucy filter, he conditional density is usually to complicated to be solved analytically. However it can happen that the solution of the conditional density is dependent on just a few parameters because the initial condition is chosen in a suitable way. It is a general feature of conditional density equations which involve unknown parameters which do not change in time that the support of the conditional density does not increase as time evolves. If one knows a priori that \( a \) is an element of the set \( S \subset \mathbb{R}^1 \), then the same is true for all time, regardless of the observation history. In particular, if we know that \( a \) is either -1 or -1.5, then the conditional density \( \rho(t,a,x) \) can be computed rather easily. A simple example should help to fix ideas. Consider the equation

\[
dx = axdt + dw_1 \quad ; \quad dy = xdt + dw_2\
\]

together with the stochastic equation

\[
da = 0
\]

What does this mean? Of course \( a \) is a constant but that does not mean that \( a \) is known! What this models is the situation where \( a \) is a random variable as opposed to being a full-fledged stochastic process. The initial distribution for \((a,x)\) expresses our a priori knowledge and the observation of \( y \) allows us to condition this.

The conditional density equation gives us a method to compute the joint conditional probability \( \rho(t,a,x) \). In order to obtain the conditional probability for \( a \) alone, it is necessary to integrate with respect to \( x \). Applying the general results of the previous chapter we have

\[
\frac{\partial \rho(t,a,x)}{\partial t} = \left( -\frac{\partial}{\partial x} ax + \frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{1}{2} x^2 \right) \rho(t,a,x) + \frac{dy}{dt} x \rho(t,a,x)
\]

One tractable class of identification problems goes as follows. Suppose that we know that the system is one of a finite set of possible models and that these models are linear and time invariant, say of the form

\[
dx_i = A_i x_i dt + B_i dw \quad ; \quad m = c_i x
\]

for \( i = 1, 2, ..., k \). Further assume that we can observe \( dy = cxdt + dv \) We can construct as above, the conditional density equation for the joint probability of the value of \( x \) and the value of \( i \). Instead of thinking of this as \( \rho(t,x,i) \) however, we can think of it as a vector with components \( \rho_i(t,x) \). In any case, the conditional density equation takes the form

\[
\frac{\partial \rho_i}{\partial t} = L_i \rho_i - \frac{1}{2} (c^T x)^2 \rho_i + \frac{dy}{dt} c x \rho_i
\]
with $L_i$ being the Fokker-Planck operator. We seek a solution of the form

$$\rho_i(t, x) = \frac{\alpha_i}{\sqrt{(2\pi)^n \det P_i}} e^{-\frac{1}{2}(x - \bar{x})^T P_i^{-1}(x - \bar{x})}$$

Differentiating this with respect to time gives a family of equations

$$\frac{\partial \rho_i}{\partial t} = \left( \frac{\dot{\alpha}_i}{\alpha} - \langle P_i^{-1}, \dot{P}_i \rangle + (x - \bar{x})^T P_i^{-1} \dot{P}_i P_i^{-1}(x - \bar{x}) + \left( \frac{d\bar{x}}{dt} \right)^T P_i^{-1}(x - \bar{x}) \right) \rho_i$$

### 6.10 Baum-Welch Identification

Consider the application of the maximum likelihood method to the problem of identifying the infinitesimal generator of a continuous time jump process. We assume that the given data consists of a doubly indexed array $T_{ij}$ consisting of the various times that the system has spent in the state $i$ before exiting, together with a second array $N_{ij}$ consisting of the total number of times the system left state $i$ to go to state $j$. We will show that the maximum likelihood values for the $a_{ij}$ based on these data are

$$a_{ii} = -\frac{N_i}{\sum T_{ij}}; \quad a_{ij} = -a_{ii} \frac{N_{ij}}{\sum_j N_{ij}}$$

where

$$N_i = \sum_j N_{ij}$$

The calculations go as follows. As we have noted previously, the probability distribution for the time spent in state $i$ is $a_{ii} e^{-a_{ii}t}$. The maximum likelihood choice for $a_{ii}$ is then given by the formula of section 1.6. On the other hand, given that the state is exiting from state $i$ the relative sizes of the $a_{ij}$ are simply the ratios of the observed transitions.

Propagating the conditional probability, while based on sound principles, is not necessarily the most practical way to do system identification because of the computational difficulties encountered. More common are methods based on maximum likelihood methods. Even though these do not exactly fit with the idea of formulating all problems in terms of conditional probabilities, it is important to know about these ideas.

Applying this type of reasoning to a discrete-time Markov chain, we see that if we have an observed sequence of states, corresponding to times $1, 2, ..., n$, say $x(1), x(2), ..., x(n)$, then we can categorize them by forming

$$N = (n_{ij})$$

where $n_{ij}$ is the number of transitions from state $j$ to state $i$. Now assume a model of the form $p(k + 1) = Ap(k)$ and find constants $a_{ij}$ which to make these observations as likely as possible. Observe that the constants in any given column of $A$ must sum to one and that the off-diagonals must be nonnegative. Moreover, entries in distinct columns refer to probabilistically independent situations and thus the columns be considered independently.
Suppose we look at the $j^{th}$ column. We have the integers $n_{ij}$ which represent the number of times the sample path was in state $j$ and jumped to state $i$. We wish to maximize over the possible choices of $a_{ij}$ the quantity

$$L = \prod_{i=1}^{m} a_{ij}^{n_{ij}} ; \sum_{i=1}^{n} a_{ij} = 1$$

Alternatively, we can maximize the logarithm of this

$$LL = n_{ij} \ln(a_{ij}) ; \sum_{i=1}^{n} a_{ij} = 1$$

Treating this with a Lagrange multiplier we see that the best values for $a_{ij}$ are determined by

$$\frac{n_{ij}}{a_{ij}} = \mu$$

Thus we see that for each $j$

$$a_{ij} = \frac{n_{ij}}{\sum_{j=1}^{n} n_{ik}}$$

Now to treat a continuous time problem we could sample the time axis using very small intervals and determine the best $A$ and then take a logarithm. As the sampling interval goes to zero this can be re-expressed in terms of the total time spent in the $j^{th}$ state, denote this by $T_j$ and the the number of of times state $i$ transitioned to state $j$. The expression for $A$ in the continuous time case is then

$$a_{ij} = \frac{1}{T_i} \frac{n_{ij}}{\sum_{j=1}^{n} n_{ij}}$$

If we have a partially observable Markov processes we can insert an state estimator to generate a most probable state sequence and use this. If the observations are noisy then we can smooth the observations to get the best estimate possible and use these as if they were the real thing. The problem is that the the best estimator/smoker depends on a knowledge of $A$. This leads to an iterative process that lies at the heart of Baum-Welch. The idea is to make a guess for $A$ and then generate an improvement based on the data.

6.11 Exercises Chapter 6

1. By a “Wiener filter” one usually means the steady state filter (state estimator) associated with a Gauss-Markov process observed with an additive noise term which is also a Gauss-Markov process. In terms of state variable models, we have

$$dx = Axdt + bdw_1$$
$$dn = Fndt + gdw_2$$
$$dy = cxdt + hndt + dw_3$$
6.11. EXERCISES CHAPTER 6

1. Set up the appropriate Riccati equation and estimation equation for the conditional mean (also least squares) estimate. Show that under appropriate assumptions about controllability, the relevant Riccati equation approaches a steady state as $t$ goes to infinity.

2. Show that if $x(0)$ is distributed according to an arbitrary smooth density and satisfies the stochastic equation

$$dx = Ax dt + b d w_1$$

then provided that the eigenvalues of $A$ have negative real parts, the probability density approaches a Gaussian density as $t$ goes to infinity and this Gaussian density is independent of the initial density. Establish a similar result for the conditional density that goes along with the observation equation

$$dy = x dt + d w_2$$

3. Consider $2n$ by $2n$ dimensional matrices of the form

$$\begin{bmatrix} A & Q \\ R & -A^T \end{bmatrix}$$

with $Q$ and $R$ symmetric. Show that these form a Lie algebra.

4. At $t = 0$ one knows that $x = \dot{x} = 0$ and over the interval $0 \leq t \leq t_0$ we observe $x(t) + \dot{v}(t)$ where $v(t)$ is a unity-variance Wiener process. Suppose that

$$\ddot{x}(t) + x(t) = \dot{\omega}(t)$$

where $\omega$ is also a unity variance Wiener process. Find a filter which computes an unbiased, minimum variance estimate of $x$ and $\dot{x}$, based on the observation $x(t) + \dot{v}(t)$. Examine the limiting case $t_0 \rightarrow \infty; \; t \rightarrow t_0$.

5. Suppose that $A \in \mathbb{R}^{n \times m}$ is of rank $m$. Suppose that $x$ is a zero mean Gaussian random variable. Let $y$ be a given $n$-vector. Find $v$ such that

$$\eta = \mathcal{E} \|Av - y + x\|^2$$

is minimized. Find the minimizing value of $\eta$.

6. Given that $x$ and $y$ are uncorrelated Gaussian random variables with mean $\bar{x}$ and $\bar{y}$ and variance $\Sigma_{xx}$ and $\Sigma_{yy}$, find the mean and variance for $x + y$.

7. Consider the process $x$ generated by

$$dx = Ax dt + B d w + \xi(t) dt \quad ; \quad x(0) = \text{Gaussian}$$

Where $w$ is a standard $m$-dimensional Wiener process and $\xi$ is a known function of time. If we observe

$$dy = cx dt + d \nu$$

With $\nu$ an $n$-dimensional Wiener process independent of $w$, write out in full detail a set of stochastic differential equations for the conditional mean given the observations. (These will look like the Kalman-Bucy equations with a suitable modification.)

