COURSE ON LMI
PART II.1

LMIs IN SYSTEMS CONTROL
STATE-SPACE METHODS
STABILITY ANALYSIS

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State-space methods

Developed by Kalman and colleagues in the 1960s as an alternative to frequency-domain techniques (Bode, Nichols..)

Starting in the 1980s, numerical analysts developed powerful linear algebra routines for matrix equations: numerical stability, low computational complexity, large-scale problems

Matlab launched by Cleve Moler (1977-1984) heavily relies on LINPACK, EISPACK & LAPACK packages

Pseudospectrum of a Toeplitz matrix
Linear systems stability

The continuous-time linear time invariant (LTI) system

$$\dot{x}(t) = Ax(t) \quad x(0) = x_0$$

where $x(t) \in \mathbb{R}^n$ is asymptotically stable, meaning

$$\lim_{t \to \infty} x(t) = 0 \quad \forall x_0$$

if and only if

- there exists a quadratic Lyapunov function $V(x) = x^*Px$ such that

$$V(x(t)) > 0$$
$$\dot{V}(x(t)) < 0$$

along system trajectories
- equivalently, matrix $A$ satisfies

$$\max_i \text{real } \lambda_i(A) < 0$$
Lyapunov stability

Note that \( V(x) = x^*Px = x^*(P + P^*)x/2 \)
so that Lyapunov matrix \( P \) can be chosen symmetric without loss of generality.

Since \( \dot{V}(x) = \dot{x}^*Px + x^*P\dot{x} = x^*A^*Px + x^*PAx \) positivity of \( V(x) \) and negativity of \( \dot{V}(x) \) along system trajectories can be expressed as an LMI:

\[
A^*P + PA \prec 0 \quad P \succ 0
\]

Matrices \( P \) satisfying Lyapunov’s LMI with \( A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \)
Lyapunov equation

The Lyapunov LMI can be written equivalently as the Lyapunov equation

\[ A^*P + PA + Q = 0 \]

where \( Q \succ 0 \)

The following statements are equivalent

- the system \( \dot{x} = Ax \) is asymptotically stable
- for some matrix \( Q \succ 0 \) the matrix \( P \) solving the Lyapunov equation satisfies \( P \succ 0 \)
- for all matrices \( Q \succ 0 \) the matrix \( P \) solving the Lyapunov equation satisfies \( P \succ 0 \)

The Lyapunov LMI can be solved numerically without IP methods since solving the above equation amounts to solving a linear system of \( n(n+1)/2 \) equations in \( n(n+1)/2 \) unknowns
Alternative to Lyapunov LMI

Recall the theorem of alternatives for LMI

\[ F(x) = F_0 + \sum_i x_i F_i \]

Exactly one statement is true
- there exists \( x \) s.t. \( F(x) \succ 0 \)
- there exists a nonzero \( Z \succeq 0 \) s.t.
  \[ \text{trace } F_0 Z \leq 0 \text{ and trace } F_i Z = 0 \text{ for } i > 0 \]

Alternative to Lyapunov LMI

\[ F(x) = \begin{bmatrix} -A^* P - PA & 0 \\ 0 & P \end{bmatrix} \succ 0 \]

is the existence of a nonzero matrix

\[ Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \succeq 0 \]

such that

\[ Z_1 A^* + AZ_1 - Z_2 = 0 \]
Alternative to Lyapunov LMI (proof)

Suppose that there exists such a matrix $Z \neq 0$ and extract Cholesky factor

$$Z_1 = UU^*$$

Since $Z_1 A^* + AZ_1 \succeq 0$ we must have

$$A U U^* = U S U^*$$

where $S = S_1 + S_2$ and $S_1 = -S_1^*$, $S_2 \succeq 0$

It follows from

$$A U = U S$$

that $U$ spans an invariant subspace of $A$ associated with eigenvalues of $S$, which all satisfy real $\lambda_i(S) \geq 0$

Conversely, suppose $\lambda_i(A) = \sigma + j \omega$ with $\sigma \geq 0$ for some $i$ with eigenvector $v$

Then rank-one matrices

$$Z_1 = vv^* \quad Z_2 = 2\sigma vv^*$$

solve the alternative LMI
Discrete-time Lyapunov LMI

Similarly, the discrete-time LTI system

\[ x_{k+1} = Ax_k \]

is asymptotically stable iff

- there exists a **quadratic Lyapunov function**

\[ V(x) = x^*Px \]

such that

\[
\begin{align*}
V(x_k) &> 0 \\
V(x_{k+1}) - V(x_k) &< 0
\end{align*}
\]

along system trajectories

- equivalently, matrix \( A \) satisfies

\[
\max_i |\lambda_i(A)| < 1
\]

Here too this can be expressed as an **LMI**

\[ A^*PA - P < 0 \quad P > 0 \]
More general stability regions

Let

$$D = \{ s \in \mathbb{C} : \begin{bmatrix} 1 \\ s \end{bmatrix}^* \begin{bmatrix} d_0 & d_1 \\ d_1^* & d_2 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0 \}$$

with $d_0, d_1, d_2 \in \mathbb{C}^3$ be a region of the complex plane (half-plane or disk)

Matrix $A$ is said $D$-stable when its spectrum $\sigma(A) = \{ \lambda_i(A) \}$ belongs to region $D$

Equivalent to generalized Lyapunov LMI

$$\begin{bmatrix} I \\ A \end{bmatrix}^* \begin{bmatrix} d_0 P & d_1 P \\ d_1^* P & d_2 P \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} \prec 0 \quad P > 0$$
LMI stability regions

We can consider $\mathcal{D}$-stability in LMI regions

$$\mathcal{D} = \{ s \in \mathbb{C} : D(s) = D_0 + D_1s + D_1^*s^* < 0 \}$$

such as

<table>
<thead>
<tr>
<th>$\mathcal{D}$ dynamics</th>
<th>$\text{real}(s) &lt; -\alpha$ dominant behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{real}(s) &lt; -\alpha$, $</td>
<td>s</td>
</tr>
<tr>
<td>$\alpha_1 &lt; \text{real}(s) &lt; \alpha_2$ bandwidth</td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>\text{imag}(s)</td>
</tr>
<tr>
<td>$\text{real}(s) \tan \theta &lt; -</td>
<td>\text{imag}(s)</td>
</tr>
</tbody>
</table>

or intersections thereof

Example for the cone

$$D_0 = 0 \quad D_1 = \begin{bmatrix} \sin \theta & \cos \theta \\ \cos \theta & \sin \theta \end{bmatrix}$$
Lyapunov LMI for LMI stability regions

Matrix $A$ has all its eigenvalues in the region

\[ \mathcal{D} = \{ s \in \mathbb{C} : D_0 + D_1 s + D_1^* s^* < 0 \} \]

if and only if the following LMI is feasible

\[
D_0 \otimes P + D_1 \otimes A P + D_1^* \otimes P A^* \prec 0 \quad P > 0
\]

where $\otimes$ denotes the Kronecker product

Litterally replace $s$ with $A$!

Can be extended readily to quadratic matrix inequality stability regions

\[ \mathcal{D} = \{ s \in \mathbb{C} : D_0 + D_1 s + D_1^* s^* + D_2 s^* s < 0 \} \]

parabolae, hyperbolae, ellipses etc convex ($D_2 \succeq 0$) or not
Uncertain systems and robustness

When modeling systems we face several sources of uncertainty, including

- **non-parametric (unstructured) uncertainty**
  - unmodeled dynamics
  - truncated high frequency modes
  - non-linearities
  - effects of linearization, time-variation..

- **parametric (structured) uncertainty**
  - physical parameters vary within given bounds
  - interval uncertainty ($l_{\infty}$)
  - ellipsoidal uncertainty ($l_2$)
  - diamond uncertainty ($l_1$)

How can we overcome uncertainty?

- model predictive control
- adaptive control
- **robust control**

A control law is robust if it is valid over the whole range of admissible uncertainty (can be designed off-line, usually cheap)
Uncertainty modeling

Consider the continuous-time LTI system

\[ \dot{x}(t) = Ax(t) \quad A \in A \]

where matrix \( A \) belongs to an uncertainty set \( A \)

For unstructured uncertainties we consider norm-bounded matrices

\[ A = \{ A + B\Delta C : \|\Delta\|_2 \leq \mu \} \]

For structured uncertainties we consider polytopic matrices

\[ A = \text{conv} \{ A_1, \ldots, A_N \} \]

There are other more sophisticated uncertainty models not covered here

Uncertainty modeling is an important and difficult step in control system design!
Robust stability

The continuous-time LTI system
\[ \dot{x}(t) = Ax(t) \quad A \in \mathcal{A} \]

is robustly stable when it is asymptotically stable for all \( A \in \mathcal{A} \).

If \( \mathcal{S} \) denotes the set of stable matrices, then robust stability is ensured as soon as \( \mathcal{A} \subset \mathcal{S} \). Unfortunately, \( \mathcal{S} \) is a non-convex cone!
Symmetry

If dynamic systems were symmetric, i.e

\[ A = A^* \]

continuous-time stability \( \max_i \text{real } \lambda_i(A) < 0 \) would be equivalent to

\[ A + A^* < 0 \]

and discrete-time stability \( \max_i |\lambda_i(A)| < 1 \) to

\[ A^*A < I \iff \begin{bmatrix} -I & A \\ A^* & -I \end{bmatrix} < 0 \]

which are both LMIs !

We can show that stability of a symmetric linear system can be always proven with the Lyapunov matrix \( P = I \)

Fortunately, the world is not symmetric !
Robust and quadratic stability

Because of non-convexity of the cone of stable matrices, robust stability is sometimes \textit{difficult} to check numerically, meaning that computational cost is an exponential function of the number of system parameters.

Remedy:
The continuous-time LTI system $\dot{x}(t) = Ax(t)$ is \textit{quadratically stable} if its robust stability can be guaranteed with the \textit{same} quadratic Lyapunov function for all $A \in \mathcal{A}$.

Obviously, quadratic stability is more pessimistic, or \textit{more conservative} than robust stability:

Quadratic stability $\implies$ Robust stability

but the converse is not always true.
Quadratic stability for polytopic uncertainty

The system with polytopic uncertainty

\[ \dot{x}(t) = Ax(t) \quad A \in \text{conv} \{A_1, \ldots, A_N\} \]

is quadratically stable iff there exists a matrix \( P \) solving the LMI

\[ A_i^T P + PA_i \prec 0 \quad P \succ 0 \]

Proof by convexity

\[ \sum_i \lambda_i (A_i^T P + PA_i) = A^T(\lambda) P + PA(\lambda) \prec 0 \]

for all \( \lambda_i \geq 0 \) such that \( \sum_i \lambda_i = 1 \)

This is a vertex result: stability of a whole family of matrices is ensured by stability of the vertices of the family

Usually vertex results ensure computational tractability
Quadratic and robust stability: example

Consider the uncertain system matrix

\[
A(\delta) = \begin{bmatrix} -4 & 4 \\ -5 & 0 \end{bmatrix} + \delta \begin{bmatrix} -2 & 2 \\ -1 & 4 \end{bmatrix}
\]

with real parameter \( \delta \) such that \( |\delta| \leq \mu \) = polytope with vertices \( A(-\mu) \) and \( A(\mu) \)

<table>
<thead>
<tr>
<th>stability</th>
<th>max ( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>quadratic</td>
<td>0.7526</td>
</tr>
<tr>
<td>robust</td>
<td>1.6666</td>
</tr>
</tbody>
</table>

![Graph](image-url)
Quadratic stability for norm-bounded uncertainty

The system with norm-bounded uncertainty

\[ \dot{x}(t) = (A + B\Delta C)x(t) \quad \|\Delta\|_2 \leq \mu \]

is quadratically stable iff there exists a matrix \( P \) solving the LMI

\[
\begin{bmatrix}
A^*P + PA + C^*C & PB \\
B^*P & -\gamma^2 I
\end{bmatrix} \prec 0 \quad P \succ 0
\]

with \( \gamma^{-1} = \mu \)

This is called the bounded-real lemma proved next with the S-procedure

We can maximize the level of allowed uncertainty by minimizing scalar \( \gamma \)
Norm-bounded uncertainty as feedback

Uncertain system

\[
\dot{x} = (A + B\Delta C)x
\]

can be written as the feedback system

\[
\begin{align*}
\dot{x} &= Ax + Bw \\
z &= Cx \\
w &= \Delta z
\end{align*}
\]

so that for the Lyapunov function \( V(x) = x^*Px \) we have

\[
\dot{V}(x) = 2x^*P\dot{x} = 2x^*P(Ax + Bw) = x^*(A^*P + PA)x + 2x^*PBw
\]

\[
\begin{bmatrix} x \\ w \end{bmatrix}^* \begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}
\]
Norm-bounded uncertainty as feedback (2)

Since $\Delta^* \Delta \preceq \mu^2 I$ it follows that

$$w^*w = z^* \Delta^* \Delta z \preceq \mu^2 z^*z$$

$$w^*w - \mu^2 z^*z = \begin{bmatrix} x \\ w \end{bmatrix}^* \begin{bmatrix} -C^*C & 0 \\ 0 & \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \leq 0$$

Combining with the quadratic inequality

$$\dot{V}(x) = \begin{bmatrix} x \\ w \end{bmatrix}^* \begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0$$

and using the S-procedure we obtain

$$\begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} \preceq \begin{bmatrix} -C^*C & 0 \\ 0 & \gamma^2 I \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} A^*P + PA + C^*C & PB \\ B^*P & -\gamma^2 I \end{bmatrix} \preceq 0 \quad P \succ 0$$
Norm-bounded uncertainty: generalization

Now consider the feedback system

\[
\begin{align*}
\dot{x} &= Ax + Bw \\
z &= Cx + Dw \\
w &= \Delta z
\end{align*}
\]

with additional feedthrough term \( Dw \)

We assume that matrix \( I - \Delta D \) is non-singular = well-posedness of feedback interconnection so that we can write

\[
\begin{align*}
w &= \Delta z = \Delta(Cx + Dw) \\
(I - \Delta D)w &= \Delta Cx \\
w &= (I - \Delta D)^{-1} \Delta Cx
\end{align*}
\]

and derive the linear fractional transformation (LFT) uncertainty description

\[
\dot{x} = Ax + Bw = (A + B(I - \Delta D)^{-1} \Delta C)x
\]
Norm-bounded LFT uncertainty

The system with norm-bounded LFT uncertainty

\[
\dot{x} = \left( A + B(I - \Delta D)^{-1} \right) x \quad \|\Delta\|_2 \leq \mu
\]

is quadratically stable iff there exists a matrix \( P \) solving the LMI

\[
\begin{bmatrix}
A^*P + PA + C^*C & PB + C^*D \\
B^*P + D^*C & D^*D - \gamma^2 I
\end{bmatrix} \prec 0 \quad P \succ 0
\]

Notice the lower right block \( D^*D - \gamma^2 I \prec 0 \) which ensures non-singularity of \( I - \Delta D \) hence well-posedness

LFT modeling can be used more generally to cope with rational functions of uncertain parameters, but this is not covered in this course.
Sector-bounded uncertainty

Consider the feedback system

\[
\begin{align*}
\dot{x} &= Ax + Bw \\
z &= Cx + Dw \\
w &= f(z)
\end{align*}
\]

where vector function \( f(z) \) satisfies

\[ z^* f(z) \leq 0 \quad f(0) = 0 \]

which is a sector condition

\( f(z) \) can also be considered as an uncertainty but also as a non-linearity
Quadratic stability for sector-bounded uncertainty

We want to establish quadratic stability with the quadratic Lyapunov matrix $V(x) = x^*Px$ whose derivative

$$
\dot{V}(x) = 2x^*P(Ax + Bf(z)) = \begin{bmatrix} x \\ f(z) \end{bmatrix}^* \begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} \begin{bmatrix} x \\ f(z) \end{bmatrix}
$$

must be negative when

$$
2z^*f(z) = 2(Cx + Df(z))^*f(z) = \begin{bmatrix} x \\ f(z) \end{bmatrix}^* \begin{bmatrix} 0 & C^* \\ C & D + D^* \end{bmatrix} \begin{bmatrix} x \\ f(z) \end{bmatrix}
$$

is non-positive, so we invoke the S-procedure to derive the LMI

$$
\begin{bmatrix}
A^*P + PA & PB - C^* \\
B^*P - C & -D - D^*
\end{bmatrix} \prec 0 \quad P \succ 0
$$

This is called the positive-real lemma
Parameter-dependent Lyapunov functions

Quadratic stability:
- fast variation of parameters
- computationally tractable
- conservative, or pessimistic (worst-case)

Robust stability:
- no variation of parameters
- computationally difficult (in general)
- exact (is it really relevant?)

Is there something in between?

For example, given an LTI system affected by box, or interval uncertainty

\[
\dot{x}(t) = A(\lambda(t))x(t) = (A_0 + \sum_{i=1}^{N} \lambda_i(t)A_i)x(t)
\]

where

\[
\lambda \in \Lambda = \{\lambda_i \in [\lambda_i, \lambda_i]\}
\]

we may consider parameter-dependent Lyapunov matrices, such as

\[
P(\lambda(t)) = P_0 + \sum_i \lambda_i(t)P_i
\]
Polytopic Lyapunov certificate

Quadratic Lyapunov function $V(x) = x^* P(\lambda) x$ must be positive with negative derivative along system trajectory hence

$$P(\lambda) \succ 0 \quad \forall \lambda \in \Lambda$$

and

$$A^*(\lambda) P(\lambda) + P(\lambda) A^*(\lambda) + \dot{P}(\lambda) \prec 0 \quad \forall \lambda \in \Lambda$$

We have to solve parametrized LMIs

Parametrized LMIs feature non-linear terms in $\lambda$ so it is not enough to check vertices of $\Lambda$, denoted by vert$\Lambda$

$$\lambda_1^2 - \lambda_1 + \lambda_2 \geq 0 \text{ on } \text{vert } \Delta$$

but not everywhere on $\Delta = [0, 1] \times [0, 1]$
Parametrized LMIs

Central problem in robustness analysis: find $x$ such that

$$F(x, \lambda) = \sum_\alpha \lambda^\alpha F_\alpha(x) \succ 0, \quad \forall \lambda \in \Lambda$$

where $\Lambda$ is a compact set, typically the unit simplex or the unit ball

Convex but infinite-dimensional problem which is difficult in general

Matrix extensions of polynomial positivity conditions, for which various hierarchies of LMIs are available:

- Pólya’s theorem
- Schmüdgen’s representation
- Putinar representation

See EJC 2006 survey by Carsten Scherer
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**$H_2$ norm**

The $H_2$ norm is the energy ($l_2$ norm) of the impulse response $h(t)$ of a system $G$

$$\|G\|_2 = \left( \int_{0}^{\infty} h^*(t)h(t) dt \right)^{1/2} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)H^*(j\omega) d\omega \right)^{1/2}$$

For a continuous-time system

$$\dot{x} = Ax + Bu$$  
$$y = Cx + Du$$

with transfer function $G(s) = C(sI - A)^{-1}B + D$ we must assume $D = 0$ to have $\|G\|_2$ finite
Computing the $H_2$ norm

Let $h_i(t) = Ce^{At}B_i$ denote the $i$-th column of the impulse response of $G$, then

$$\|G\|_2^2 = \sum_i \|h_i\|_2^2$$

$$= \sum_i \int_0^\infty B_i^* e^{A^*t} C^* C e^{At} B_i dt$$

$$= \text{trace } B^* \left( \int_0^\infty e^{A^*t} C^* C e^{At} dt \right) B$$

Matrix

$$P_o = \int_0^\infty e^{A^*t} C^* C e^{At} dt$$

is the observability Grammian solution to the Lyapunov equation

$$A^* P_o + P_o A + C^* C = 0$$

and hence

$$\|G\|_2^2 = \text{trace } B^* P_o B$$

If $(A, C)$ observable then $P_o \succ 0$
Dual and LMI computation of the $H_2$ norm

Defining the controllability Grammian

$$P_c = \int_0^\infty e^{At} BB^* e^{A^*t} dt$$

solution to the Lyapunov equation

$$AP_c + P_c A^* + BB^* = 0$$

we have a dual expression

$$\|G\|_2^2 = \text{trace } C P_c C^*$$

Dual lyapunov equations formulated as LMIs

$$\|G\|_2^2 = \min \text{ trace } B^* P B$$
\[\text{s.t. } \begin{align*}
    A^* P + P A + C^* C &\preceq 0 \\
    P &\succeq 0
\end{align*} \]

$$\|G\|_2^2 = \min \text{ trace } C Q C^*$$
\[\text{s.t. } \begin{align*}
    A Q + Q A^* + B B^* &\preceq 0 \\
    Q &\succeq 0
\end{align*} \]
$H_\infty$ norm

The $H_\infty$ norm is the induced energy gain ($l_2$ to $l_2$)

$$\|G\|_\infty = \sup_{\|x\|_2=1} \|Gx\|_2 = \sup_\omega \|G(j\omega)\|$$

It is the worst case gain.
Computing the $H_\infty$ norm

In contrast with the $H_2$ norm, computation of the $H_\infty$ norm is iterative.

For the continuous-time linear system
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]
with transfer function $G(s) = C(sI-A)^{-1}B+D$ we have $\|G(s)\|_\infty < \gamma$ iff $R = \gamma^2 I - D^*D \succ 0$ and the Hamiltonian matrix
\[
\begin{bmatrix}
A + BR^{-1}D^*C & BR^{-1}B^* \\
-C^*(I + DR^{-1}D^*)C & -(A + BR^{-1}D^*C)^*
\end{bmatrix}
\]
has no eigenvalues on the imaginary axis.

We can then design a bisection algorithm with guaranteed quadratic convergence to find the minimum value of $\gamma$ such that the Hamiltonian has no imaginary eigenvalues.
LMI computation of the $H_\infty$ norm

Refer to the part of the course on norm-bounded uncertainty

$$\sup_{\|z\|_2=1} \|w\| = \|\Delta\| < \gamma^{-1}$$

to prove that for the continuous-time system

$$\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}$$

with transfer function $G(s) = C(sI-A)^{-1}B+D$ we have $\|G(s)\|_\infty < \gamma$ iff there exists a matrix $P$ solving the LMI

$$\begin{bmatrix}
A^*P + PA + C^*C & PB + C^*D \\
B^*P + D^*C & D^*D - \gamma^2 I
\end{bmatrix} \prec 0 \quad P \succ 0$$

Using the Schur complement and a change of variables this can be expanded to

$$\begin{bmatrix}
A^*P + PA & PB & C^* \\
B^*P & -\gamma I & D^* \\
C & D & -\gamma I
\end{bmatrix} \prec 0 \quad P \succ 0$$
Linear systems design

Open-loop continuous-time LTI system

\[ \dot{x} = Ax + Bu \]

with state-feedback controller

\[ u = Kx \]

produces closed-loop system

\[ \dot{x} = (A + BK)x \]

Applying Lyapunov LMI stability condition

\[ (A + BK)^* P + P(A + BK) \prec 0 \quad P \succ 0 \]

we get bilinear terms.

Bilinear Matrix Inequalities (BMIs) are non-convex in general!
State-feedback design:  
linearizing change of variables

Project BMI onto \( P^{-1} \succ 0 \)

\[
(A + BK)^* P + P(A + BK) \prec 0 \iff P^{-1} [(A + BK)^* P + P(A + BK)] P^{-1} \prec 0 \iff P^{-1} A^* + P^{-1} K^* B^* + A P^{-1} + B K P^{-1} \prec 0
\]

Denoting

\[
Q = P^{-1} \quad Y = K P^{-1}
\]

we derive a state-feedback design LMI

\[
AQ + QA^* + BY + Y^* B^* \prec 0 \quad Q \succ 0
\]

We obtained an LMI thanks to a one-to-one linearizing change of variables

Simple but very useful trick..
Finsler’s lemma

A very useful trick in robust control..