8. Suppose that we have

$$dx = dw \quad ; \quad dy = cs dt + d w + d \nu$$

where $w$ and $\nu$ are independent Wiener processes. Derive the values of the coefficients of the unbiased, minimum variance estimator assuming the optimal filter to be one dimensional and linear.
9. Consider the system
\[ dx = \alpha xd\tau + dw + bu d\tau \quad ; \quad dy = xd\tau + dv \]
where \( \alpha \) is either -1 or -2 and \( x(0) \) is Gaussian, independent of \( \alpha \). Suppose that the a priori probability is 1/2 that \( \alpha = -1 \) and 1/2 that \( \alpha = -2 \). If \( u \) is zero, find a finite set of equations which accept input \( dy \) and propagate the conditional probability of \( \alpha \).

(a) Find equations for the conditional mean of \( x \) given that \( u(t) \) is zero.

(b) Consider letting \( u(t) = k(t)z(t) \) where
\[ dz(t) = -\beta(t)z(t)dt + \gamma(t)dy \]
Write an equation for the conditional mean of \( \alpha \).

(c) Discuss qualitatively which choices of \( k, \beta \) and \( \gamma \) make it easiest to determine \( \alpha \).

10. Write down the differential equations for the conditional mean and the conditional variance for
\[ dx = Axd\tau + \alpha Bdw \quad ; \quad dy = \beta Cxd\tau + dv \quad ; \quad \alpha \beta > 0 \]
This is standard except for the appearance of the real parameters \( \alpha \) and \( \beta \). Assume that rank \( (B,AB,...A_{n-1}B) \) and rank \( (C;CA,...CA_{n-1}) \) are \( n = \text{dim} x \). Show that the steady state conditional variance \( \Sigma(\infty) \) is a monotone increasing function of \( \alpha \) and a monotone decreasing functions of \( \beta \) where \( \Sigma_1 \geq \Sigma_2 \) means that the symmetric matrix \( \Sigma_1 - \Sigma_2 \) is nonnegative definite. (Please avoid heuristic arguments.)

11. Let \( x(t) \) be a Markov process which takes on values in the set of \( \{1,2,3,4\} \) and let \( p_i(t) \) be the probability that \( x(t) = i \); assume that \( \dot{p} = Ap \) describes the evolution of the probabilities. Suppose that at \( t = 1,2,3,... \) one makes an observation of \( x \) and through this observation learns if \( x \) is larger than 2.5 or smaller than 2.5 (there is no uncertainty about the fact). Give a rule for propagating the conditional probability in this case.

12. Consider the setup
\[ dx_1 \quad dx_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} 0 \\ u \end{bmatrix} dt + \begin{bmatrix} 0 \\ dw \end{bmatrix} \]
\[ dy = x_1 dt + dv \]
Suppose that \( x(0) \) is Gaussian. Find the control law which minimizes
\[ \eta = \mathcal{E} \int_0^1 x_1^2(t) + u^2(t) dt \]
Give all the details.

13. Show that the Lie algebra generated by the operators
\[ A = \begin{bmatrix} \frac{\partial^2}{\partial x^2} - x^2 - \frac{\partial}{\partial x} \lambda + \frac{\partial}{\partial x} x \\ \lambda \end{bmatrix} \]
\[ B = \begin{bmatrix} x \\ 0 \end{bmatrix} \]
is infinite dimensional by showing that it contains the two operators
\[ C = \begin{bmatrix} \frac{\partial}{\partial x} + x & 0 \\ 0 & \frac{\partial}{\partial x} - x \end{bmatrix} \]
\[ D = \begin{bmatrix} x + \frac{\partial}{\partial x} & 0 \\ 0 & x - \frac{\partial}{\partial x} \end{bmatrix} \]
14. Let $e$ be a random variable taking on values in $(-\infty, \infty)$. Let $\psi(e)$ be the probability density of $e$. Let $x$ be a random variable taking on values in the set $\{-2, -1, 0, 1, 2\}$. Suppose that the probability that $x = -2$ is $p_{-2}$, $x = -1$ is $p_{-1}$, $x = 0$ is $p_0$ and $x = 1$ is $p_1$ and $x = 2$ is $p_2$. Added to this a priori information is the statement $x + e = b$. Find the probability distribution for $x$ conditioned on this additional information. In particular, explain which aspects of $\psi(\cdot)$ matter and which are irrelevant.

15. If $\rho(t, x)$ satisfies the unnormalized conditional density equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial^2 \rho}{\partial x^2} - \frac{1}{2} x^2 \rho + \frac{dy}{dt} \rho$$

What equation does $\psi(x)\rho(t, x) = \eta(t, x)$ satisfy? Explain the relevance for the propagation of the conditional density for

$$dx = f(x)dt + dw$$

with observation

$$dy = xdt + d\nu$$

16. Consider the pair of equations

$$dz = -2zdN; \quad z(0) \in \{1, -1\}$$
$$dx = -(2 + z)xdt + dw$$

and the observation

$$dy = xdt + d\nu$$

where $N$ is a Poisson counter of rate $\lambda$ and $\nu$ and $w$ are standard Wiener processes. If $\lambda$ is very small, then the coefficient in the equation for $x$ is unlikely to change very frequently. Thus it is likely to be either

$$dx = -3xdt + dw$$

or

$$dx = -\lambda xdt + dw$$

for long periods of time. If we knew that $z$ never switched between these values and just took on one or the other of them, then we would use two Kalman-Bucy filters to decide which it is. What would be the equations of these filters?

17. Consider the stochastic process $y$ generated by

$$dx = f(x)dt + g(x)dw$$
$$dy = h(x)dt + d\nu$$

Suppose that an inspired engineer has found that the equation

$$dz = a(z)dt + b(z)d\nu$$

is such that

$$m(t) = \eta(z(t))$$

is a good estimate of $h(x)$. In fact, the engineer claims that it is the least squares estimate of $x$ in the sense that for any other past measurable function of $y$, say $n$,

$$\mathcal{E}(h(x(t)) - n(t))^2 \geq \mathcal{E}(h(x(t)) - m(t))^2$$

Assuming the claim is correct, show that

$$\mathcal{E}(h(x(t)) - m(t)) \cdot r(t) = 0$$

if $r(t)$ satisfies any equation of the form

$$dv = \gamma(r)dt + \delta(r)d\nu$$
18. Verify that there exists constants $a, b, c$ and $f$ such that the fundamental solution for
\[
\frac{\partial \rho(t, x)}{\partial t} = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} - x^2 \right) \rho(t, x)
\]
is given by
\[
\rho(t, x) = \frac{a}{\sqrt{\sinh bt}} e^{-cx^2} e^{-ft^2}
\]
19. If $y: [a, b] \to \mathbb{R}$ is given them in some sense
\[
F(\omega) = \int_a^b e^{-i\omega t} y(t) dt
\]
is an “approximate Fourier Transform” of $y$. If, on the other hand, we wish to construct a periodic extension of $y$ we can evaluate
\[
b_n = \int_a^b y(t) \cos \left( \frac{2\pi t}{b-a} \right) dt
\]
\[
c_n = \int_a^b y(t) \sin \left( \frac{2\pi t}{b-a} \right) dt
\]
If $y$ is a stochastic process generated by
\[
dx = Ax dt + bd\omega \quad ; \quad y = cx
\]
then we may wish to define the short term spectral content by
\[
F(\omega) = \int_{-\infty}^t e^{-t+\sigma} e^{i\omega \sigma} d\sigma
\]
20. Consider the linear system
\[
dx = Ax dt + Bd\omega \quad ; \quad dy = Cx dt + d\nu
\]
Find a means to generate the conditional probability density for $x$ given the observations but, unlike the standard Kalman-Bucy assumptions, in this case the initial probability distribution for $x$ is the sum of two Gaussians.
\[
\rho_1(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma_a}} e^{-\frac{1}{2} (x-x_a)^T \Sigma_a^{-1} (x-x_a)}
\]
\[
\rho_2(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma_b}} e^{-\frac{1}{2} (x-x_b)^T \Sigma_b^{-1} (x-x_b)}
\]
\[
\rho(0, x) = .3\rho_1(x) + .7\rho_2(x)
\]
Hint: The sufficient statistics can be generated by two filters of the type you know about and possibly a little more.
21. Consider a three state Markov process taking on the values 0, 1, 2 and having a transition matrix $A$. Suppose that it is possible to observe a counting process $N$ whose rate depends on the state of the Markov process, with the rate being 1 if the process is in state 1, 2 if the process is in state 2 and 3 if the process is in state 3. Find a set of differential equations which will generate the conditional density of the process, given the observed process. Describe the main differences between this estimator and the estimator you would use if one observed the state plus white noise.
22. Suppose that $y$ is generated by
\[ dx_1 = -2x_1 dt + dw_1 ; \quad dx_2 = -4x_2 dt + dw_2 ; \quad dy = (x_1 + x_2) dt \]
Find a set of equations to generate the conditional density for $x_1$ given $y$. This is an example of the so-called “colored noise” problem because the observation $y$ consists of the signal plus a noise term whose autocorrelation function shows that the noise is not independent from one moment to the next.

23. You will have noticed that the operators such as
\[ L = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} - x^2 \right) \]
show up in the conditional density equation. Verify that there exists constants $a, b, c$ and $f$ such that the fundamental solution for
\[ \frac{\partial \rho(t, x)}{\partial t} = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} - x^2 \right) \rho(t, x) \]
is given by
\[ \rho(t, x) = \frac{a}{\sqrt{b}} e^{-ct^2} \]
and sketch the solution for two or three values of $t$.