The following statements are equivalent

1. $x^* Ax > 0$ for all $x \neq 0$ s.t. $Bx = 0$
2. $\tilde{B}^* A \tilde{B} > 0$ where $\tilde{B} \tilde{B} = 0$
3. $A + \lambda \tilde{B}^* \tilde{B} > 0$ for some scalar $\lambda$
4. $A + XB + B^* X^* > 0$ for some matrix $X$

Paul Finsler
(1894 Heilbronn - 1970 Zurich)
State-feedback design: null-space projection

We can also use item 2 of Finsler’s lemma, projecting onto the (full column rank) null-space $\tilde{B}$ of $B^*$

\[ B^*\tilde{B} = 0 \]

so that BMI

\[ A^*P + PA + K^*B^*P + PBK - 0 \]

is equivalent to the projected LMI

\[ \tilde{B}^*(AQ + QA^*)\tilde{B} < 0 \quad Q > 0 \]

Feedback $K$ can be recovered from Lyapunov matrix $Q$ as

\[ K = -\lambda B^*Q^{-1} \]

where $\lambda$ is a suitably large scalar

Here we obtained an LMI thanks to a projection onto a null-space
State-feedback design: Riccati inequality

We can also use item 3 of Finsler’s lemma to convert BMI

\[ A^*P + PA + K^*B^*P + PBK \prec 0 \]

into

\[ A^*P + PA - \lambda PBB^*P \prec 0 \]

where \( \lambda \geq 0 \) is an unknown scalar

Now replacing \( P \) with \( \lambda P \) we get

\[ A^*P + PA - PBB^*P \prec 0 \]

which is equivalent to the Riccati equation

\[ A^*P + PA - PBB^*P + Q = 0 \]

for some matrix \( Q \succ 0 \)

Shows equivalence between state-feedback LMI stabilizability and the linear quadratic regulator (LQR) problem
Robust state-feedback design for polytopic uncertainty

Open-loop system \( \dot{x} = Ax + Bu \) with polytopic uncertainty

\[
(A, B) \in \text{conv} \{(A_1, B_1), \ldots, (A_N, B_N)\}
\]

and robust state-feedback controller \( u = Kx \)

In order to derive synthesis condition, we start with analysis conditions

\[
(A_i + B_iK)^*P + P(A_i + B_iK) \prec 0 \quad \forall i \quad Q \succ 0
\]

and we obtain the quadratic stabilizability LMI

\[
A_iQ + QA_i^* + B_iY + Y*B_i^* \prec 0 \quad \forall i \quad Q \succ 0
\]

with the linearizing change of variables

\[
Q = P^{-1} \quad Y = KP^{-1}
\]
State-feedback $H_2$ control

Continuous-time LTI open-loop system

\[
\begin{align*}
\dot{x} &= Ax + B_w w + B_u u \\
z &= C_z x + D_{zw} w + D_{zu} u
\end{align*}
\]

with state-feedback controller

\[u = K x\]

yields closed-loop system

\[
\begin{align*}
\dot{x} &= (A + B_u K)x + B_w w \\
z &= (C_z + D_{zu} K)x + D_{zw} w
\end{align*}
\]

with transfer function

\[G(s) = D_{zw} + (C_z + D_{zu} K)(sI - A - B_u K)^{-1}B_w\]

between performance signals $w$ and $z$

$H_2$ performance specification

\[\|G(s)\|_2 < \gamma\]

We must have $D_{zw} = 0$ (finite gain)
**$H_2$ design LMIs**

As usual, start with analysis condition: there exists $K$ such that $\|G(s)\|_2 < \gamma$ iff

\[
\text{trace } (C_z + D_{zu}K)Q(C_z + D_{zu}K)^* < \gamma \\
(A + B_uK)Q + Q(A + B_uK)^* + BB^* < 0
\]

The trace inequality can be written as

\[
\text{trace}(C_z + D_{zu}K)Q(C_z + D_{zu}K)^* < \text{trace } W < \gamma
\]

for some matrix $W$ such that

\[
\begin{bmatrix}
W & (C_z + D_{zu}K)Q \\
Q(C_z + D_{zu}K)^* & Q
\end{bmatrix} \succ 0
\]

We obtain the overall LMI formulation

\[
\begin{bmatrix}
W & C_zQ + D_{zu}Y \\
QC_z^* + Y^*D_{zu}^* & Q
\end{bmatrix} \succ 0
\]

\[
AQ + QA^* + B_uY + Y^*B_u^* + B_wB_w^* < 0
\]

with resulting $H_2$ suboptimal state-feedback

\[
K = YQ^{-1}
\]
State-feedback $H_\infty$ control

Similarly, with closed-loop system
\[
\dot{x} = (A + B_u K)x + B_w w \\
z = (C_z + D_{zu} K)x + D_{zw} w
\]
and $H_\infty$ performance specification
\[
\|G(s)\|_\infty < \gamma
\]
on transfer function between $w$ and $z$ we obtain the design LMI
\[
\begin{bmatrix}
AQ + QA^* + B_u Y + Y^* B_u^* + B_w B_w^* \\
C_z Q + D_{zu} Y + D_{zw} B_w^* \\
Q \succ 0 \\
\end{bmatrix} \prec 0
\]
with resulting $H_\infty$ suboptimal state-feedback
\[
K = Y Q^{-1}
\]
Optimal $H_\infty$ control: minimize $\gamma$
Mixed $H_2/H_\infty$ control

State-feedback controller system with two performance channels

\[
\dot{x} = (A + B_uK)x + B_w w \\
\begin{align*}
z_\infty &= (C_\infty + D_{\infty u}K)x + D_{\infty w} w \\
z_2 &= (C_2 + D_{2 u}K)x
\end{align*}
\]

and mixed performance specifications

\[
\|G_\infty(s)\|_\infty < \gamma_\infty \quad \|G_2(s)\|_2 < \gamma_2
\]
on transfer functions from $w$ to $z_\infty$ and $z_2$ respectively

Formulation of $H_\infty$ constraint

\[
\begin{bmatrix}
AQ_\infty + B_uKQ_\infty + (\ast) + B_wB^*_w \\
C_\infty Q_\infty + D_{\infty u}KQ_\infty + D_{\infty w}B^*_w \\
Q_\infty \hat{} & D_{\infty w}D_{\infty w}^* - \gamma_\infty^2 I
\end{bmatrix} \hat{} 0
\]

BMI formulation of $H_2$ constraint

\[
\begin{bmatrix}
W & C_2Q_2 + D_{2 u}KQ_2 \\
\ast & Q_2 \\
AQ_2 + B_uKQ_2 + (\ast) + B_wB^*_w \hat{} 0
\end{bmatrix} \hat{} 0
\]

Problem:

We cannot linearize simultaneously terms $KQ_\infty$ and $KQ_2$!
Mixed $H_2/H_\infty$ control design LMI

Remedy:

Enforce $Q_2 = Q_\infty = Q$!

Conservative but useful.. Always trade-off between conservatism and tractability

Resulting mixed $H_2/H_\infty$ design LMI

\[
\begin{bmatrix}
AQ + B_uY + (\ast) + B_wB_w^* \\
C_\infty Q + D_\infty uY + D_\infty wB_w^* \\
\end{bmatrix}
\begin{bmatrix}
\ast \\
D_\infty wD_w^* - \gamma_2^2 I \\
\end{bmatrix} < 0
\]

\[
\text{trace } W < \gamma_2
\]

\[
\begin{bmatrix}
W & C_2Q + D_2uY \\
\ast & Q \\
A Q + B_u Y + (\ast) + B_w B_w^* \\
\end{bmatrix} > 0
\]

Guaranteed cost mixed $H_2/H_\infty$:

given $\gamma_\infty$ minimize $\gamma_2$

Can be used for $H_2$ design of uncertain systems with norm-bounded uncertainty
**Mixed $H_2/H_\infty$ control: example**

**Active suspension system (Weiland)**

\[
m_2 \ddot{q}_2 + b_2 (\dot{q}_2 - \dot{q}_1) + k_2 (q_2 - q_1) + F = 0
\]
\[
m_1 \ddot{q}_1 + b_2 (\dot{q}_1 - \dot{q}_2) + k_2 (q_1 - q_2) + k_1 (q_1 - q_0) + b_1 (\dot{q}_1 - \dot{q}_0) + F = 0
\]

\[
z = \begin{bmatrix} q_1 - q_0 \\ \dot{q}_2 \\ \dot{q}_2 - q_1 \end{bmatrix} \quad y = \begin{bmatrix} \ddot{q}_2 \\ q_2 - q_1 \end{bmatrix} \quad w = q_0 \quad u = F
\]

\[
G_\infty(s) \text{ from } q_0 \text{ to } [q_1 - q_0 \ F]
\]
\[
G_2(s) \text{ from } q_0 \text{ to } [\dot{q}_2 \ \dot{q}_2 - q_1]
\]

**Trade-off** between $\|G_\infty\|_\infty \leq \gamma_1$ and $\|G_2\|_2 \leq \gamma_2$
Dynamic output-feedback

Continuous-time LTI open-loop system

\[
\begin{align*}
\dot{x} &= Ax + B_ww + B_uu \\
z &= C_zx + D_zww + D_zuu \\
y &= C_yx + D_yww
\end{align*}
\]

with dynamic output-feedback controller

\[
\begin{align*}
\dot{x}_c &= A_cx_c + B_cy \\
u &= C_cx_c + D_cy
\end{align*}
\]

Denote closed-loop system as

\[
\begin{align*}
\dot{x} &= \tilde{A}\tilde{x} + \tilde{B}w \\
z &= \tilde{C}\tilde{x} + \tilde{D}w
\end{align*}
\]

with \( \tilde{x} = \begin{bmatrix} x \\ x_c \end{bmatrix} \)

\[
\begin{align*}
\tilde{A} &= \begin{bmatrix}
A + B_uD_cC_y & B_uC_c \\
B_cC_y & A_c
\end{bmatrix} \\
\tilde{B} &= \begin{bmatrix}
B_w + B_uD_cD_yw \\
B_cD_yw
\end{bmatrix} \\
\tilde{C} &= \begin{bmatrix}
C_z + D_zuD_cC_y & D_zuC_c
\end{bmatrix} \\
\tilde{D} &= D_zw + D_zuD_cD_yw
\end{align*}
\]

Affine expressions on controller matrices
\(H_2\) output feedback design

\(H_2\) design conditions

\[
\begin{align*}
\text{trace } W & < \gamma \\
W & \begin{bmatrix} \tilde{C} \tilde{Q} \\ * \end{bmatrix} > 0 \\
\begin{bmatrix} \tilde{A} \tilde{Q} + \tilde{Q} \tilde{A}^* & \tilde{B} & \tilde{B}^* \\ \end{bmatrix} & < 0
\end{align*}
\]

can be linearized with a specific change of variables

Denote

\[
\tilde{Q} = \begin{bmatrix} Q & \tilde{Q}^* \\ \tilde{Q} & \times \end{bmatrix} \quad \tilde{P} = \tilde{Q}^{-1} = \begin{bmatrix} P & \tilde{P} \\ \tilde{P}^* & \times \end{bmatrix}
\]

so that \(\tilde{P}\) and \(\tilde{Q}\) can be obtained from \(P\) and \(Q\) via relation

\[
PQ + \tilde{P} \tilde{Q} = I
\]

Always possible when controller has same order than the open-loop plant.
Linearizing change of variables for \( H_2 \) output-feedback design

Then define

\[
\begin{bmatrix}
X & U \\
Y & V
\end{bmatrix} = \begin{bmatrix}
\bar{P} & PB_u \\
0 & I
\end{bmatrix} \begin{bmatrix}
A_c & B_c \\
C_c & D_c
\end{bmatrix} \begin{bmatrix}
\bar{Q} & 0 \\
C_y & I
\end{bmatrix} + \begin{bmatrix}
P \\
0
\end{bmatrix} A \begin{bmatrix}
Q & 0
\end{bmatrix}
\]

which is a one-to-one affine relation with converse

\[
\begin{bmatrix}
A_c & B_c \\
C_c & D_c
\end{bmatrix} = \begin{bmatrix}
\bar{P}^{-1} - \bar{P}^{-1} PB_u & 0 \\
0 & I
\end{bmatrix} \begin{bmatrix}
X - PAQ U \\
Y & V
\end{bmatrix} \begin{bmatrix}
\bar{Q}^{-1} & 0 \\
-C_y & \bar{Q}^{-1} I
\end{bmatrix}
\]

We derive the following \( H_2 \) design LMI

\[
\begin{align*}
\text{trace } W &< \gamma \\
D_{zw} + D_{zu} V D_{yw} &= 0 \\
\begin{bmatrix}
W & C_z Q + D_{zu} Y & C_z + D_{zu} V C_y \\
* & Q & I \\
* & * & P
\end{bmatrix} &> 0 \\
\begin{bmatrix}
AQ + B_u Y + (\ast) & A + B_u V C_y + X^* & B_w + B_u V D_{yw} \\
* & PA + U C_y + (\ast) & PB_w + U D_{yw} - I \\
* & * & *
\end{bmatrix} &< 0
\end{align*}
\]

in decision variables \( Q, P, W \) (Lyapunov) and \( X, Y, U, V \) (controller)

Controller matrices are obtained via the relation

\[
PQ + \bar{P} \bar{Q} = I
\]

(tedious but straightforward linear algebra)
$H_\infty$ output-feedback design

Similarly two-step procedure for full-order $H_\infty$ output-feedback design:
• solve LMI for Lyapunov variables $Q, P, W$ and controller variables $X, Y, U, V$
• retrieve controller matrices via linear algebra

For reduced-order controller of order $n_c < n$
there exists a solution $\bar{P}, \bar{Q}$ to the equation

$$PQ + \bar{P}\bar{Q} = I$$

iff

$$\text{rank } (PQ - I) = n_c \iff \text{rank } \begin{bmatrix} Q & I \\ I & P \end{bmatrix} = n + n_c$$

Static output feedback iff $PQ = I$

**Difficult** rank constrained LMI!
$H_{\infty}$ output-feedback design

Alternative LMI formulation via projection onto null-spaces (recall Finsler's lemma)

$$
\begin{align*}
N^* &\begin{bmatrix}
AQ + QA^* & QC_z^* & B_w \\
* & -\gamma I & D_{zw} \\
* & * & -\gamma I
\end{bmatrix} N \prec 0 \\
M^* &\begin{bmatrix}
A^*P + PA & PB_w & C_z^* \\
* & -\gamma I & D_{zw}^* \\
* & * & -\gamma I
\end{bmatrix} M \prec 0

\begin{bmatrix}
Q & I \\
I & P
\end{bmatrix} \succeq 0
\end{align*}
$$

where $N$ and $M$ are null-space basis

$$
\begin{bmatrix}
B_u^* & D^*_{zu} & 0
\end{bmatrix} N = 0 \quad \begin{bmatrix}
C_u & D_{yw} & 0
\end{bmatrix} M = 0
$$

Controller coefficients retrieved afterwards with a similar change of variables

Alleviate assumptions of Riccati approach ($D_{zu}$ and $D_{yw}$ full rank, no imaginary zeros)
COURSE ON LMI
PART II.3

LMIs IN SYSTEMS CONTROL
ROBUSTNESS ANALYSIS
POLYNOMIAL METHODS

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Polynomial methods

Based on the algebra of polynomials and polynomial matrices, typically involve
• linear Diophantine equations
• quadratic spectral factorization

Pioneered in central Europe during the 70s mainly by Vladimír Kučera from the former Czechoslovak Academy of Sciences

Network funded by the European commission

www.utia.cas.cz/europoly

Polynomial matrices also occur in Jan Willems' behavioral approach to systems theory

Alternative to state-space methods developed during the 60s most notably by Rudolf Kalman in the USA, rather based on
• linear Lyapunov equations
• quadratic Riccati equations
Ratio of polynomials

A scalar transfer function can be viewed as the ratio of two polynomials.