24. Consider the scalar system
\[ dx = -x dt + dw ; \quad dy = x dt + d\nu \]
with $w$ and $\nu$ independent, standard Wiener processes, $\mathcal{E}x(0) = 3, \mathcal{E}(x(0) - 3)^2 = 1$. Describe an algorithm that will accept as input the observed values of $y$ on $[0, 1]$ and produce the conditional expectation of $x(0)$.

25. Consider the difference equation
\[ x(k + 1) = ax(k) + n(k) ; \quad y(k) = x(k) + \nu(k) \]
where $n(k)$ and $\nu(k)$ are sequences of Gaussian random variables with zero mean, unity variance and independent from each other and from one time to another. Derive a pair of equations that will propagate the conditional mean of $x$, given that $x(0)$ has a Gaussian distribution of mean $a$ and variance $b$.

26. Suppose that you are given a realization $y$ of a stochastic process and told that $y$ was generated by either
\[ dx = -x dt + dw ; \quad dy = x dt + d\nu \]
or by
\[ dx = -3x dt + dw ; \quad dy = x dt + d\nu \]
that $x(0)$ is Gaussian with mean $a$, variance $b$ and that a priori either model is equally likely. Find a set of equations for generating the conditional probability of the state and the model.

27. Let $\Sigma$ be a symmetric, positive definite matrix partitioned into four blocks as follows.
\[ \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} \]
Assume $\Sigma_{11}$ and $\Sigma_{22}$ are square. Verify the formula
\[ \Sigma^{-1} = \begin{bmatrix} (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)^{-1} & \Sigma_{12}^{-1}(\Sigma_{22} - \Sigma_{12}\Sigma_{11}^{-1}\Sigma_{12}) \\ \Sigma_{12}^{-1}(\Sigma_{22} - \Sigma_{12}\Sigma_{11}^{-1}\Sigma_{12}) & (\Sigma_{22} - \Sigma_{12}\Sigma_{11}^{-1}\Sigma_{12})^{-1} \end{bmatrix} \]
And show that if \( \Sigma \) is positive definite the various inverses appearing actually exist. What is the conditional probability of \( y \) given \( x \) assuming that \((x,y)\) is Gaussian with mean \( (\bar{x}, \bar{y}) \) and variance \( \Sigma \)? Specialize the above formula for the inverse to the case where \( \Sigma_{22} \) is a scalar and therefore \( B_{12} \) is a column vector.

28. Consider a random process \( x \) taking on values \( \pm 1 \). Suppose that it is known that \( x \) is generated by an equation of the form \( dx = -2xdN \) with \( N \) a Poisson counter of constant rate but that the rate of the counter is not known. Describe a procedure, not necessarily optimal in any sense, that will correctly estimate the rate in the limit as the observation record becomes infinite.

29. Let \( A \) be an infinitesimal generator of the form

\[
A = \begin{bmatrix}
-a & b \\
a & -b
\end{bmatrix}
\]

Show that \( e^{At} \) is given by

\[
e^{At} = \begin{bmatrix}
1 - \frac{a}{a+b} \left(1 - e^{-(a+b)t}\right) & \frac{b}{a+b} \left(1 - e^{-(a+b)t}\right) \\
\frac{a}{a+b} \left(1 - e^{-(a+b)t}\right) & 1 - \frac{b}{a+b} \left(1 - e^{-(a+b)t}\right)
\end{bmatrix}
\]

Now suppose that one observes a two state continuous time Markov process over some period and observes that the times between jumps from state one to state two and state two to state one are \( t_1, t_2, \ldots, t_n \) that the times between jumps from state two to state one and jumps from state one to state two are \( s_1, s_2, \ldots, s_m \). Show that the maximum likelihood values for \( a \) and \( b \) maximize

\[
L(a,b) = \prod_i \frac{b}{a+b} \left(1 - e^{-(a+b)t_i}\right) \frac{a}{a+b} \left(1 - e^{-(a+b)s_i}\right)
\]

or, equivalently, the logarithm

\[
LL = \ln a + \ln b - \ln(a + b) + \ln \left(1 - e^{-(a+b)t_i}\right) + \ln \left(1 - e^{-(a+b)s_i}\right)
\]

30. Consider the system

\[
dx = -3x\,dt + 2\,dw; \quad dy = x\,dt + d\nu
\]

with \( x(0) \) being Gaussian with mean 1 and variance 5. Suppose we have observations on the interval \([0,2]\) and wish to find the conditional mean of \( x \) at \( t = 1 \). Describe how to do this.

31. Maximum likelihood is an approach for determining parameter values in a probability distribution on the basis of sampled values. Show that if we are to associate an exponential density \( \theta e^{-\theta x} \) with a given set of nonnegative numbers \( x_1, x_2, \ldots, x_n \) in such a way as to maximize

\[
L = \prod_{i=1}^{n} \theta e^{-\theta x_i}
\]

then the best value of \( \theta \) is

\[
\theta = \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{-1}
\]

In deriving this we assumed that the \( x_i \) were chosen independently of each other, thus leading to the product in the expression for \( L \). Now suppose that the \( x_i \) are not available but what is available is \( x_i + \nu_i \) where the \( \nu_i \) are independent random variables distributed with a density \( q \). We seek to maximize

\[
\mathcal{E}L = \mathcal{E} \prod_{i=1}^{n} \theta e^{-\theta (x_i + \nu_i)}
\]
where the expectation is taken with respect to the density \( q \). Does the presence of the \( \nu_i \) change anything? Is there a role in your analysis for the term

\[
\gamma(\theta) = \ln E e^{-\langle \theta \rangle \sum_{i=1}^{n} \nu_i}
\]

If so, is there a choice for \( q \) that makes \( \gamma \) independent of \( \theta \)?

### 6.12 Notes and References

Our basic approach follows the ideas described in


This path leading to the association of a Lie algebra with an estimation problem is described in


The identification of an explicit class of solvable nonlinear filters is due to Benes


The smoothing solution was given Bryson and Frazier and also by Mayne in


**Section 6**

There is an alternative approach to estimation theory which is based on the idea that an estimate can be optimal in the least squares sense only if the difference between it and the true signal, i.e., the estimation error is orthogonal to any signal which can be generated from the observable quantities. This point of view is explored in

Chapter 7

Stochastic Control

Decision making under uncertainty is ubiquitous. The issues may be momentous such as selecting the right breaking thrust for a soft landing on the moon or inconsequential as selecting a seat on the bus, but have in common the fact that whatever the action taken, it has an uncertain outcome because of uncertainty about the information available and/or the uncertain effect of the choice to be made. If the uncertainties can be described in probabilistic terms and if there is a well defined objective function expressed in terms of expected values, then such questions belong to the provence of stochastic control in the sense we adopt here. The variety and depth of such problems is such that the results to be discussed here will build on almost everything in the preceding chapters. The nature of the results, and the methods used to justify them, depend on the type of information available when making the decision. The organizational principle used here is to distinguish between classes of problems on the basis of what observations are available. This approach places in evidence the type of feedback that can be used and brings out the remarkable differences in technique required to solve problems that might have appeared to be quite similar.

Example 1: Suppose that a store has in its inventory $n$ perishable items which, if not sold over the next $T$ units of time become worthless. Let $r$ be the nominal retail price of each of these items. To stimulate sales, the manager can use advertising. For $t \in [0,T]$, let $x(t) \in (\mathbb{Z}^+ \cup 0)$ be the number of items remaining to be sold. Let $u(t) \geq 0$ be the rate of spending on advertising. Based on past experience the manager models this situation as a Markov process with probability law

\[
\dot{p} = Ap + \phi(u)B_1p
\]

with $p(t)$ being a vector whose $i^{th}$ component is the probability that there are $i$ items remaining to be sold at time $t$. The goal is to minimize the cost plus the cost of the worthless items left at time $T$,

\[
\eta = \int_0^T udt + cX(T)
\]
The inventory is assumed to be known at all times and so $u$ can be taken to be a function of $x$ and $t$.

**Example 2:** Let $x(t)$ represent the alignment between two parts of a machine used in a production line. Suppose $x$ takes on the values $\{-2, -1, 0, 1, 2\}$ with $x = 0$ representing the desired alignment. Suppose, further, that when the parts are out of alignment the machine produces parts of less value. Experience shows that if there is no intervention it is reasonable to model the evolution of $x$ as a Markov process with $\dot{p} = Ap$. However, there is a possibility of applying a realignment procedure that acts to restore the alignment, albeit probabilistically, giving $\dot{p} = Ap + uBp$. The generators $A$ and $B$ are given by

$$A = \begin{bmatrix} -1 & 0.2 & 0 & 0 & 0 \\ 0.1 & -0.4 & 0.2 & 0 & 0 \\ 0 & 0.4 & -0.8 & 0.2 & 0 \\ 0 & 0 & 0.2 & -0.4 & -0.1 \\ 0 & 0 & 0 & 0.2 & -0.1 \end{bmatrix}; \quad B = \begin{bmatrix} -0.4 & 0.4 & 0 & 0 & 0 \\ 0.4 & -0.8 & 0.4 & 0 & 0 \\ 0 & 0.2 & -0.1 & 0.2 & 0 \\ 0 & 0 & 0.2 & -0.4 & -0.4 \\ 0 & 0 & 0 & 0.2 & -0.4 \end{bmatrix}$$

The application of the corrective action requires some down time and hence it has an associated cost. The performance measure to be minimized is

$$\eta = \int_0^T e^T x + u \, dt$$

### 7.1 Stochastic Control with Perfect Observations

#### 7.1.1 Generalities

Suppose that $x$ evolves according to an equation of the form

$$dx = f(x)dt + b(x)udt + g(x)dw$$

with $u$ being a control and suppose we wish to minimize a loss function of the form

$$\eta = \mathcal{E} \int_{t_0}^{t_1} L(x(t), u(t))dt + \mathcal{E} \phi(x(t_1))$$

If we observe the state $x$ we can attempt to reduce the expected value of a loss function by suitable selection of $u$ as a function of $x$ and $t$. Because $u$ enters the the Fokker-Planck equation as in

$$\frac{\partial \rho(t, x)}{\partial t} = \mathcal{L} \rho(t, x) + \sum \frac{\partial b_i(x)u(t, x)\rho(t, x)}{\partial x_i}$$

with $\mathcal{L}$ being the Fokker Planck operator with $u = 0$, and because we can express $\eta$ as

$$\eta = \int_{t_0}^{t_1} \int_X L(x, t)\rho(t, x)dt \, dx + \int_X \phi(x(t_1))\rho(t_1, x)dx$$

the optimal control problem can be stated as a problem about controlling a deterministic partial differential equation with no reference to the sample path equation.
7.1. STOCHASTIC CONTROL WITH PERFECT OBSERVATIONS

To see what this involves in a concrete problem, consider
\[ dx = -x dt + u dt + dw \; ; \; x(0) = 0 \]
and the performance measure
\[ \eta = \mathcal{E} \int_0^T x^2 + u^2 dt \]
Suppose that \( u = u(t, x) \). In this case the Fokker-Planck equation for the density \( \rho(t, x) \) can be expressed in terms of \( u \) and takes the form
\[ \frac{\partial \rho}{\partial t} = \left( \frac{\partial}{\partial x} (x - u(t, x)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) \rho \]
If we express the performance measure in terms of \( \rho \) we get
\[ \eta = \int_0^T \int_{-\infty}^{\infty} \rho(t, x)(x^2 + u^2(t, x)) dx dt \]
Contributing to the difficulty inherent in such problems are the fact that the optimization is over functions of both \( x \) and \( t \) and the fact that the evolution equation is a partial differential equation. As we will see in shortly, this integral is minimized by letting \( u \) satisfy the equation
\[ u(t, x) = k(t)x(t) \]
where \( k \) satisfies a certain differential equation, but we will use different methods to establish this.

In some cases there is an underlying steady state problem that comes about if \( t \) drops out of the picture because all transient effects have gone away. In this situation matters are somewhat simpler. Consider the equation
\[ dx = -f(x) dt + b(x) u dt + g(x) dw \; ; \; x(0) = 0 \]
and the performance measure
\[ \eta = \lim_{t \to \infty} \mathcal{E} \phi(x(t), u(t)) \]
Suppose that \( u = u(x) \) so that the the Fokker-Planck equation for the density \( \rho(t, x) \) takes the form
\[ \frac{\partial \rho}{\partial t} = \left( \frac{\partial}{\partial x} (f(x) - b(x)u(x)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) g^2(x) \rho \]
If we express the performance measure in terms of \( \rho \) we get
\[ \eta = \int_{-\infty}^{\infty} \rho(t, x)(x^2 + u^2(x)) dx \]
The minimization of this integral can be approached using the calculus of variations. Setting the time derivative of \( \rho \) equal to zero gives
\[ \left( \frac{\partial}{\partial x} (f(x) - b(x)u(x)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) g^2(x) \rho = 0 \]
which eliminates \( t \) from the problem.

In general, problems of this type are not easy to solve and so other approaches are adopted.
7.1.2 The Linear, Quadratic, Gaussian Problem

Consider the model
\[ dx = Ax dt + Budt + Gdw ; \quad \rho(0, x) = \rho_0(x) \]

together with the performance measure
\[ \eta = \mathbb{E} \int_0^T x^T L x + u^T u dt \]

Assumptions we make are \( x(t) \in \mathbb{R}^n \), \( w \) is a standard \( m \)-dimensional wiener process, the pair \((A, B)\) is controllable, and \( L = L^T \geq 0 \). Our objective is to find \( u = u(t, x) \) such as to minimize \( \eta \). We proceed as follows. Observe that if \( K \) is a symmetric matrix and if \( x \) satisfies the given Itô equation, then
\[
0 = -x^T(t)K(t)x(t)\bigg|_0^T + \\
\int_0^T x^T \dot{K} x + x^T (A^T K + KA) x + x^T Kdw + (dw)^T K x + trB^T KB dt
\]

with the last term arising from the Itô formula. Now if we add the right-hand side of this equation to the expression for \( \eta \) and take advantage of the expectation formula we have
\[
\eta = -\mathbb{E} x^T(t)K(t)x(t)\bigg|_0^T + \\
\mathbb{E} \int_0^T x^T(L + A^T K + KA)x + u^T u dt + trB^T KB
\]

Now observe that we can consider the integrand to be a quadratic form in the vector \((x, u)\), i.e.,
\[
\begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} \dot{K} + A^T K + KA + L & KB \\ KB^T K & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}
\]

Now observe that if we impose the condition
\[
\dot{K} = -A^T K - KA - L + KBB^T K
\]

then we can write \( \eta \) as
\[
\eta = \mathbb{E} \int_0^T \|B^T K x + u\|^2 dt + \mathbb{E} trB^T KB + \mathbb{E} x^T(t)K(t)x(t)
\]

Finally, if we impose the boundary condition on \( K \) in the form \( K(T) = 0 \). then we see that the minimizing control is \( u = -B^T K x \) and the minimum cost is
\[
\eta^* = \mathbb{E} x^T(0)K(0)x(0) + \mathbb{E} trB^T KB
\]

\textbf{Example 1:} Consider the control problem
\[
dx = -x dt + ud t + dw ; \quad x(0) = 0
\]
and the performance measure
\[ \eta = \mathcal{E} \int_0^T x^2 + u^2 \, dt \]
Following the theory just given, we see that we must solve the equation
\[ \dot{k} = 2k - 1 + k^2 ; \quad k(T) = 0 \]
and that the optimal control is
\[ u(t, x) = -k(t)x(t) \]

**Example 2:** In some situations the corrective control signals must be delivered in the form of pulses whose magnitude can be controlled but whose timing is somewhat uncertain. Consider the model
\[
\begin{bmatrix}
  dx_1 \\
  dx_2
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 \\
  -1 & -1
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
dt +
\begin{bmatrix}
  0 \\
  0
\end{bmatrix}
udN
\]
with \( N \) a Poisson process of rate 3. Assume that \( x_1 \) and \( x_2 \) can be measured without error and that we seek a control law \( u(x_1, x_2) \) which will minimize
\[ \eta = \mathcal{E} \int_0^1 x_1^2 \, dt \]
If we decide to content ourselves with the best linear control law, the problem becomes that of finding the values of \( k_1 \) and \( k_2 \) in
\[ u(x, y) = k_1 x_1 + k_2 x_2 \]
so as to minimize \( \eta \). To get some insight, observe that \( k \) \( x \) \( dN \) with \( N \) of rate 3 delivers on the average as much signal as \( 3k \) \( x \) and that this approximation is better when \( x \) is changing slowly and is less easy to defend when \( x \) is changing rapidly. In the absence of any control \( x(t) \) is a lightly damped version of \( x(0) \) \( \sin t \). Thus with three spikes per unit time it may not be too bad to think in terms of this approximation.

Begin the actual analysis by noting that if we use the given form for \( u \) we have
\[
\begin{bmatrix}
  dx_1 \\
  dx_2
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 \\
  -1 & -1
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
dt +
\begin{bmatrix}
  0 & 0 \\
  k_1 & k_2
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
dN
\]
However, it is more to the point to work with the larger set of equations that describe the quadratic terms
\[
\begin{bmatrix}
  dx_1^2 \\
  dx_2^2 \\
  dx_1x_2 \\
  dx_1x_2
\end{bmatrix} = M
\begin{bmatrix}
  x_1^2 \\
  x_1x_2 \\
  x_2^2
\end{bmatrix}
\]
where
\[
M =
\begin{bmatrix}
  0 & 2dt & 0 \\
  -1dt + k_1 dN & -1 dt + (2k_2 + k_2^2) dN & dt \\
  k_1^2 dN & (k_1 + 2k_1 k_2) dN - 2 dt - .1 dt & -.2 dt + (1 + k_2^2) dN
\end{bmatrix}
\]
Taking the expectation of both sides we get
\[
\frac{d}{dt} \mathcal{E}
\begin{bmatrix}
  x_1^2 \\
  x_1x_2 \\
  x_2^2
\end{bmatrix} =
\begin{bmatrix}
  0 & 2 & 0 \\
  -1 + 3k_1 & -1 + 2k_2 + k_2^2 & 1 \\
  3k_1^2 & 3(k_1 + 2k_1 k_2) - 2.1 & -2.8 + 3k_2^2
\end{bmatrix}
\mathcal{E}
\begin{bmatrix}
  x_1^2 \\
  x_1x_2 \\
  x_2^2
\end{bmatrix}
\]
7.1.3 Multiplicative Noise

Consider the following class of models with multiplicative noise terms
\[ dx = Axdt + Budt + dwGx \quad ; \quad \eta = \mathcal{E} \int_0^T x^T Lx + u^T udt \]

As above, we introduce \( K \) in the form
\[ 0 = -x^T(t)K(t)x(t)|_0^T + \int_0^T x^T K(Ax + Bu)dt + (Ax + Bu)^T K x dt + \]
\[ x^T KGx dw + x^T G^T K x dw + x^T G^T KGx dt \]

We can complete the square as above to get an equation for \( K \) of the form
\[ \dot{K} = -A^T K + KA - L + KB^T BK - G^T KG \]

If this equation is solved with \( K(T) = 0 \) then the optimal feedback solution is \( u(t) = -B^T K x \) as above.