**Example**
Consider the mechanical system

\[ \begin{align*}
  y & \text{ displacement} \\
  u & \text{ external force} \\
  k_1 & \text{ viscous friction coeff} \\
  k_2 & \text{ spring constant} \\
  m & \text{ mass}
\end{align*} \]

Neglecting static and Coulomb frictions, we obtain the linear transfer function

\[ G(s) = \frac{y(s)}{u(s)} = \frac{1}{ms^2 + k_1 s + k_2} \]
Ratio of polynomial matrices

Similarly, a MIMO transfer function can be viewed as the ratio of polynomial matrices

\[ G(s) = N_R(s)D_R^{-1}(s) = D_L^{-1}(s)N_L(s) \]

the so-called matrix fraction description (MFD)

Lightly damped structures such as oil derricks, regional power models, earthquakes models, mechanical multi-body systems, damped gyroscopic systems are most naturally represented by second order polynomial MFDs

\[(D_0 + D_1 s + D_2 s^2)y(s) = N_0 u(s)\]

Example
The (simplified) oscillations of a wing in an air stream is captured by properties of the quadratic polynomial matrix [Lancaster 1966]

\[
D(s) = \begin{bmatrix}
121 & 18.9 & 15.9 \\
0 & 2.7 & 0.145 \\
11.9 & 3.64 & 15.5
\end{bmatrix} + \begin{bmatrix}
7.66 & 2.45 & 2.1 \\
0.23 & 1.04 & 0.223 \\
0.6 & 0.756 & 0.658
\end{bmatrix} s + \begin{bmatrix}
17.6 & 1.28 & 2.89 \\
1.28 & 0.824 & 0.413 \\
2.89 & 0.413 & 0.725
\end{bmatrix} s^2
\]
First-order polynomial MFD

Example

RCL network

\[ \begin{bmatrix} 1 & -Ls \\ Cs & 1 + RCs \end{bmatrix} \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ Cs \end{bmatrix} u(s) \]

and thus the first-order left system MFD

\[ G(s) = \begin{bmatrix} 1 \\ Cs \end{bmatrix} \begin{bmatrix} 1 & -Ls \\ Cs & 1 + RCs \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ Cs \end{bmatrix}. \]
Second-order polynomial MFD

Example
mass-spring system

Vibration of system governed by 2nd-order differential equation $M \ddot{x} + C \dot{x} + Kx = 0$ where e.g. $n = 250$, $m_i = 1$, $\kappa_i = 5$, $\tau_i = 10$ except $\kappa_1 = \kappa_n = 10$ and $\tau_1 = \tau_n = 20$

Quadratic matrix polynomial

$$D(s) = Ms^2 + Cs + K$$

with

$$M = I$$

$$C = \text{tridiag}(-10, 30, -10)$$

$$K = \text{tridiag}(-5, 15, -5).$$
Another second-order polynomial MFD

Example
Inverted pendulum on a cart

Linearization around the upper vertical position yields the left polynomial MFD

\[
\begin{bmatrix}
(M + m)s^2 + bs \\
lms^2
\end{bmatrix}
\begin{bmatrix}
lms^2 \\
(J + l^2m)s^2 + ks - lmg
\end{bmatrix}
\begin{bmatrix}
x(s) \\
\phi(s)
\end{bmatrix}
= \begin{bmatrix}
1 \\
0
\end{bmatrix} f(s)
\]

With \( J = mL^2/12 \), \( l = L/2 \) and \( g = 9.8 \), \( M = 2 \), \( m = 0.35 \), \( l = 0.7 \), \( b = 4 \), \( k = 1 \), we obtain the denominator polynomial matrix

\[
D(s) = \begin{bmatrix}
5s + 3s^2 & 0.35s^2 \\
0.35s^2 & -3.4 + s + 0.16s^2
\end{bmatrix}
\]
More examples of polynomial MFDs

Higher degree polynomial matrices can also be found in aero-acoustics (3rd degree) or in the study of the spatial stability of the Orr-Sommerfeld equation for plane Poiseuille flow in fluid mechanics (4rd degree)

For more info see Nick Higham’s homepage at www.ma.man.ac.uk/~higham
Stability analysis for polynomials

Well established theory - LMIs are of no use here!

Given a continuous-time polynomial

\[ p(s) = p_0 + p_1 s + \cdots + p_{n-1} s^{n-1} + p_n s^n \]

with \( p_n > 0 \) we define its \( n \times n \) Hurwitz matrix

\[
H(p) = \begin{bmatrix}
p_{n-1} & p_{n-3} & 0 & 0 \\
p_n & p_{n-2} & \vdots & \vdots \\
0 & p_{n-1} & 0 & 0 \\
0 & p_n & p_0 & 0 \\
& \vdots & \vdots & p_1 & 0 \\
0 & 0 & p_2 & p_0
\end{bmatrix}
\]

Hurwitz stability criterion: Polynomial \( p(s) \) is stable iff all principal minors of \( H(p) \) are > 0

Adolf Hurwitz
(Hanover 1859 - Zürich 1919)
Robust stability analysis for polynomials

Analyzing stability robustness of polynomials is a little bit more interesting..

Here too computational complexity depends on the uncertainty model

In increasing order of complexity, we will distinguish between

- single parameter uncertainty $q \in [q_{\text{min}}, q_{\text{max}}]$
- interval uncertainty $q_i \in [q_{i\text{min}}, q_{i\text{max}}]$
- polytopic uncertainty $\lambda_1 q_1 + \cdots + \lambda_N q_N$
- multilinear uncertainty $q_0 + q_1 \cdot q_2 \cdot q_3$

LMIs will not show up very soon..
..just basic linear algebra
Single parameter uncertainty and eigenvalue criterion

Consider the uncertain polynomial

$$p(s, q) = p_0(s) + qp_1(s)$$

where

- $p_0(s)$ nominally stable with positive coefs
- $p_1(s)$ such that $\text{deg } p_1(s) < \text{deg } p_0(s)$

The largest stability interval

$$q \in ]q_{\text{min}}, q_{\text{max}}[$$

such that $p(s, q)$ is robustly stable is given by

$$
\begin{align*}
q_{\text{max}} &= 1/\lambda_{\text{max}}^+(-H_0^{-1}H_1) \\
q_{\text{min}} &= 1/\lambda_{\text{min}}^-(-H_0^{-1}H_1)
\end{align*}
$$

where $\lambda_{\text{max}}^+$ is the max positive real eigenvalue
$\lambda_{\text{min}}^-$ is the min negative real eigenvalue
$H_i$ is the Hurwitz matrix of $p_i(s)$
Higher powers of a single parameter

Now consider the continuous-time polynomial

\[ p(s, q) = p_0(s) + qp_1(s) + q^2p_2(s) + \cdots + q^m p_m(s) \]

with \( p_0(s) \) stable and \( \deg p_0(s) > \deg p_i(s) \)

Using the zeros (roots of determinant) of the polynomial Hurwitz matrix

\[ H(p) = H(p_0) + qH(p_1) + q^2H(p_2) + \cdots + q^m H(p_m) \]

we can show that

\[ q_{\min} = \frac{1}{\lambda_{\min}(M)} \]
\[ q_{\max} = \frac{1}{\lambda_{\max}(M)} \]

where

\[
M = \begin{bmatrix}
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & I & 0 \\
0 & \cdots & 0 & I \\
-H_0^{-1}H_m & \cdots & -H_0^{-1}H_2 & -H_0^{-1}H_1
\end{bmatrix}
\]

is a block companion matrix
MIMO systems

Uncertain multivariable systems are modeled by uncertain polynomial matrices

\[ P(s, q) = P_0(s) + qP_1(s) + q^2P_2(s) + \cdots + q^mP_m(s) \]

where \( p_0(s) = \det P_0(s) \) is a stable polynomial

We can apply the scalar procedure to the determinant polynomial

\[ \det P(s, q) = p_0(s) + qp_1(s) + q^2p_2(s) + \cdots + q^rp_r(s) \]

Example

MIMO design on the plant with left MFD

\[
A^{-1}(s, q)B(s, q) = \begin{bmatrix}
  s^2 & q \\
  q^2 + 1 & s
\end{bmatrix}^{-1} \begin{bmatrix}
  s + 1 & 0 \\
  q & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  s^2 + s - q^2 & -q \\
  qs^2 - (q^2 + 1)s - (q^2 + 1) & s^2
\end{bmatrix} \\
= \begin{bmatrix}
  s^3 - q^2 - q \\
  s^3 - q^2 - q
\end{bmatrix}
\]

with uncertain parameter \( q \in [0, 1] \)
MIMO systems: example

Using some design method, we obtain a controller with right MFD

\[ Y(s)X^{-1}(s) = \begin{bmatrix} 94 - 51s & -18 + 17s \\ -55 & 100 \end{bmatrix} \begin{bmatrix} 55 + s & -17 \\ -1 & 18 + s \end{bmatrix} \]

Closed-loop system with characteristic denominator polynomial matrix

\[ D(s, q) = A(s, q)X(s) + B(s, q)Y(s) = D_0(s) + qD_1(s) + q^2D_2(s) \]

Nominal system poles: roots of \( \det D_0(s) \)

Applying the eigenvalue criterion on \( \det D(s, q) \) yields the stability interval

\[ q \in ] - 0.93, 1.17 [ \supset [0, 1] \]

so the closed-loop system is robustly stable
Independent uncertainty

So far we have studied polynomials affected by a single uncertain parameter

\[ p(s, q) = (6 + q) + (4 + q)s + (2 + q)s^2 \]

However in practice several parameters can be uncertain, such as in

\[ p(s, q) = (6 + q_0) + (4 + q_1)s + (2 + q_2)s^2 \]

Independent uncertainty structure: each component \( q_i \) enters into only one coefficient

Interval uncertainty: independent structure and uncertain parameter vector \( q \) belongs to a given box, i.e. \( q_i \in [q_i^-, q_i^+] \)

Example
Uncertain polynomial

\[ (6 + q_0) + (4 + q_1)s + (2 + q_2)s^2, \quad |q_i| \leq 1 \]

has interval uncertainty, also denoted as

\[ [5, 7] + [3, 5]s + [1, 3]s^2 \]

Some coefficients can be fixed, e.g.

\[ 6 + [3, 5]s + 2s^2 \]
Kharitonov’s polynomials

Associated with the interval polynomial

\[ p(s, q) = \sum_{i=0}^{n} [q_i^-, q_i^+] s^i \]

are four Kharitonov’s polynomials

\[ p^{--}(s) = q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + \ldots \]
\[ p^{-+}(s) = q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + q_4^- s^4 + q_5^+ s^5 + \ldots \]
\[ p^{+-}(s) = q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + q_4^+ s^4 + q_5^- s^5 + \ldots \]
\[ p^{++}(s) = q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + q_4^+ s^4 + q_5^+ s^5 + \ldots \]

where we assume \( q_n^- > 0 \) and \( q_n^+ > 0 \)

Example

Interval polynomial

\[ p(s, q) = [1, 2] + [3, 4]s + [5, 6]s^2 + [7, 8]s^3 \]

Kharitonov’s polynomials

\[ p^{--}(s) = 1 + 3s + 6s^2 + 8s^3 \]
\[ p^{-+}(s) = 1 + 4s + 6s^2 + 7s^3 \]
\[ p^{+-}(s) = 2 + 3s + 5s^2 + 8s^3 \]
\[ p^{++}(s) = 2 + 4s + 5s^2 + 7s^3 \]
Kharitonov’s theorem

In 1978 the Russian researcher Vladimír Kharitonov proved the following fundamental result:

A continuous-time interval polynomial is robustly stable iff its four Kharitonov polynomials are stable.

Instead of checking stability of an infinite number of polynomials we just have to check stability of four polynomials, which can be done using the classical Hurwitz criterion.

Peter and Paul fortress in St Petersburg
Affine uncertainty

Sadly, Kharitonov’s theorem is valid only
- for continuous-time polynomials
- for independent interval uncertainty
so that we have to use more general tools in practice

When coefficients of an uncertain polynomial \( p(s, q) \) or a rational function \( n(s, q)/d(s, q) \) depend affinely on parameter \( q \), such as in

\[
a^T q + b
\]

we speak about affine uncertainty

The above feedback interconnection

\[
\frac{n(s, q)x(s)}{d(s, q)x(s) + n(s, q)y(s)}
\]

preserves the affine uncertainty structure of the plant
Polytopes of polynomials

A family of polynomials $p(s, q)$, $q \in Q$ is said to be a polytope of polynomials if

- $p(s, q)$ has an affine uncertainty structure
- $Q$ is a polytope

There is a natural isomorphism between a polytope of polynomials and its set of coefficients

**Example**

$p(s, q) = (2q_1 - q_2 + 5) + (4q_1 + 3q_2 + 2)s + s^2$, $|q_i| \leq 1$

Uncertainty polytope has 4 generating vertices

- $q^1 = [-1, -1]$
- $q^2 = [-1, 1]$
- $q^3 = [1, -1]$
- $q^4 = [1, 1]$

Uncertain polynomial family has 4 generating vertices

- $p(s, q^1) = 4 - 5s + s^2$
- $p(s, q^2) = 2 + s + s^2$
- $p(s, q^3) = 8 + 3s + s^2$
- $p(s, q^4) = 6 + 9s + s^2$

Any polynomial in the family can be written as

$$p(s, q) = \sum_{i=1}^{4} \lambda_i p(s, q^i), \sum_{i=1}^{4} \lambda_i = 1, \lambda_i \geq 0$$
Interval polynomials

Interval polynomials are a special case of polytopic polynomials

\[ p(s, q) = \sum_{i=0}^{n} [q_i^-, q_i^+] s^i \]

with at most \(2^{n+1}\) generating vertices

\[ p(s, q^k) = \sum_{i=0}^{n} q_i^k s^i, \quad q_i^k = \begin{cases} q_i^- & 1 \leq k \leq 2^{n+1} \text{ or} \\ q_i^+ \end{cases} \]

**Example**
The interval polynomial

\[ p(s, q) = [5, 6] + [3, 4]s + 5s^2 + [7, 8]s^3 + s^4 \]

can be generated by the \(2^3 = 8\) vertex polynomials

\[
\begin{align*}
p(s, q^1) &= 5 + 3s + 5s^2 + 7s^3 + s^4 \\
p(s, q^2) &= 6 + 3s + 5s^2 + 7s^3 + s^4 \\
p(s, q^3) &= 5 + 4s + 5s^2 + 7s^3 + s^4 \\
p(s, q^4) &= 6 + 4s + 5s^2 + 7s^3 + s^4 \\
p(s, q^5) &= 5 + 3s + 5s^2 + 8s^3 + s^4 \\
p(s, q^6) &= 6 + 3s + 5s^2 + 8s^3 + s^4 \\
p(s, q^7) &= 5 + 4s + 5s^2 + 8s^3 + s^4 \\
p(s, q^8) &= 6 + 4s + 5s^2 + 8s^3 + s^4 
\end{align*}
\]
The edge theorem

Let \( p(s, q), q \in Q \) be a polynomial with invariant degree over polytopic set \( Q \).

Polyomial \( p(s, q) \) is robustly stable over the whole uncertainty polytope \( Q \) iff \( p(s, q) \) is stable along the edges of \( Q \).

In other words, it is enough to check robust stability of the single parameter polynomial

\[
\lambda p(s, q^{i_1}) + (1 - \lambda)p(s, q^{i_2}), \quad \lambda \in [0, 1]
\]

for each pair of vertices \( q^{i_1} \) and \( q^{i_2} \) of \( Q \).

This can be done with the eigenvalue criterion.
Interval feedback system

**Example**
We consider the interval control system

\[ K(n(s, q), d(s, q)) \]

with \( n(s, q) = [6, 8]s^2 + [9.5, 10.5] \), \( d(s, q) = s(s^2 + [14, 18]) \) and characteristic polynomial

\[ K[9.5, 10.5] + [14, 18]s + K[6, 8]s^2 + s^3 \]

For \( K = 1 \) we draw the 12 edges of its root set

The closed-loop system is **robustly stable**
More about uncertainty structure

In typical applications, uncertainty structure is more complicated than interval or affine.

Usually, uncertainty enters highly non-linearly in the closed-loop characteristic polynomial.

We distinguish between

- **multilinear** uncertainty, when each uncertain parameter $q_i$ is linear when other parameters $q_j$, $i \neq j$ are fixed
- **polynomic** uncertainty, when coefficients are multivariable polynomials in parameters $q_i$

We can define the following hierarchy on the uncertainty structures:

\[ \text{interval} \subset \text{affine} \subset \text{multilinear} \subset \text{polynomic} \]
Examples of uncertainty structures

Examples
The uncertain polynomial
\[(5q_1 - q_2 + 5) + (4q_1 + q_2 + q_3)s + s^2\]
has affine uncertainty structure

The uncertain polynomial
\[(5q_1 - q_2 + 5) + (4q_1q_3 - 6q_1q_3 + q_3)s + s^2\]
has multilinear uncertainty structure

The uncertain polynomial
\[(5q_1 - q_2 + 5) + (4q_1 - 6q_1 - q_3^2)s + s^2\]
has polynomic (here quadratic) uncertainty structure

The uncertain polynomial
\[(5q_1 - q_2 + 5) + (4q_1 - 6q_1q_3^2 + q_3)s + s^2\]
has polynomic uncertainty structure
Multilinear uncertainty

We will focus on multilinear uncertainty because it arises in a wide variety of system models such as:

- multiloop systems

\[
\begin{align*}
G_1 + G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3
\end{align*}
\]

- state-space models with rank-one uncertainty

\[
\dot{x} = A(q)x, \quad A(q) = \sum_{i=1}^{n} q_i A_i, \quad \text{rank } A_i = 1
\]

and characteristic polynomial

\[
p(s, q) = \det(sI - A(q))
\]

- polynomial MFDs with MIMO interval uncertainty

\[
G(s) = A^{-1}(s, q)B(s, q), \quad C(s) = Y(s)X^{-1}(s)
\]

and closed-loop characteristic polynomial

\[
p(s, q) = \det(A(s, q)X(s) + B(s, q)Y(s))
\]
Robust stability analysis for multilinear and polynomic uncertainty

Unfortunately, there is no systematic computational tractable necessary and sufficient robust stability condition

On the one hand, sufficient condition through polynomial value sets, the zero exclusion condition and the mapping theorem

On the other hand, brute-force method: intensive parameter gridding, expensive in general

No easy trade-off between computational complexity and conservatism
Polynomial stability analysis: summary

Checking robust stability can be
• easy (polynomial-time algorithms) or more
• difficult (exponential complexity)
depending namely on the uncertainty model

We focused on polytopic uncertainty:
• Interval scalar polynomials
  Kharitonov’s theorem (ct only)
• Polytope of scalar polynomials
  (affine polynomial families)
  Edge theorem
• Interval matrix polynomials
  (multiaffine polynomial families)
  Mapping theorem
• Polytopes of matrix polynomials
  (polynomic polynomial families)
Lessons from robust analysis:
lack of extreme point results

Ensuring robust stability of the parametrized polynomial

\[ p(s, q) = p_0(s) + qp_1(s) \]
\[ q \in [q_{\text{min}}, q_{\text{max}}] \]

amounts to ensuring robust stability of the whole segment of polynomials

\[ \lambda p(s, q_{\text{min}}) + (1 - \lambda)p(s, q_{\text{max}}) \]
\[ \lambda = \frac{q_{\text{max}} - q}{q_{\text{max}} - q_{\text{min}}} \in [0, 1] \]

A natural question arises: does stability of two vertices imply stability of the segment?

Unfortunately, the answer is no

Example
First vertex: \( 0.57 + 6s + s^2 + 10s^3 \) stable
Second vertex: \( 1.57 + 8s + 2s^2 + 10s^3 \) stable
But middle of segment:
\( 1.07 + 7s + 1.50s^2 + 10s^3 \) unstable
Lessons from robust analysis: 
lack of edge results

In the same way there is lack of vertex results for affine uncertainty, there is a lack of edge results for multilinear uncertainty.

Example
Consider the uncertain polynomial

\[
p(s, q) = (4.032q_1q_2 + 3.773q_1 + 1.985q_2 + 1.853) \\
+ (1.06q_1q_2 + 4.841q_1 + 1.561q_2 + 3.164)s \\
+ (q_1q_2 + 2.06q_1 + 1.561q_2 + 2.871)s^2 \\
+ (q_1 + q_2 + 2.56)s^3 + s^4
\]

with multilinear uncertainty over the polytope \(q_1 \in [0, 1], q_2 \in [0, 3]\), corresponding to the state-space interval matrix

\[
p(s, q) = \det(sI - \begin{bmatrix}
-1.5 & -0.5 & -12.06 & -0.06 & 0 \\
-0.25 & -0.03 & 1 & 0.5 \\
0.25 & -4 & -1.03 & 0 \\
0 & 0.5 & 0 & [-4, 1]
\end{bmatrix})
\]

The four edges of the uncertainty bounding set are stable, however for \(q_1 = 0.5\) and \(q_2 = 1\) polynomial \(p(s, q)\) is unstable.
Non-convexity of stability domain

Main problem: the stability domain in the space of polynomial coefficients $p_i$ is non-convex in general

Discrete-time stability domain in $(q_1, q_2)$ plane for polynomial $p(z, q) = (-0.825 + 0.225q_1 + 0.1q_2) + (0.895 + 0.025q_1 + 0.09q_2)z + (-2.475 + 0.675q_1 + 0.3q_2)z^2 + z^3$

How can we overcome the non-convexity of the stability conditions in the coefficient space?
Handling non-convexity

Basically, we can pursue two approaches:

- we can **approximate** the non-convex stability domain with a convex domain (segment, polytope, sphere, ellipsoid, LMI)

- we can address the non-convexity with the help of **non-convex optimization** (global or local optimization)
Stability polytopes

Largest hyper-rectangle around a nominally stable polynomial

\[ p(s) + r \sum_{i=0}^{n} [-\varepsilon_i, \varepsilon_i] s^i \]

obtained with the eigenvalue criterion applied on the 4 Kharitonov polynomials

In general, there is no systematic way to obtain more general stability polytopes, namely because of computational complexity

(no analytic formula for the volume of a polytope)

Well-known candidates:

- ct LHP: outer approximation (necessary stab cond) positive cone \( p_i > 0 \)

- dt unit disk: inner approximation (sufficient stab cond) diamond \( |p_0| + |p_1| + \cdots + |p_{n-1}| < 1 \)
Stability region (second degree)

**Necessary** stab cond in dt: convex hull of stability domain is a polytope whose $n + 1$ vertices are polynomials with roots $+1$ or $-1$

**Example**

When $n = 2$: triangle with vertices

$$(z + 1)(z + 1) = 1 + 2z + z^2$$

$$(z + 1)(z - 1) = -1 + z^2$$

$$(z - 1)(z - 1) = 1 - 2z + z^2$$
Stability region (third degree)

Example
Third degree dt polynomial: two hyperplanes and a non-convex hyperbolic paraboloid with a saddle point at $p(z) = p_0 + p_1 z + p_2 z^2 + z^3 = z(1 + z^2)$

\[
\begin{align*}
(z + 1)(z + 1)(z + 1) &= 1 + 3z + 3z^2 + z^3 \\
(z + 1)(z + 1)(z - 1) &= -1 - z + z^2 + z^3 \\
(z + 1)(z - 1)(z - 1) &= 1 - z - z^2 + z^3 \\
(z - 1)(z - 1)(z - 1) &= -1 + 3z - 3z^2 + z^3
\end{align*}
\]
Stability hyper-spheres

Largest hyper-sphere around a nominally stable polynomial

\[ p(s) + \sum_{i=0}^{n} q_i s^i, \|q\| \leq r \]

has radius

\[ r_{\text{max}} = \min \left\{ |p_0|, |p_n|, \inf_{\omega > 0} \sqrt{\frac{(\text{Re} \, p(j\omega))^2}{1 + w^4 + w^8 + \ldots} + \frac{(\text{Im} \, p(j\omega))^2}{w^2 + w^6 + \ldots}} \right\} \]

Example

\((2 + q_0) + (1.4 + q_1)s + (1.5 + q_2)s^2 + (1 + q_3)s^3, \|q\| \leq r\)

\[ r_{\text{max}} = \min \{2, 1, \inf_{\omega > 0} f(w)\} = 1.08 \cdot 10^{-3} \]
Stability ellipsoids

A weighted and rotated hyper-sphere is an ellipsoid

We are interested in inner ellipsoidal approximations of stability domains

\[ E = \{ p : (p - \bar{p})^* P (p - \bar{p}) \leq 1 \} \]

where

- \( p \) coef vector of polynomial \( p(s) \)
- \( \bar{p} \) center of ellipsoid
- \( P \) positive definite matrix
Hermite stability criterion

The polynomial \( p(s) = p_0 + p_1 s + \cdots + p_n s^n \) is stable if and only if

\[
H(x) = \sum_{i} \sum_{j} p_i p_j H_{ij} > 0
\]

where matrices \( H_{ij} \) are given and depend on the root clustering region only.