7.1.4 Markov Decision Problems

Let \( u \in \mathbb{R}^m \) and let \( \{N_{ij}\} \) be a set of Poisson counters with rates \( \lambda_{ij}(u) = \alpha_i + \sum \beta_{ij} u_j \).

Consider an optimization problem in which the evolution of \( x \in \mathbb{R}^n \) is governed by
\[ dx = \sum G_{ij} x dN_{ij} \quad ; \quad x(t) \in \{e_1, e_2, ..., e_n\} \]

and the goal is to minimize the performance measure
\[ \eta = \mathcal{E} \int_0^T c^T(t)x(t) + u^T(t)Fx(t) + u^T(t)Ru(t)dt + (\phi, x(T)) \]

where \( R = R^T \) is positive definite and \( \phi \in \mathbb{R}^n \) defines a terminal cost. We have shown earlier that the model for \( x \) is a completely general model for finite state, continuous time jump processes. To solve the optimization problem we again focus on the minimum return function, but now, in contrast to what was done in the previous two subsections, we postulate a linear function of the state \( k^T x \). Evaluating its derivative along sample paths gives
\[ dk^T x = k^T x dt + k^T \sum G_{ij} x dN_{ij} \]

or the equivalent,
\[ 0 = -k^T(t)x(t)|_0^T + \int_0^T k^T x dt + \int_0^T (\sum k^T G_{ij} x) dN_{ij} \]

Adding this to the performance measure we get
\[ \eta = -\mathcal{E} k^T(t)x(t)|_0^T + \mathcal{E} \int_0^T c^T(t)x(t) + u^T(t)Ru(t) + u^T F x + k^T x dt + \mathcal{E} \int_0^T k^T G_{ij} x dN_{ij} \]
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We now rewrite this expression by subtracting a term of the form $\lambda_i(u)dt$ from $dN_i$ to put in evidence $dN_i - \lambda_i dt$, whose expected value is zero. Upon taking expectations the previous equation yields

$$\eta = -\mathcal{E}k^T(t)x(t)|_0^T + \mathcal{E} \int_0^T \dot{k}^T x + c^T x + u^T Ru + u^T Fx + \left( \sum k^T G_{ij} x \lambda_j(u) \right) dt + \phi^T x(T)$$

where we stop showing the $t$ dependency to improve readability.

The next step is to develop a more useful expression for the terms involving $\lambda_i(u)$. Recall that we have assumed that the rates are of the form $\lambda_i(u) = \alpha_i + \sum \beta_{ij} u_j$ and thus we can define infinitesimally stochastic matrices $A$ and $B$ such that

$$\sum G_{i}(\alpha_i + \sum \beta_{ij} u_j) = (A + \sum u_i B_i)$$

In terms of this notation we have

$$\eta = -\mathcal{E}k^T(t)x(t)|_0^T + \mathcal{E} \int_0^T k^T x + c^T x + k^T (A + \sum u_i B_i) x + u^T Ru + u^T Fx + \sum k^T B_i u_i x \; dt + \phi^T x(T)$$

Now comes a small trick. Rearrange the trilinear term $k^T \sum B_i u_i x$ as $u^T D(k)x = u^T E(x) k$ and observe the identity

$$u^T Ru + u^T (F + D(k)) x = -\frac{1}{4} x^T (F + D(k))^T R^{-1} (F + D(k)) x +$$

$$\left( u + \frac{1}{2} R^{-1} (F + D(k)) x \right)^T R \left( u + \frac{1}{2} R^{-1} (F + D(k)) x \right)$$

Notice that because $x$ only takes on the values in the set $\{e_i\}$, we can write

$$x^T (F + D(k))^T R^{-1} (F + D(k)) x = x^T f(k)$$

with $f$ being given by

$$\begin{bmatrix} f_1(k) \\ f_2(k) \\ \vdots \\ f_n(k) \end{bmatrix} = \sum \begin{bmatrix} e_1^T (F + D(k))^T R^{-1} (F + D(k)) e_1 \\ e_2^T (F + D(k))^T R^{-1} (F + D(k)) e_2 \\ \vdots \\ e_n^T (F + D(k))^T R^{-1} (F + D(k)) e_n \end{bmatrix}$$

Now impose the condition on $k$

$$\dot{k} + c + A^T k = \frac{1}{4} f(k) \; ; \; k(T) = \phi$$

Under this assumption the expression for $\eta$ takes the form

$$\eta = \mathcal{E}k^T(0)x(0) + \mathcal{E} \int_0^T \left| \frac{1}{2} R^{-1} (F + D(k)) x + u \right|^2 dt$$
which makes it clear that the best choice for $u$ is $u = -\frac{1}{2}\sqrt{R} (F + D(k)) x$ and that if this choice is made then the minimizing value of $\eta$ is
\[
\eta^* = E(k^T(0)x_0)
\]

If it happens that there is no time dependence in the $A, B, c, F$ and if $A$ is irreducible, then the equation
\[
\dot{k} + c + A^T k - \frac{1}{4} ((F + D(k))^T k)^2 = 0
\]
can be expected to admit a solution of the form $k(t) = \alpha e^t + \beta$ with $e$ being the vector of all ones and $\beta$ a point in $\mathbb{R}^n$. For this to be the case, we need
\[
\alpha e = A^T \beta + c - \frac{1}{4} ((F + D(k))^T \beta)^2
\]
Because $e$ lies in the null space of both $A^T$ and $(F + D(k))^T$ we see that we may suppose without loss of generality that the sum of the entries in $\beta$ is zero. With this assumption in place this equation represents $n$ quadratic equations in $n$ unknowns. The steady state value of the system of equations
\[
\dot{k} = A^T k + c - \frac{1}{4} ((F + D(k))^T k)^2 - \gamma e
\]
\[
\dot{\gamma} = \langle e, k \rangle
\]
will be $(\beta, -\alpha)$ provided that $k(0)$ is suitably chosen.

**Example 1:** We consider first the general two state model with controls $(u, v)$, whose probability law
\[
\frac{d}{dt} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -f & g \\ f & -g \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} -u \\ u \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}
\]
Without loss of generality we can let $c = [m, -m]^T$ to get
\[
\eta = \int_0^T \langle m(e_1 - e_2), x \rangle + u^2 + v^2 dt
\]
In this case the equation for $k$, running backwards in time, is
\[
\frac{d}{dt} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} f & -f \\ g & -g \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} - \begin{bmatrix} m \\ -m \end{bmatrix} + \frac{1}{4} \begin{bmatrix} (1 + r^2)(k_1 - k_2)^2 \\ (1 + r^2)(k_1 - k_2)^2 \end{bmatrix}
\]
Letting $\alpha = k_1 - k_2$ this simplifies to
\[
\dot{\alpha} = (f - g)\alpha - 2m + \frac{1}{4}(1 - r^2)\alpha^2
\]
The optimal control is $u = -k_1(x_1 - x_2), v = -rk_2(x_1 - x_2^2).$
Example 2: Consider a three state Markov chain evolving in the set \( e_1, e_2, e_3 \), unit vectors in \( \mathbb{R}^n \), and whose sample path equation is

\[
\begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt} \\
\frac{dx_3}{dt}
\end{bmatrix} = \begin{bmatrix}
-1 & 1 & 0 \\
1 & -1 & 1 \\
0 & 0 & -1
\end{bmatrix} dN_1 + \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 1
\end{bmatrix} dN_2 + \begin{bmatrix}
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 1 & 0
\end{bmatrix} dN_3 \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]

with \( N_1 \) being a Poisson counter of rate .5, \( N_2 \) having rate .5 and \( N_3 \) having rate \( u \). Consider the problem of minimizing \( \eta \), given by

\[
\eta = \mathbb{E} \int_0^T x_1 + u^2 dt
\]

The differential equation for the probability law depends on a control \( u \) as described by the following equation

\[
\frac{d}{dt} \begin{bmatrix}
p_1 \\
p_2 \\
p_3
\end{bmatrix} = \begin{bmatrix}
-1 & .5 & 0 \\
.5 & -1 & .5 \\
.5 & 0 & -1
\end{bmatrix} \begin{bmatrix}
p_1 \\
p_2 \\
p_3
\end{bmatrix} + u \begin{bmatrix}
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
p_1 \\
p_2 \\
p_3
\end{bmatrix}
\]

Notice that the eigenvalue equation, relevant for \( u \) being a constant, is

\[
\det \begin{bmatrix}
s + 1 & -5 & 0 \\
-5 & s + .5 + .5u & -5 \\
-5 & -5u & s + .5
\end{bmatrix} = s^3 + (2 + u/2)s^2 - (1 + u)s/4
\]

and the eigenvector corresponding to the zero eigenvalue satisfies \( p_1 = 2p_2 \) and \( p_3 = p_1 + up_2 \), so it is \((p_1, 2p_1, (1 + 2u)p_1)\). Thus we have in steady state

\[
p_1(\infty) = \frac{1}{4 + 2u}
\]

According to the general theory developed above, we need to find the vector \( f(k) = (k^T Bx)^2 \) which in this case

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
-5x_2 \\
5x_3
\end{bmatrix} ; \quad f_j(k) = \frac{1}{4} \begin{bmatrix}
0 \\
(k_2 - k_3)^2
\end{bmatrix}
\]

Thus the differential equation for \( k \), \( \dot{k} = -A^T k - c + f(k) \), is

\[
\dot{k} = \begin{bmatrix}
1 & -5 & -5 \\
-5 & .5 & 0 \\
0 & -5 & .5
\end{bmatrix} \begin{bmatrix}
k_1 \\
k_2 \\
k_3
\end{bmatrix} - \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} + \frac{1}{4} \begin{bmatrix}
0 \\
(k_2 - k_3)^2
\end{bmatrix} ; \quad k(T) = 0
\]

The optimal control is

\[
u = \frac{1}{4}(k_2 - k_3)x_2
\]
Figure 7.1: The growth of the performance as a function of the time back from the end point, together with the effective gain, $k_2 - k_3$, shown on the right.

Of course the expected value of $\eta$ is given by

$$\eta^* = \mathcal{E}k^T(0)x(0)$$

In this case there is no equilibrium for these differential equations in that the cost keeps growing as a function of the length of the interval of interest. However, that growth approaches a linear growth and if that growth vector is denoted by $at$ then we see that the asymptotic situation is described by the algebraic equations

$$a_1 = -k_1 + .5k_2 + .5k_3 + 1$$
$$a_2 = .5k_1 - .5k_2 - (k_2 - k_3)^2/4$$
$$a_3 = .5k_2 - .5k_3$$

subject to the constraint that $a_2 - a_3 = k_2 - k_3$.

**Example 3:** Consider the steady state analysis of a three state markov chain modified from the previous example. The differential equation for the probability law depends on a control $u$ in accordance with

$$\frac{d}{dt} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + u \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

The problem is to minimize

$$\eta = \lim_{T \to \infty} \frac{1}{T} \mathcal{E} \int_0^T q(\langle e_1, x_1 \rangle + \langle e_3, x_3 \rangle + u^2 dt$$

According to the steady state theory developed above, we need to find a constant $\alpha$ and a vector $\beta$ such that

$$\alpha \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} q \\ 0 \\ q \end{bmatrix} - \frac{1}{4} \begin{bmatrix} (\beta_1 - \beta_2)^2 \\ 0 \\ (\beta_3 - \beta_2)^2 \end{bmatrix}$$
7.2. STOCHASTIC CONTROL WITH NOISY OBSERVATIONS

Clearly \( \beta_1 \) and \( \beta_3 \) enter this equation symmetrically and so we can look for solutions in which they are equal. We let \( a = \beta_1 - \beta_2 \). From the second equation we have \( \alpha = 6a \) and from the first

\[
2a = -a + q - (1/4)a^2 \quad \Rightarrow \quad a = -6(1 \pm \sqrt{1 + q/9})
\]

Because \( \alpha \) defines the steady state performance, and because this is positive when \( q \) is positive, we need to select the minus sign in the expression for \( a \)

\[
\alpha = 6a = 36 \sqrt{1 + q/9} - 36
\]

From the relations \( \beta_1 - \beta_2 = a \) and \( \beta_1 + \beta_2 + \beta_3 = 0 \) we see that \( \beta_1 = -2 + 2 \sqrt{1 + q/9} \).

The steady state feedback control law can be expressed as

\[
u(t) = (3 - 3 \sqrt{1 + q/9})(\langle e_1, x_1 \rangle + \langle e_3, x_3 \rangle)
\]

The values of \( q \) for which this solution remains valid are \( -9 \leq q < \infty \). If \( q \) is larger than -9 this problem has a minimum, reflecting the fact that even if there is a modest reward for being in the states \( x_1 \) and \( x_3 \), the cost of the \( u \) required to be there out weights the advantaged gained.

A second interpretation of this example is that of a three state process that takes on the values +1, 0, -1 with the performance measure penalizing a sum of terms measuring the deviation from the state 0 plus the cost of the control \( u^2(t) \). In this interpretation the system might be compared with the regulator problem

\[
\dot{x}(t) = -cx(t) + ud(t); \quad \eta = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T qx^2 + u^2 dt
\]

which has the optimal control

\[
u(t) = cx - \sqrt{c^2 + q^2}x
\]

7.2 Stochastic Control with Noisy Observations

If the state \( x \) is not directly available and, instead, must be estimated from an observation, matters or significantly more complicated. We will discuss only the simplest situations.

By far the most tractable class of stochastic control problems involve linear systems, Wiener processes and the minimization of the expectation of a quadratic form. Consider

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)\hat{w}(t)
\]

where \( \hat{w} \) is \( m \)-dimensional, unity variance, white noise and where \( x(t) \in \mathbb{R}^n \). We assume that we are given

\[
\mathcal{E}x(t_0) = \bar{x}(t_0)
\]

\[
\mathcal{E}(x(t_0) - \bar{x}(t_0))(x(t_0) - \bar{x}(t_0))^T = \Sigma_{cc}(t_0)
\]

and a performance measure of the form

\[
\eta = \mathcal{E} \int_{t_0}^{t_1} x^T(t)L(t)x(t) + u^T(t)u(t)dt + \mathcal{E}x^T(t_1)Qx(t_1)
\]
Our goal is to find a control law to minimize $\eta$. Making full use of the structure given we can find the optimal linear feedback control law for this system as follows.

Let $\Pi(t, Q, t_1)$ be the solution of the Riccati equation

$$\dot{K}(t) = -A^T(t)K(t) - K(t)A(t) + K(t)B(t)B^T(t)K(t) - L(t) ; \quad K(t_1) = Q$$

If $G$ were zero and if we knew $x$ exactly, then the optimal control law would be $u = -B^T K x$.

In the present setting we can express $\eta$, through a completion of the square argument. The first step is to observe that

$$0 = \mathcal{E}x^T(t)K(t)x(t) - \mathcal{E}x^T(0)K(0)x(0) - \mathcal{E}\int_0^t \frac{d}{d\sigma}x^T(\sigma)K(\sigma)x(\sigma)d\sigma$$

Subtracting this from the right-hand side of the equation

$$\eta = \mathcal{E}\int_{t_0}^{t_1} x^T(t)L(t)x(t) + u^T(t)u(t)dt + \mathcal{E}x^T(t_1)Qx(t_1)$$

allows us to complete the square to get

$$\eta = \mathcal{E}\int_{t_0}^{t_1} \|u(t) + B^T(t)\Pi(t, Q, t_1)x(t)\|^2 + tr(Q\Sigma(t_1))$$

$$\quad - \int_{t_0}^{t_1} tr[\dot{\Pi}(t, Q, t_1)\Sigma(t)] + tr[\Pi(t, Q, t_1)\dot{\Sigma}(t)]dt$$

$$\quad + \int_{t_0}^{t_1} tr\Pi(t, Q, t_1)G(t)G^T(t)dt$$

Integrating $tr(\dot{\Pi}\Sigma + \Pi\dot{\Sigma})$ gives $\Pi\Sigma$. So we have

$$\eta = \mathcal{E}\int_{t_0}^{t_1} \|u(t) + B^T(t)\Pi(t, Q, t_1)x(t)\|^2dt + tr(\Pi(t_0, Q, t_1)\Sigma(t_0))$$

$$\quad + \int_{t_0}^{t_1} tr\Pi(t, Q, t_1)G(t)G^T(t)dt$$

Since all quantities are of fixed value except the first integral, we see that the optimal control law is the same as in the deterministic case

$$u(t) = -B^T(t)\Pi(t, Q, t_1)x(t)$$

and the minimum return is

$$\eta = tr[\Pi(t_0, Q, t_1)\Sigma(t_0)] + \int_{t_0}^{t_1} tr[\Pi(t, Q, t_1)G(t)G^T(t)]dt$$

Now consider a more difficult problem where the state is not measured exactly. We have the formula above, which is valid for all linear control laws applied to

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)\dot{w}(t)$$
If we cannot observe \( x \) itself but only \( cx + \dot{v} \), we write \( x = \hat{x} + e \) where \( \hat{x} \) is the minimum variance unbiased estimate. Then

\[
\eta = \mathbb{E} \int_{t_0}^{t_1} u(t) + B^T(t)\Pi(t, Q, t_1)\hat{x}(t)^2 dt + \mathbb{E} \int_{t_0}^{t_1} u^T(t)B^T(t)\Pi(t, Q, t_1)\hat{x}(t)dt + \mathbb{E} \int_{t_0}^{t_1} [\hat{x}(t) - x(t)]^T\Pi(t, Q, t_1)B(t)B^T(t)\Pi(t, Q, t_1)[\hat{x}(t) - x(t)] dt + \int_{t_0}^{t_1} \text{tr} \{\Pi(t, Q, t_1)G(t)G^T(t)\} dt + \text{tr} \{\Pi(t_0, Q, t_1)\Sigma(t_0)\}
\]

The control choice \( u = F\hat{x} \) gives

\[
\eta = \mathbb{E} \int_{t_0}^{t_1} \|u(t) + B^T(t)\Pi(t, Q, t_1)x(t)\|^2 dt + \mathbb{E} \int_{t_0}^{t_1} x^T(t)F^T(t)B^T(t)\Pi(t, Q, t_1)[x(t) - \hat{x}(t)] dt + \int_{t_0}^{t_1} \text{tr} \{\Pi(t, Q, t_1)B(t)B^T(t)\Pi(t, Q, t_1)\Sigma(t)\} dt + \int_{t_0}^{t_1} \text{tr} \{\Pi(t, Q, t_1)G(t)G^T(t)\} dt
\]

But on using the fact that \( \mathbb{E}[x(t) - \hat{x}(t)]\hat{x}^T(t) = 0 \) we can drop out the second integral. Thus the best choice of \( u \) is

\[
u(t) = -B^T(t)\Pi(t, Q, t_1)\hat{x}(t)\]

The figure below shows the implementation of the separation principle for

\[
\eta = \mathbb{E} \int_{t_0}^{t_1} x^T L x + u^T u dt + x^T(t_1)Qx(t_1).
\]

The defining equations are

\[
\dot{\Sigma} = \Sigma A^T + \Sigma A^T C C^T \Sigma + GG^T \quad ; \quad \Sigma(t_0) = \Sigma_0 \quad \dot{K} = -A^T K - K A + KBB^T K - L \quad ; \quad K(t_1) = Q
\]

Figure 7.2: The implementation of the separation principle.