Examples for \( n = 3 \):

continuous-time stability
\[
H(p) = \begin{bmatrix}
2p_0 p_1 & 0 & 2p_0 p_3 \\
0 & 2p_1 p_2 - 2p_0 p_3 & 0 \\
2p_0 p_3 & 0 & 2p_2 p_3
\end{bmatrix}
\]

discrete-time stability
\[
H(p) = \begin{bmatrix}
p_3^2 - p_0^2 & p_2 p_3 - p_0 p_1 & p_1 p_3 - p_0 p_2 \\
p_2 p_3 - p_0 p_1 & p_2^2 + p_3^2 - p_0^2 - p_1^2 & p_2 p_3 - p_0 p_1 \\
p_1 p_3 - p_0 p_2 & p_2 p_3 - p_0 p_1 & p_3^2 - p_0^2
\end{bmatrix}
\]
Our objective is then to find $\bar{p}$ and $P$ such that the ellipsoid

$$E = \{ p : (p - \bar{p})^* P (p - \bar{p}) \leq 1 \}$$

is a convex inner approximation of the actual non-convex stability region

$$S = \{ p : H(p) \succ 0 \}$$

that is to say

$$E \subset S$$

Naturally, we will try to **enlarge the volume** of the ellipsoid as much as we can

Using the Hermite matrix formulation, we can derive (details omitted) a suboptimal **LMI** formulation (not given here) with decision variables $\bar{p}$ and $P$
**Stability ellipsoids**

**Example**
Discrete-time second degree polynomial

\[ p(z) = p_0 + p_1 z + z^2 \]

We solve the LMI problem and we obtain

\[
\begin{bmatrix}
1.5625 & 0 \\
0 & 1.2501
\end{bmatrix}
\quad \begin{bmatrix}
0.2000 \\
0
\end{bmatrix}
\]

which describes an ellipse \( E \) inscribed in the exact triangular stability domain \( S \).
Stability ellipsoids

**Example**
Discrete-time third degree polynomial

\[ p(z) = p_0 + p_1 z + p_2 z^2 + z^3 \]

We solve the LMI problem and we obtain

\[
P = \begin{bmatrix}
2.3378 & 0 & 0.5397 \\
0 & 2.1368 & 0 \\
0.5397 & 0 & 1.7552
\end{bmatrix} \quad \bar{x} = \begin{bmatrix}
0 \\
0.1235 \\
0
\end{bmatrix}
\]

which describes a **convex** ellipse \( E \) inscribed in the exact stability domain \( S \) delimited by the **non-convex** hyperbolic paraboloid.

Very simple scalar convex sufficient stability condition

\[
2.4166p_0^2 + 2.2088p_1^2 + 1.8143p_2^2 - 0.5458p_1 + 1.1158p_0p_2 \leq 1
\]
Volume of stability ellipsoid

In the discrete-time case, the well-known sufficient stability condition defines a diamond

\[ D = \{ p : |p_0| + |p_1| + \cdots + |p_{n-1}| < 1 \} \]

For different values of degree \( n \), we compared volumes of exact stability domain \( S \), ellipsoid \( E \) and diamond \( D \)

<table>
<thead>
<tr>
<th></th>
<th>( n = 2 )</th>
<th>( n = 3 )</th>
<th>( n = 4 )</th>
<th>( n = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stability domain ( S )</td>
<td>4.0000</td>
<td>5.3333</td>
<td>7.1111</td>
<td>7.5852</td>
</tr>
<tr>
<td>Ellipsoid ( E )</td>
<td>2.2479</td>
<td>1.4677</td>
<td>0.7770</td>
<td>0.3171</td>
</tr>
<tr>
<td>Diamond ( D )</td>
<td>2.0000</td>
<td>1.3333</td>
<td>0.6667</td>
<td>0.2667</td>
</tr>
</tbody>
</table>

\( E \) is “larger” than \( D \), yet very small wrt \( S \)

In the last part of this course, we will propose better LMI inner approximations of the stability domain
COURSE ON LMI
PART II.4

LMIs IN SYSTEMS CONTROL
ROBUST CONTROL DESIGN
POLYNOMIAL METHODS

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Fixed-order robust design: a difficult problem

In this last part of course, we study robust stabilization with a fixed-order controller affected by parametric uncertainty.

A difficult problem in general because
- fixed-order controller means non-convexity of the design space
- parametric uncertainty means highly structured uncertainty and exponential (combinatorial) complexity

In the literature: a lot of analysis results, but very few design results.

Low-order controllers are required in embedded devices.
Once more: uncertainty models

We can consider several uncertainty models, in decreasing order of complexity:

- **Polytopic uncertainty**, where plant (numerator and/or denominator) polynomial $p(s)$ belongs to a polytope with given vertices
  
  \[ p(s) = \lambda_1(s + 1)^3 + \lambda_2(s + 1)(s + 2)^2 \]  
  with $\lambda_1 + \lambda_2 = 1$

  Very general, but often leads to intractable analysis/design problems

- **Interval uncertainty**, where each of the plant parameters are assumed to vary independently in given intervals
  
  \[ p(s) = 1 + [2, 3]s + [-4, 1]s^2 + s^3 \]

  Leads to nice analysis results (Kharitonov) but intractable design problems

- **Ellipsoidal uncertainty**, also called rank-one (LFT), or norm-bounded
  
  \[ p(s) = p_0 + p_1s + p_2s^2 + s^3 \]  
  with $3p_0^2 + p_0p_1 + 2p_1^2 + p_2^2 \leq 1$

  Less structured, but more realistic, naturally arises in the context of parameter estimation for process identification, ellipsoid = covariance matrix
Existing results

- $H_\infty$ design methods, state-space techniques
- Critical direction (Nyquist), convex optimization with cutting-plane algorithms
- Infinite-dimensional Youla-Kučera parametrization generally leading to high-order controllers
- Use of linear programming with polytopic sufficient conditions for stability

In this course

- Use of polynomial techniques
- Use of LMI optimization
- Controllers of fixed (hence low) order
Nominal Pole placement

We consider the SISO feedback system

\[
\begin{align*}
\text{Closed-loop transfer function} \\
\frac{bx}{ax + by}
\end{align*}
\]

In the absence of hidden modes (a and b coprime polynomials), pole placement amounts to finding polynomials \(x\) and \(y\) solving the Diophantine equation (from Diophantus of Alexandria 200-284)

\[
ax + by = c
\]

where \(c\) is a given closed-loop characteristic polynomial capturing the desired system poles
Pole placement: numerical aspects

The polynomial Diophantine equation

\[ ax + by = c \]

is linear in unknowns \( x \) and \( y \), and denoting

\[
a(s) = a_0 + a_1 s + \cdots + a_{da} s^{da} \\
x(s) = x_0 + x_1 s + \cdots + x_{db} s^{db}
\]

eetc..

we can identify powers of the indeterminate \( s \) to build a linear system of equations

\[
\begin{bmatrix}
a_0 & b_0 \\
a_1 & b_1 \\
\vdots & \vdots \\
a_{da} & b_{da}
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
\vdots \\
x_{da}
\end{bmatrix}
= 
\begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{dc}
\end{bmatrix}
\]

The above matrix is called **Sylvester matrix**, it has a special Toeplitz banded structure that can be exploited when solving the equation

James J Sylvester
(1814 London - 1897 London)

Otto Toeplitz
(1881 Breslau - 1940 Jerusalem)
Pole placement for MIMO systems

Pole placement can be performed similarly for a plant left MFD

\[ A^{-1}(s)B(s) \]

with a controller right MFD

\[ Y(s)X^{-1}(s) \]

The Diophantine equation to be solved is now over polynomial matrices

\[ A(s)X(s) + B(s)Y(s) = C(s) \]

and right hand-side matrix \( C(s) \) captures information on invariant polynomials and eigenstructure

For example \( C(s) \) may contain \( H_2 \) or \( H_\infty \) optimal dynamics (obtained with spectral factorization)
Robust pole placement

Now assume that the plant transfer function

\[ \frac{b(q)}{a(q)} \]

contains some uncertain parameter \( q \)

The problem of robust pole placement will then consists in finding a controller

\[ \frac{y}{x} \]

such that the uncertain closed-loop character polynomial

\[ a(q)x + b(q)y = c(q) \]

is robustly stable

How can we find \( x, y \) to ensure robust stability of \( c(q) \) for all admissible uncertainty \( q \)?

Coefficients of \( c \) are linear in \( x \) and \( y \), but we saw that stability conditions are non-linear and highly non-convex in \( c \).
Robust pole placement

One possible remedy is a suitable convex approximation of the stability region.

Then we can perform design with:
- linear programming (polytopes)
- quadratic programming (spheres, ellipsoids)
- semidefinite programming (LMIs)

Complexity of design algorithm increases.
Conservatism of control law decreases.
Robust design via polytopic approximation

MIMO plant with right MFD

\[ B(s)A^{-1}(s) = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s + 1 & 0 \\ 0 & s + 1 \end{bmatrix}^{-1} \]

with uncertainty in parameter \( b \in [0.5, 1.5] \)

We seek a proper first order controller

\[ X^{-1}(s)Y(s) = \begin{bmatrix} s + x_1 & x_2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} y_1s + y_2 & y_3s + y_4 \\ 0 & y_5 \end{bmatrix} \]

assigning robustly the closed-loop polynomial matrix

\[ C(s) = \begin{bmatrix} s^2 + \alpha s + \beta & \delta(s) \\ 0 & s + \gamma \end{bmatrix} \]

whose coefficients live in the polytope

\[
\begin{bmatrix}
-14 & 1 & 0 \\
16 & -2 & 0 \\
-2 & 1 & 0 \\
0 & 0 & -1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix}
> \begin{bmatrix}
-196 \\
56 \\
-4 \\
2 \\
-14
\end{bmatrix}
\]

These specifications amounts to assigning the poles within the disk

\[ |s + 8| < 6 \]
Robust design via polytopic approximation (2)

Equating powers of indeterminate $s$ in the polynomial matrix Diophantine equation

$$X(s)A(s) + Y(s)B(s) = C(s)$$

we obtain the design inequalities

$$\begin{bmatrix}
-13 & -7 & 0.5 \\
14 & 8 & -1 \\
-1 & -1 & 0.5 \\
-13 & -21 & 1.5 \\
14 & 24 & -3 \\
-1 & -3 & 1.5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
y_1 \\
y_2
\end{bmatrix}
>
\begin{bmatrix}
-182 \\
40 \\
-2 \\
-182 \\
40 \\
-2
\end{bmatrix}$$

characterizing all parameters $x_1$, $y_1$ and $y_2$ of admissible robust controllers

Corresponding polytope with 9 vertices
Robust design via ellipsoidal approximation

Closed-loop characteristic polynomial

\[ c(s) = a(s)x(s) + b(s)y(s) \]

\[ = c_0 + c_1 s + \cdots + c_{d-1} s^{d-1} + s^d \]

whose coefficients are given by the LSE

\[
\begin{bmatrix}
  c_0 \\
  c_1 \\
  \vdots \\
  c_{d-1}
\end{bmatrix}
= 
\begin{bmatrix}
  y_0 & \cdots & x_0 \\
  y_1 & \cdots & x_1 \\
  \vdots & \ddots & \vdots \\
  y_{m-1} & y_0 & x_{m-1} & x_1 \\
  y_m & \cdots & 1 & \cdots \\
  \vdots & \cdots & y_{m-1} & \cdots & x_{m-1} \\
  \vdots & \cdots & y_m & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
  b_0 \\
  b_1 \\
  \vdots \\
  b_{n-1} \\
  a_0 \\
  a_1 \\
  \vdots \\
  a_{n-1}
\end{bmatrix}
+ 
\begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  0 \\
  x_0 \\
  x_1 \\
  \vdots \\
  x_{m-1}
\end{bmatrix}
\]

\[ c = S(x, y)p + e(x) \]

It is assumed that uncertain plant parameters belong to the ellipsoid

\[ E_p = \{p : (p - \bar{p})^* P (p - \bar{p}) \leq 1\} \]

where \( \bar{p} \) is a given nominal plant vector and \( P \) is a given positive definite covariance matrix

Robust control problem:

Find controller coefficients \( x, y \) robustly stabilizing plant \( a, b \) subject to ellipsoidal uncertainty \( p \in E_p \)
Robust design via ellipsoidal approximation

Recall that by solving an LMI we could approximate from the interior the non-convex stability region with an ellipsoid

\[ E_q = \{ q : (q - \bar{q})^* Q (q - \bar{q}) \leq 1 \} \]

In other words, \( q \in E_q \) implies \( q(s) \) stable

Using conditions for inclusion of an ellipsoid into another we can show that finding \( x \) and \( y \) such that \( q \in E_q \) for all \( p \in E_p \) amounts to solving another LMI problem (not given here)

Coefficients \( x, y \) are such that the controller \( y(s)/x(s) \) robustly stabilizes plant \( b(s)/a(s) \)
Ellipsoidal robust design: example

We consider the two mixing tanks arranged in cascade with recycle stream

\[ P \]
\[ Fa \]
\[ Fa \]
\[ Fb \]
\[ Fa+Fb \]
\[ Ta \]
\[ Tb \]
\[ P \]

The controller must be designed to maintain the temperature \( T_b \) of the second tank at a desired set point by manipulating the power \( P \) delivered by the heater located in the first tank

The only available measurement is temperature \( T_b \)
Ellipsoidal robust design: example (2)

The identification of the nominal plant model is carried out using a standard least-squares method.