**Example:** Consider the scalar system

\[
dx = (-x + u)dt + dw \quad ; \quad dy = x dt + dv
\]

with \( x(0) \) being distributed according to a Gaussian law with mean \( \bar{x} \) and variance \( \sigma(0) = \sigma_0 \). Suppose we wish to minimize

\[
\eta = \mathbb{E} \int_{0}^{t_1} (3x)^2(\tau) + \mu^2(\tau) d\tau + \mathbb{E} x^2(t_1)
\]
According to the separation theorem we need to compute both the optimal filter to recover the best estimate of $x$ and the optimal control assuming we know $x$. The latter involves solving

$$\dot{k} = 2k + k^2 - 9 ; \quad k(1) = 1$$

We denote the solution of this equation by $\pi$.

We also need to solve for the optimal estimate of $x_0 - x_f$, where $x_f = \int_0^t e^{-t+\tau} u(\tau) d\tau$. If we denote this by $\hat{x}$ then for

$$\dot{\sigma} = -2\sigma + 1 - \sigma^2 \quad ; \quad \sigma(0) = \sigma_0$$

we would have

$$d\hat{x} = -2\hat{x} + \sigma(t)(dy - \hat{x}dt)$$

if $u$ were 0.

Putting this together we have

$$dx = -xdt - \pi(t)\hat{x}(t)dt + dw$$

$$d\hat{x} = (-2 - \sigma(t))\dot{\hat{x}}dt + \sigma(t)dy$$

$$u(t, x) = -\pi(t)\dot{\hat{x}}$$

$$\pi(t) = 2\pi(t) + \pi^2(t) - 9 \quad ; \quad \pi(1) = 1$$

$$\dot{\pi} = 2\pi + \pi^2 - 9 \quad \pi(1) = 1 \quad \text{Final cond.}$$

$$\dot{\sigma} = -2\sigma + 1 - \sigma^2 \quad ; \quad \sigma(0) = \sigma_0 \quad \text{Initial Cond.}$$

It may be enlightening to see this as a system

$$d\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} -1 & -\pi \\ \sigma c & -2 - \sigma \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} dt + \begin{bmatrix} dw \\ k(t)dv \end{bmatrix}$$

$$\dot{\pi} = 2\pi + \pi^2 - 9 \quad \pi(1) = 1 \quad \text{Final cond.}$$

$$\dot{\sigma} = -2\sigma + 1 - \sigma^2 \quad ; \quad \sigma(0) = \sigma_0 \quad \text{Initial Cond.}$$

If we work with $\hat{x}$ and $e = x - \hat{x}$ then

$$d\begin{bmatrix} e \\ \hat{x} \end{bmatrix} = \begin{bmatrix} 2 - k & 0 \\ k & 2 - k \end{bmatrix} \begin{bmatrix} e \\ \hat{x} \end{bmatrix} dt + \begin{bmatrix} dw \\ kdv \end{bmatrix}$$
If we are given a system of the form
\[ dx = f(x, u)dt + \sum g_i(x, u)d\omega_i \]
with no observations of \( x \) available, what choice of \( u \), possibly dependent on \( t \), will minimize a performance measure of the expected value form? That is to say, how can we choose \( u \) so as to minimize
\[ \mathcal{E}\phi(x(t)) = \int \rho(x, t)\phi(x)dx \]
Of course it may turn out that the best choice for \( u \) is some constant value. For example, if we have
\[ dx = -xdt + udt + d\omega \]
Then
\[ \frac{d}{dt}\mathcal{E}x = -\mathcal{E}x + u \]
and
\[ \frac{d}{dt}\mathcal{E}(x - \mathcal{E}x)^2 = -2\mathcal{E}(x - \mathcal{E}x)^2 \]
In this case \( u \) has no effect on the variance of \( x \). Unless one knows the sign of \( \mathcal{E}x(0) \) it is impossible to choose \( u \) so as to reduce \( \mathcal{E}x(t) \).

This situation is not universal. There are classes of models for which open loop control produces interesting effects. One such class is modeled on the study of the thermodynamics of heat engines. If we combine the equation of a Nyquist-Johnson resistor with that of a linear capacitor having the capacitance \( c \), the result is an Itô equation describing the voltage across the capacitance,
\[ dcv = -gvdt + \sqrt{2gT}d\omega \]
The steady state value of \( \mathcal{E}(v^2) \) is easily seen to be
\[ \mathcal{E}v^2 = T/c \]
and so, in steady state, the expected value of the energy stored in the capacitor, \( \mathcal{E}v^2/2 \), is just \( T/2 \). Notice that it does not depend on the values of \( g \) and \( c \). This is, a very special form of the equipartition of energy theorem discussed in chapter 5.

We now investigate a stochastic control analog of the problem of extracting mechanical work from a heat bath. We do so by setting up a thermodynamic cycle based on a variable capacitance interacting with resistors at different temperatures.

We are interested in analyzing the possibility of extracting energy from the system using control laws which depend on average values only and not properties of sample paths. One way to get mechanical energy (i.e., work) out of such a system is by changing the capacitance when a charge is present. Because the energy stored in a capacitor is \( \mathcal{E}v^2/2 = q^2/2c \) where \( v \) is the voltage on the capacitor and \( q \) is the charge, we see that when we change the capacitance we change the energy stored in the capacitor. The nature of the change depends on the
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Figure 7.3: Extraction of work from an electrical circuit with noisy resistors and a variable capacitor.

electrical connection in effect while the change is being made. We have the alternative expressions

\[
\dot{E} = \frac{d}{dt} \frac{cV^2}{2} = \frac{d}{dt} \frac{q^2}{2c}
\]

Thus we see that the effect is very different in the case of changing \( c \) at constant charge versus changing \( c \) at constant voltage. The force required to change the value of the capacitance, is expressible as the negative of the derivative of the potential energy, \((f = -\partial V/\partial x)\)

\[
f = -\frac{d}{dx} \frac{q^2}{2c} = \frac{c_x q^2}{c^2} \quad ; \quad c_x = \frac{dc}{dx}
\]

The work done on the capacitor when it changes from an initial value of \( c_i \) to a final value of \( c_f \) at constant charge can be computed by integrating \( f dx \) but it more direct to observe that it is just the difference in the potential energy at the end points

\[
W = \frac{q^2}{2c_f} - \frac{q^2}{2c_i}
\]

From the stochastic equation for the circuit we get a variance equation

\[
\dot{\sigma} = -2(\dot{c}/c)\sigma - 2g(T_i/c^2) \sigma + 2g_i T_i/c \]

with \( T_i \) being the temperature of the resistor connected to the capacitor. This can also be written as

\[
\frac{d}{dt} \sigma = -\frac{\dot{c}}{c} \sigma - 2\frac{g}{c} \sigma + 2g_i T_i/c
\]

This equation clearly expresses the flow of energy. Along a path on which \( \mathcal{E} v^2 \) is constant, the integral of the left-hand side is zero, the integral of the first term on the right is the work while the last two terms describe the flow of the heat supplied to the resistor. In view of this, we may say that the expected value of the work done when a capacitor changes slowly at an equilibrium condition associated with a resistor at temperature \( T \) is

\[
\mathcal{E} W = \int_a^b \frac{c_x T dx}{2c} = \frac{T}{2} \ln \frac{c(b)}{c(a)}
\]

We now discuss the Carnot cycle using figure 7.3 as a guide. In passing from 1 to 2 the capacitor is connected to the resistor at temperature \( T_1 = T_i \). Along this path work is done by the capacitor in the amount

\[
\mathcal{E} W_{12} = \frac{T_i}{2} \ln \frac{c_2}{c_1} \quad \text{(negative number)}
\]
7.3. STOCHASTIC CONTROL WITH NO OBSERVATIONS

Figure 7.4: Constant charge paths and constant energy paths.

The next step is to follow the path from point 2 to point 3 which is a constant charge path. The capacitor is not connected to anything and work is done on the capacitor as required to change \( c \) at constant charge.

\[
\mathcal{E}W_{23} = \frac{q^2}{2c_3} - \frac{q^2}{2c_2} = T_3 - T_2
\]

where we have used \( T = \mathcal{E}q^2/2c \). In passing from point 3 to point 4, the capacitor is attached to the resistor with temperature \( T_h \) along this path the capacitor does work in the amount

\[
\mathcal{E}W_{34} = \frac{T_4}{2} \ln \frac{c_4}{c_3} \quad \text{(positive number)}
\]

Finally, the capacitor is disconnected again and follows the constant charge path from point 4 to point 1. Along this path the capacitor does work on the environment in the amount

\[
\mathcal{E}W_{41} = \frac{q^2}{2c_1} - \frac{q^2}{2c_4} = T_1 - T_4
\]

Notice that because \( T_1 = T_2 \) and \( T_3 = T_4 \), the magnitude of the work done along the path from 2 to 3 is equal to the magnitude of the work done along the path from 4 to 1. They are of opposite sign and therefore cancel in the overall accounting of the work. To evaluate the performance of this system we examine the heat supplied at temperature \( T_h = T_4 = T_3 \) and compare it to the heat rejected at temperature \( T_l = T_1 = T_2 \). Defining the thermodynamic efficiency as \( \eta = W/Q \) we see that because the magnitudes of the heat supplied along the paths 1-2 or 3-4 equals the magnitudes of the work done, we have

\[
\eta = \frac{W}{Q_h} = \frac{W_{34} + W_{12}}{W_{34}} = \frac{T_2}{2} \ln \frac{c_3}{c_2} - \frac{T_1}{2} \ln \frac{c_2}{c_1}
\]

we proceed to evaluate \( \eta \). The key step in evaluating \( \eta \) is to establish the fact that along the closed path we have defined

\[
\frac{c_1}{c_2} = \frac{c_4}{c_3}
\]