A second-order discrete-time model
\[ p(z) = \frac{b_0 + b_1 z}{a_0 + a_1 z + z^2} \]
is obtained with nominal plant vector
\[ \bar{p} = \begin{bmatrix} 0.0038 & 0.0028 & 0.2087 & -1.1871 \end{bmatrix}^* \]

The positive definite matrix \( P \) characterizing the uncertainty ellipsoid
\[ E_p = \{ p : (p - \bar{p})^* P (p - \bar{p}) \leq 1 \} \]
is readily available as a by-product of the identification technique
\[ P = 10^5 \begin{bmatrix} 2.4179 & 0.0568 & 0.0069 & 0 \\ 0.0568 & 2.4121 & 0.0045 & 0.0062 \\ 0.0069 & 0.0045 & 0.0015 & 0.0014 \\ 0 & 0.0062 & 0.0014 & 0.0015 \end{bmatrix} \]
Ellipsoidal robust design: example (3)

Solving the LMI analysis problem we obtain first an inner ellipsoidal approximation

\[ E_q = \{ q : (q - \bar{q})^*Q(q - \bar{q}) \leq 1 \} \]

of the non-convex stability region, with

\[
Q = \begin{bmatrix}
2.3378 & 0 & 0.5397 \\
0 & 2.1368 & 0 \\
0.5397 & 0 & 1.7552
\end{bmatrix}, \quad \bar{q} = \begin{bmatrix}
0 \\
0.1235 \\
0
\end{bmatrix}
\]

Then we solve the design LMI to obtain the first-order robustly stabilizing controller

\[
y(z) \frac{1}{x(z)} = \frac{0.3377+166.0z}{0.6212+z}
\]
Strict positive realness

Let

$$\mathcal{D} = \{s : \begin{bmatrix} 1 \\ s \end{bmatrix}^* \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0\}$$

be a stability region in the complex plane where Hermitian matrix $H$ has inertia $(1, 0, 1)$

Let $\partial \mathcal{D}$ denote the 1-D boundary of $\mathcal{D}$

Standard choices are

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad H = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

for the left half-plane and the unit disk resp.

We say that a rational matrix $G(s)$ is strictly positive real (SPR for short) when

$$\text{Re } G(s) \succ 0 \quad \text{for all} \quad s \in \partial \mathcal{D}$$
Stability and strict positive realness

Consider two square polynomial matrices of size $n$ and degree $d$

$$\begin{align*}
N(s) &= N_0 + N_1 s + \cdots + N_d s^d \\
D(s) &= D_0 + D_1 s + \cdots + D_d s^d
\end{align*}$$

Polynomial matrix $N(s)$ is **stable** iff there is a stable polynomial $D(s)$ such that rational matrix $N(s)D^{-1}(s)$ is **strictly positive real**

**Proof**

From the definition of SPRness, $N(s)D^{-1}(s)$ SPR with $D(s)$ stable implies $N(s)$ stable

Conversely, if $N(s)$ is stable then the choice $D(s) = N(s)$ makes rational matrix $N(s)D^{-1}(s) = I$ obviously SPR

It turns out that this condition can be characterized by an LMI..
**SPRness as an LMI**

Let $N = [N_0 \ N_1 \cdots N_d]$, $D = [D_0 \ D_1 \cdots D_d]$ and

$$\Pi = \begin{bmatrix}
I & 0 \\
\vdots & \ddots & \ddots \\
0 & \cdots & I \\
0 & \cdots & \cdots & I
\end{bmatrix}$$

Given a stable $D(s)$, $N(s)$ ensures SPRness of $N(s)D^{-1}(s)$ iff there exists a matrix $P = P^*$ of size $dn$ such that

$$D^*N + N^*D - H(P) \succeq 0$$

where

$$H(P) = \Pi^*(S \otimes P)\Pi = \Pi^* \begin{bmatrix} aP & bP \\ b^*P & cP \end{bmatrix} \Pi$$

**Proof**

Similar to the proof on positivity of a polynomial, based on the decomposition as a sum-of-squares with lifting matrix $P$. 
LMI condition for design

Given a stable polynomial matrix $D(s)$, polynomial matrix $N(s)$ is stable if there is a matrix $P$ satisfying the LMI

$$D^*N + N^*D - H(P) \succ 0$$

(it follows then that $P \succ 0$)

- New convex inner approximation of stability domain
- Shape described by an LMI
- Depends on the particular choice of $D(s)$
- More general than polytopes and ellipsoids

Useful for design because linear in $N$

Polynomial $D(s)$ will be referred to as the central polynomial
LMI condition for analysis

The LMI condition of SPRness can be used also for design if we interchange the respective roles played by $N(s)$ and $D(s)$

If there is a matrix $P$ and a polynomial matrix $D(s)$ satisfying the LMI

\[
D^*N + N^*D - H(P) \succ 0 \\
P \succ 0
\]

then polynomial matrix $N(s)$ is stable (it follows that $D(s)$ is stable as well)

Useful for (robust) analysis because linear in $D$

Note that, in contrast with the design LMI, we have here to enforce $P \succ 0$
Consider the discrete polynomial

\[ n(z) = n_0 + n_1 z + z^2 \]

We will study the shape of the LMI stability region for the following central polynomial

\[ d(z) = z^2 \]

We can show that non-strict feasibility of the LMI is equivalent to existence of a matrix \( P \) satisfying

\[
\begin{align*}
p_{00} + p_{11} + p_{22} &= 1 \\
p_{10} + p_{01} + p_{21} + p_{22} &= n_1 \\
p_{20} + p_{02} &= n_0 \\
P &\succeq 0
\end{align*}
\]

which is an LMI in the primal SDP form.
Second-degree discrete-time polynomial (2)

Infeasibility of primal LMI is equivalent to the existence of a vector satisfying the dual LMI

\[ y_0 + n_1y_1 + n_0y_2 < 0 \]
\[ Y = \begin{bmatrix} y_0 & y_1 & y_2 \\ y_1 & y_0 & y_1 \\ y_2 & y_1 & y_0 \end{bmatrix} \succeq 0 \]

The eigenvalues of Toeplitz matrix \( Y \) are

\[ y_0 - y_2 \quad \text{and} \quad (2y_0 + y_2 \pm \sqrt{y_2^2 + 8y_1^2})/2 \]

so it is positive definite iff \( y_1 \) and \( y_2 \) belong to the interior of a bounded parabola scaled by \( y_0 \)

The corresponding values of \( n_0 \) and \( n_1 \) belong to the interior of the envelope generated by the curve

\[(2\lambda_2-1)n_0 + (2\lambda_1-1)\sqrt{\lambda_2}n_1 + 1 > 0 \quad 0 \leq \lambda_i \leq 1 \]
Second-degree discrete-time polynomial (3)

The implicit equation of the envelope is

$$(2n_0 - 1)^2 + \left(\frac{\sqrt{2}}{2}n_1\right)^2 = 1$$

a scaled circle

The LMI stability region is then the union of the interior of the circle with the interior of the triangle delimited by the two lines

$$n_0 \pm n_1 + 1 = 0$$

tangent to the circle, with vertices $[-1, 0]$, $[1/3, 4/3]$ and $[1/3, -4/3]$. 

![Graph showing the LMI stability region and the envelope equation](image-url)
Application to robust stability analysis

Assume that \( N(s, \lambda) \) is a polynomial matrix with multi-linear dependence in a parameter vector \( \lambda \) belonging to a polytope \( \Lambda \)

Denote by \( N_i(s) \) the vertices obtained by enumerating each vertex in \( \Lambda \)

Polytopic polynomial matrix \( N(s, \lambda) \) is robustly stable if there exists a matrix \( D \) and matrices \( P_i \) satisfying the LMI

\[
D^* N_i + N_i^* D - H(P_i) \succ 0 \\
P_i \succ 0 \quad \forall i
\]

Proof

Since the LMI is linear in \( D \) - matrix of coefficients of polynomial matrix \( D(s) \) - it is enough to check the vertices to prove stability in the whole polytope
Robust stability of polynomial matrices

**Example**
Consider the following **mechanical system**

\[
\begin{bmatrix}
 m_1 s^2 + d_1 s + c_1 + c_{12} \\
 -c_{12} \quad m_2 s^2 + d_2 s + c_2 + c_{12}
\end{bmatrix}
\begin{bmatrix}
 x_1(s) \\
 x_2(s)
\end{bmatrix}
= \begin{bmatrix}
 0 \\
 u(s)
\end{bmatrix}
\]

It is described by the polynomial MFD

System parameters \( \lambda = [m_1 \ d_1 \ c_1 \ m_2 \ d_2 \ c_2] \) belong to the **uncertainty hyper-rectangle**

\( \Lambda = [1, \ 3] \times [0.5, \ 2] \times [1, \ 2] \times [2, \ 5] \times [0.5, \ 2] \times [2, \ 4] \)

and we set \( c_{12} = 1 \)

This mechanical system is **passive** so it must be open-loop stable (when \( u(s) = 0 \)) independently of the values of the masses, springs, and dampers
Robust stability of polynomial matrices

However, it is a non-trivial task to know whether the open-loop system is robustly D-stable in some stability region $\mathcal{D}$ ensuring a certain damping. Here we choose the disk of radius 12 centered at -12

$$\mathcal{D} = \{ s : (s + 12)^2 < 12^2 \}$$

The robust stability analysis problem amounts to assessing whether the second degree polynomial matrix in the MFD has its zeros in $\mathcal{D}$ for all admissible uncertainty in a polytope with $m = 2^6 = 64$ vertices

LMI problem is feasible – vertex zeros shown below
Polytope of polynomials

We can also check robust stability of polytopes of polynomials without using the edge theorem or the graphical value set.

**Example**

Continuous-time polytope of degree 3 with 3 vertices

\[
\begin{align*}
n_1(s) & = 28.3820 + 34.7667s + 8.3273s^2 + s^3 \\
n_2(s) & = 0.2985 + 1.6491s + 2.6567s^2 + s^3 \\
n_3(s) & = 4.0421 + 9.3039s + 5.5741s^2 + s^3
\end{align*}
\]

The LMI problem is feasible, so the polytope is robustly stable – see robust root locus below.
Interval polynomial matrices

Similarly, we can assess robust stability of interval polynomial matrices, a difficult problem in general.

**Example**
Continuous-time interval polynomial matrix of degree 2 with $2^3 = 8$ vertices

$$
\begin{bmatrix}
[7.7 - 2.3s + 4.3s^2, & 3.7 + 2.7s + 4.3s^2] \\
3.6 + 6.4s + 4.3s^2 & [-3.1 - 6s - 2.2s^2, \\
-4.1 - 7s - 2.2s^2] \\
3.2 + 11s + 8.2s^2 & 16 + 12s + 8.2s^2
\end{bmatrix}
$$

LMI is feasible so the matrix is **robustly stable**

See robust root locus below
State-space systems

One advantage of our approach is that state-space results can be obtained as simple by-products, since stability of a constant matrix $A$ is equivalent to stability of the pencil matrix

$$N(s) = sI - A$$

Matrix $A$ is stable iff there exists a matrix $F$ and a matrix $P$ solving the LMI

$$
\begin{bmatrix}
F^*A + A^*F - aP & -A^* - F^* - bP \\
-A - F - b^*P & 2I - cP
\end{bmatrix}
\succ 0
$$

$P \succ 0$

Proof
Just take $D(s) = sI - F$ and notice that the LMI can be also written more explicitly as

$$
\begin{bmatrix}
-F^* \\
I
\end{bmatrix}
\begin{bmatrix}
-A & I \\
I
\end{bmatrix}
+ 
\begin{bmatrix}
-A^* \\
I
\end{bmatrix}
\begin{bmatrix}
-F & I \\
I
\end{bmatrix}
- \begin{bmatrix}
aP & bP \\
b^*P & cP
\end{bmatrix}
\succ 0
$$
Robust stability of state-space systems

We recover the LMI stability conditions obtained by Geromel and de Oliveira in 1999.

Nice decoupling between Lyapunov matrix $P$ and additional variable $F$ allows for construction of parameter-dependent Lyapunov matrix.

Assume that uncertain matrix $A(\lambda)$ has multi-linear dependence on polytopic uncertain parameter $\lambda$ and denote by $A_i$ the corresponding vertices.

Matrix $A(\lambda)$ is robustly stable if there exists a matrix $F$ and matrices $P_i$ solving the LMI:

\[
\begin{bmatrix}
F^*A_i + A_i^*F - aP_i & -A_i^* - F^* - bP_i \\
-A_i - F - b^*P_i & 2I - cP_i
\end{bmatrix} \succ 0
\]

\[
P_i \succ 0 \quad \forall i
\]

Proof
Consider the parameter-dependent Lyapunov matrix $P(\lambda)$ built from vertices $P_i$. 
Robust design

Assume now that system matrix \( C(s, \lambda) \) comes from a polynomial Diophantine equation

\[
C(s, \lambda) = A(s, \lambda)X(s) + B(s, \lambda)Y(s)
\]

where system matrices \( A \) and \( B \) are subject to multi-linear polytopic uncertainty \( \lambda \).

In order to ensure robust SPRness of the rational matrix \( D^{-1}(s)C(s, \lambda) \) central polynomial \( D(s) \) must be close to the nominal closed-loop matrix, such as

\[
D(s) = C(s, \lambda_0)
\]

where \( \lambda_0 \) is the nominal parameter vector.

A sensible simple choice of \( D(s) \) is therefore the nominal closed-loop denominator polynomial matrix, obtained by any standard design method (pole assignment, LQ, \( H_\infty \)).
Reactor

Consider the stirred tank reactor model described by non-linear state-space equations

\[ \begin{align*}
\dot{\xi}_1 &= (\xi_2 + 0.5)e^{(E\xi_1/(\xi_1 + 2))} - (2 + u)(\xi_1 + 0.25) \\
\dot{\xi}_2 &= 0.5 - \xi_2 - (\xi_2 + 0.5)e^{(E\xi_1/(\xi_1 + 2))}
\end{align*} \]

where \( E \) is a parameter related to the activation energy

During the life of the reactor, some representative values of \( E \) are 20, 25 and 30.

Using only \( \xi_1 \) for feedback we obtain a polytopic system with 3 linearized vertex transfer functions

\[ \begin{align*}
b_1(s)/a_1(s) &= (0.5 - 0.25s)/(11 - 5s + s^2) \\
b_2(s)/a_2(s) &= (-0.5 - 0.25s)/(-2.25 - 2.25s + s^2) \\
b_3(s)/a_3(s) &= (-0.5 - 0.25s)/(-3.5 - 3.5s + s^2).
\end{align*} \]

Our objective is to stabilize the whole polytopic system with a single static output feedback controller \( y(s)/x(s) \).
For the choice
\[ d(s) = (s + 1)^2 \]
as a central polynomial, solving the LMI yields the robustly stabilizing controller
\[ y(s)/x(s) = k = -21.4409 \]

With the eigenvalue criterion we can check that \( k \) is simultaneous stabilizing iff
\[ -22 < k < -20 \]

Note however that the problem solved here is more difficult since we must stabilize the whole polytope, not only its vertices
Oblique wing aircraft

We consider the model of an experimental oblique wing aircraft

The linearized transfer function

\[
\begin{array}{c}
[90, 166] + [54, 74]s \\
[−0.1, 0.1] + [30.1, 33.9]s + [50.4, 80.8]s^2 + [2.8, 4.6]s^3 + s^4
\end{array}
\]

features 6 uncertain parameters, and must be stabilized with a PI controller

\[
\frac{y(s)}{x(s)} = K_p + \frac{K_i}{s}
\]

One can check easily that the choice \(K_p = 1\) and \(K_i = 1\) stabilizes the vertex plant

\[
\begin{array}{c}
90 + 54s \\
−0.1 + 30s + 50s^2 + 2.8s^3 + s^4
\end{array}
\]
Oblique wing aircraft (2)

With the choice

\[ d(s) = sa_1(s) + (1 + s)b_1(s) \]

as the central polynomial, the LMI problem is infeasible

But when considering another vertex plant

\[ \frac{b_2(s)}{a_2(s)} = \frac{90 + 54s}{-0.1 + 30s + 50s^2 + 4.6s^3 + s^4} \]

the choice

\[ d(s) = sa_2(s) + (1 + s)b_2(s) \]

now proves successful, the LMI is solved and we obtain the robustly stabilizing PI controller

\[ \frac{y(s)}{x(s)} = 0.8634 + \frac{0.6454}{s} \]
We consider a satellite control problem where the satellite model is two masses with the same inertia connected by a spring with torque constant $q_1$ and viscous damping constant $q_2$.

The transfer function between the control torque and the satellite angle is given by

$$\frac{b(s, q)}{a(s, q)} = \frac{q_1 + q_2 s + s^2}{s^2(2q_1 + 2q_2 s + s^2)}$$

where uncertain parameters are assumed to vary within the bounds

$$q_1 \in [0.09, 4], \ q_2 \in [0.04\sqrt{q_1}10, \ 0.2\sqrt{q_2}10]$$

With the choice

$$d(s) = (s + 1)^6$$

as a central polynomial, the LMI design method cannot stabilize the whole polytope.
Satellite (2)

However we can stabilize each vertex $a_i(s)$, $b_i(s)$ individually with controller polynomials $x_i(s)$, $y_i(s)$.

The roots of the corresponding closed-loop polynomials $c_i(s) = a_i(s)x_i(s) + b_i(s)y_i(s)$ are given below:

<table>
<thead>
<tr>
<th>$i$</th>
<th>roots of $c_i(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-0.4520 \pm j1.8129, -0.0365 \pm j0.8954, -0.0019 \pm j0.2533$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.6161 \pm j1.3829, -0.2190 \pm j0.6745, -0.0237 \pm j0.2136$</td>
</tr>
<tr>
<td>3</td>
<td>$-1.0484, -0.0917 \pm j2.1598, -0.0425 \pm j0.5019, -0.0004$</td>
</tr>
<tr>
<td>4</td>
<td>$-0.2220 \pm j2.0695, -0.1695 \pm j0.5354, -0.0709 \pm j0.0045$</td>
</tr>
</tbody>
</table>

The third vertex has poles nearest to the imaginary axis, and with the choice $d(s) = c_3(s)$ as a central polynomial, the LMI is found feasible and we obtain the robustly stabilizing controller,

$$\frac{y(s)}{x(s)} = \frac{0.0181 + 0.0170s - 1.4048s^2}{3.6082 + 1.0237s + s^2}$$

Sometimes it can be tricky to be find the central polynomial (provided one exists = the system is robustly stabilizable).
Robot

We consider the problem of designing a robust controller for the approximate ARMAX model of a PUMA robotic disk grinding process.

From the results of identification and because of the nonlinearity of the robot, the coefficients of the numerator of the plant transfer function change for different positions of the robot arm. We consider variations of up to 20% around the nominal value of the parameters.

The fourth-order discrete-time model is given by

\[
\frac{b(z^{-1}, q)}{a(z^{-1}, q)} = \frac{(0.0257 + q_1) + (-0.0764 + q_2)z^{-1} + (-0.1619 + q_3)z^{-2} + (-0.1688 + q_4)z^{-3}}{1 - 1.914z^{-1} + 1.779z^{-2} - 1.0265z^{-3} + 0.2508z^{-4}}
\]

where

\[|q_1| \leq 0.00514, |q_2| \leq 0.01528, |q_3| \leq 0.03238, |q_4| \leq 0.03376\]
Robot

Closed-loop polynomial

\[ d(z, q) = z^{12}[(1 - z^{-1})a(z^{-1}, q)x(z^{-1}) + z^{-5}b(z^{-1}, q)y(z^{-1})] \]

where the term \( 1 - z^{-1} \) is introduced in the controller denominator to maintain the steady state error to zero when parameters are changed.

With the input central polynomial \( d(z) = z^{19} \) the LMI returns the seventh-order robust controller

\[
\frac{y(z^{-1})}{x(z^{-1})} = \frac{(-0.2863 + 0.2928z^{-1} + 0.0221z^{-2} \\
-0.1558z^{-2} + 0.0809z^{-3} + 0.1420z^{-5} \\
-0.1254z^{-6} + 0.0281z^{-7})}{1 + 1.1590z^{-1} + 0.9428z^{-2} \\
+ 0.4996z^{-3} + 0.3044z^{-4} + 0.4881z^{-5} \\
+ 0.4003z^{-6} + 0.3660z^{-7})}
\]
Second-order systems

Second-order linear system

\[
(A_0 + A_1s + A_2s^2)x = Bu \\
y =Cx
\]

to be controlled by PD output-feedback controller

\[
u = -(F_0 + F_1s)y
\]

Applications: large flexible space structures, earthquake engineering, mechanical multi-body systems, damped gyroscopic systems, robotics control, vibration in structural dynamics, flows in fluid mechanics, electrical circuits

320m long Millenium footbridge over river Thames in London
PD controller

Closed-loop system behavior captured by zeros of quadratic polynomial matrix

\[ N(s) = (A_0 + BF_0C) + (A_1 + BF_1C)s + A_2s^2 \]

Zeros of \( N(s) \) must be located in some stability region \( D \) characterized as before by matrix \( H \)

**Uncertainty** can affect \( A_0 \) (stiffness) \( A_1 \) (damping) and \( A_2 \) (mass)

Given \( A_0, A_1, A_2, B, C \) find \( F_0, F_1 \) ensuring robust pole placement
Robust LMI stability condition

- **Norm-bounded** (unstructured) uncertainty
  \[ N(s) + \Delta M(s) \quad \sigma_{\text{max}}(\Delta) \leq \delta \]

LMI robust stability condition on \( N(s) \)

\[
\begin{bmatrix}
D^*N + N^*D - H(P) - \gamma D^*D & \delta M^* \\
\delta M & \gamma I
\end{bmatrix} \succ 0
\]

- **Polytopic** (structured) uncertainty
  \[ N(s) = \sum_i \lambda_i N^i(s) \quad \sum_i \lambda_i = 1 \quad \lambda_i \geq 0 \]

Vertex LMI robust stability conditions:

\[
DN^i + (N^i)^*D - H(P^i) \succ 0, \quad i = 1, 2, \ldots
\]

Parameter-dependent Lyapunov matrix
\[ P(\lambda) = \sum_i \lambda^i P^i \]
Robust design

Once central polynomial matrix $D(s)$ is fixed, robust stability condition is LMI in $N(s)$, so extension to design is straightforward.

Easy incorporation of structural constraints on controller coefficient matrices $F_0, F_1$:
- minimization of 2-norm (SOCP)
- enforcing some entries to zero (LP)
- maximization of uncertainty radius (SDP)

Key point is choice of central polynomial matrix

Good policy: set $D(s)$ to some nominal system matrix obtained by some standard design method (pole placement, LQ, $H_2$ or $H_\infty$), then try to optimize around $D(s)$
Example: five masses

Five masses linked by elastic springs controlled by two external forces

\[
A_0 = \begin{bmatrix}
2.565 & 1.080 & 0 & 0 & 1.089 \\
0.6038 & 0.8206 & 0.4766 & 0 & 0 \\
0 & 0.6009 & 1.504 & 0.4808 & 0 \\
0 & 0 & 0.4300 & 1.114 & 0.5131 \\
0.6190 & 0 & 0 & 0.4626 & 0.8352
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 1.964 \\
0 & 0 \\
0 & 0 \\
1.116 & 0
\end{bmatrix}
\]

Purely imaginary open-loop poles $\pm i 1.783, \pm i 1.380, \pm i 1.145, \pm i 0.5675$ and $\pm i 0.3507$

Nominal PD controller $F_0^0, F_1^0$ obtained with LQ design

Resulting central polynomial matrix

\[
D(s) = (A_0 + BF_0^0C) + (A_1 + BF_1^0C)s + A_2s^2
\]

Stability region $\mathcal{D} = \left\{ s : \text{Re} s < -0.1 \right\}$
Five mass example (2)

Minimizing the norm of feedback matrices $F_0$, $F_1$ over the design LMI yields

$$\| [F_0 \ F_1] \| = 0.7537 < \| [F_0^0 \ F_1^0] \| = 1.462$$

Closed-loop pseudospectrum of the five masses example