To do this we combine the four equalities

\[
c_1\mathcal{E}v_1^2 = c_2\mathcal{E}v_2^2; \quad \mathcal{E}c_2v_2^2 = \mathcal{E}c_3v_3^2; \quad \mathcal{E}c_3v_3^2 = \mathcal{E}c_4v_4^2; \quad \mathcal{E}c_4v_4^2 = \mathcal{E}c_1v_1^2
\]

These imply that

\[
1 = \frac{c_1}{c_2} \frac{c_2}{c_3} \frac{c_3}{c_4} \frac{c_4}{c_1} = \frac{\mathcal{E}v_2^2}{\mathcal{E}v_1^2} \sqrt{\frac{\mathcal{E}v_3^2}{\mathcal{E}v_2^2} \frac{\mathcal{E}v_4^2}{\mathcal{E}v_3^2}} = \sqrt{\frac{\mathcal{E}v_2^2\mathcal{E}v_3^2}{\mathcal{E}v_4^2\mathcal{E}v_1^2}}
\]
Again, using the four equations, we see that

\[
1 = \sqrt{\frac{\mathcal{E}E_v^2}{\mathcal{E}E_v^2}} = \sqrt{\frac{c_2c_4}{c_1c_3}}
\]

which establishes the desired fact. Now, returning to the efficiency, we see that

\[
\eta = \frac{T_h}{2} \ln \frac{c_4}{c_3} - \frac{T_l}{2} \ln \frac{c_1}{c_2} = \frac{T_h - T_l}{T_h}
\]

According to the Kelvin-Planck statement of the second law, it is impossible to remove work from a single heat bath using a thermodynamic cycle. In our situation we see that we get no work out if \( T_h = T_l \).

### 7.4 Exercises Chapter 7

1. Consider the Fokker-Planck equation whose right-hand side depends on a function \( u \) as follows

\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} (ax + u(t,x)\rho) + \frac{1}{2} \frac{\partial^2 \rho}{\partial x^2}
\]

Assume that \( \rho(0,x) = (1/\sqrt{2\pi a}) e^{-x^2/2a} \). Find \( u \) as a function of \( x \) and \( t \) such that the quantity

\[
\eta = \int_0^T \int_{-\infty}^{\infty} (x^2 + u^2) \rho(t,x) dx dt
\]

is as small as possible.

2. In deterministic optimal control there is no difference between the value of the optimal performance for the optimal open-loop control and the the value of the optimal performance for the closed-loop control. Revisit the previous problem and find the optimal \( u \), now under the assumption that \( u \) can depend on \( t \) and possibly \( \rho(0,x) \), but not \( x(t) \) for \( t > 0 \).

3. Consider the stochastic differential equation

\[
dx = -x dt + dw + ut; x(t) = 0
\]

Suppose we set \( u = -kx \). What value of \( k \) minimizes

\[
\lim_{t \to \infty} E(x^2(t) + u^2(t))
\]

4. Suppose that \( x \) satisfies the equation

\[
dx = -x dt + dw + u dt
\]

and suppose that one observes

\[
dy = x dt + d\nu
\]

with \( w \) and \( \nu \) being standard, independent, Weiner processes. If \( x(0) \), the initial data for \( x \), is distributed according to the probability density

\[
\rho(x(0)) = \frac{1}{\sqrt{2\pi}} e^{-x^2(0)/2}
\]
7.4. EXERCISES CHAPTER 7

1. Evaluate
   \[ \lim_{t \to \infty} E x^2(t) \]
   for \( u dt = k dy \). (Your answer will, of course, depend on \( k \).)

2. Consider the second order equation
   \[
   \begin{bmatrix}
   dx_1 \\
   dx_2
   \end{bmatrix} = \begin{bmatrix}
   0 & 1 \\
   -1 & -2
   \end{bmatrix} \begin{bmatrix}
   x_1 \\
   x_2
   \end{bmatrix} dt + \begin{bmatrix}
   0 \\
   dw
   \end{bmatrix} + \begin{bmatrix}
   0 \\
   u
   \end{bmatrix} dt
   \]
   Assume that the initial probability density for \( x \) is Gaussian with known mean and variance. Assume perfect observations. Find \( u(t) = a(t)x_1 + b(t)x_2 \) such that
   \[ \eta = E \int_0^1 u^2(t) + x_1^2(t) dt \]
   is as small as possible.

3. Consider the stochastic differential equation
   \[ dx = -x dt + dw + u dt; \quad x(t) = 0 \]
   Suppose we set \( u = -kx \). What value of \( k \) minimizes
   \[ \lim_{t \to \infty} E(x^2(t) + u^2(t)) \]

4. Consider the second order equation
   \[
   \begin{bmatrix}
   dx_1 \\
   dx_2
   \end{bmatrix} = \begin{bmatrix}
   0 & 1 \\
   -1 & -2
   \end{bmatrix} \begin{bmatrix}
   x_1 \\
   x_2
   \end{bmatrix} dt + \begin{bmatrix}
   0 \\
   dw
   \end{bmatrix} + \begin{bmatrix}
   0 \\
   u
   \end{bmatrix} dt
   \]
   Find \( u(t) = a(t)x_1 + b(t)x_2 \) such that
   \[ \eta = E \int_0^1 u^2(t) + x_1^2(t) dt \]
   is as small as possible.

5. Let \( x \) be an \( n \)-vector and let \( w \) be an \( n \)-dimensional Wiener process. Consider the Itô equations
   \[
   \begin{align*}
   dx &= dw; \quad x(0) = 0 \\
   dQ &= xx^T dt; \quad Q(0) = 0
   \end{align*}
   \]
   Compute \( E Q(t) \). Consider the deterministic control problem
   \[
   \begin{align*}
   \dot{x} &= u; \quad x(0) = 0 \\
   \dot{Q} &= xx^T; \quad Q(0) = 0
   \end{align*}
   \]
   Suppose that one wants to find the optimal control for achieving \( Q(T) = M, \ x(T) = 0 \) while minimizing
   \[ \eta = \int_0^T u^T u \ dt \]
   Of course the optimal value of \( \eta \) depends on \( M \) and so we write \( \eta(M) \). Show that \( \eta(T) \) depends on \( M \) only through the eigenvalues of \( M \). How does \( \eta \) depend on \( T \)? Write the Hamilton-Jacobi equation for this problem.

6. Write the Fokker-Planck equation for the stochastic differential equations appearing in the previous problem.
10. Consider the solution $\rho$ of the Fokker-Planck equation for $x(0) = 0, Q(0) = 0$. If $t > 0$, do you expect that $\rho(t, x, Q)$ will be positive for all $x$ and $Q = Q^T$? Will it be positive for all $x, Q$ such that $Q = Q^T > 0$?

11. Find the best open-loop $u$ and $v$ for the problem

$$dx = -xdt + ud t + dw_1$$
$$dy = -ydt + vdt + dw_2$$
$$dz = (xv - yu)zdt + dw_3$$

with

$$\eta = \mathcal{E}(x^2 + y^2 + |z| + u^2 + v^2)$$

12. Consult section 6.5 of the notes and use the theory given there to solve this problem. Suppose that $x(0) \in \{e_1, e_2, e_3\}$ where $e_i$ are the standard unit basis vectors in $\mathbb{R}^3$. Consider the three state Markov process defined by

$$d\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} dN_1 + \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} dN_2$$

Suppose one can observe $dy = c^T xdt + d\nu$ with $c^T = [1, 1, 1]$ and that this observation is available on the interval $[0, t_f]$. If before any observations are made the three states are equally probable what is the conditional probability of $x(0)$ conditioned on the observations.

13. Consult section 6.8 of the notes if necessary. Consider the scalar linear system

$$dx = axdt + dw; \ dy = xdt + d\nu$$

Suppose that before any observations are made $x(0)$ has a normal distribution with mean 1 and variance 3. Given observations $y$ on $[0, t_f]$ the variance associated with the conditional probability of $x(0)$ should be reduced. Evaluate it as a function of $t_f > 0$. Under what circumstances does it go to zero as $t_f$ goes to infinity. What differences are there between the cases $a > 0$ and $a < 0$?

14. Let $n$ be a $m$-dimensional random variable which is Gaussian with zero mean and variance $I$. Consider the discrete time model

$$x(k+1) = Ax(k) + Bn(k); \ y(k) = cx(k) + dn(k)$$

with $y$ being a scalar. Find the the conditional density $\rho(x(k+1)|y(k))$ given that $x(k)$ is Gaussian with variance $\Sigma$.

15. A store manager wants to stock an item for possible sale over the coming six weeks. The supplier in Asia has a two tier price system. For fifty units and below, the price is $an + b$, with $n$ being the number of units ordered. For more than fifty units the price structure is $cn + d$. The item can be reordered later at any time but the supplier will only promise to deliver within three weeks of when the order is placed. Past experience indicates that in every season some items are “hot” and some will languish but that this bimodal behavior can be predicted, with some error, on the basis of early sales. The manager thinks that the store is likely to sell 3 units a day if the item proves to be hot and .5 units a day if the item is not hot. The a priori probability of this unit being hot this year is judged to be .3. The manager’s performance is based on total sales and this is what is to be maximized. Discuss the role of estimation theory and optimal decision making in the context of this problem by making mathematical models for each aspect of the problem. There is no “right answer” here. What is wanted is a mathematical model and some prose that argues for its relevance.
Bibliography


