Port-Hamiltonian Systems: from Geometric Network Modeling to Control

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Part II (Tuesday afternoon): From Network Modeling to Port-Hamiltonian Systems

1. From junction structures to Dirac structures
2. Port-Hamiltonian systems
3. Examples
4. Input-state-output port-Hamiltonian systems
5. Multi-modal physical systems
6. Representations and transformations

In Part I we have seen how port-based network modeling of lumped-parameter physical systems leads to a representation of physical systems as generalized bond graphs, where each bond corresponds to a (vector) pair of flow variables $f$, and effort variables $e$.

- **Energy-storing elements:**
  \[
  \dot{x} = f_S \\
  e_S = \frac{\partial H}{\partial x}(x)
  \]

- **Power-dissipating elements:**
  \[
  R(f_R, e_R) = 0, \quad e_R^T f_R \geq 0
  \]

- **Power-conserving elements:** transformers, gyrators, (ideal) constraints.

- **0- and 1-junctions:**
  \[
  e_1 = e_2 = \cdots = e_k, \quad f_1 + f_2 + \cdots + f_k = 0 \\
  f_1 = f_2 = \cdots = f_k, \quad e_1 + e_2 + \cdots + e_k = 0
  \]

  **Note:** 0- and 1-junctions are also power-conserving:
  \[
  e_1 f_1 + e_2 f_2 + \cdots + e_k f_k = 0
  \]
• Transformers, gyrators, etc., are energy-routing devices, and may correspond to exchange between different types of energy.

• Ideal powerless constraints such as kinematic constraints.

• 0- and 1-junctions correspond to basic conservation laws such as Kirchhoff’s laws.

All power-conserving elements have the following properties in common. They are described by linear equations:

\[ Ff + Ee = 0, \quad f, e \in \mathbb{R}^l \]

satisfying

\[ e^T f = e_1 f_1 + e_2 f_2 + \cdots + e_l f_l = 0, \]

\[ \text{rank} \begin{bmatrix} F & E \end{bmatrix} = l \]

Geometric definition:

Definition 1 A (constant) Dirac structure on a finite-dimensional space \( V \) is a subspace \( D \subset V \times V^* \) such that

(i) \( e^T f = 0 \) for all \( (f, e) \in D \),

(ii) \( \dim D = \dim V \).

For any skew-symmetric map \( J : V^* \to V \) its graph \( \{(f, e) \in V \times V^* | f = J e\} \) is a Dirac structure!

Alternative, more general, definition of Dirac structure

Power is defined by

\[ P = e(f) =: < e | f > = e^T f, \quad (f, e) \in V \times V^*. \]

where the linear space \( V \) is called the space of flows \( f \) (e.g. currents), and \( V^* \) the space of efforts \( e \) (e.g. voltages).

Symmetrized form of power is the indefinite bilinear form \( \langle, \rangle \) on \( V \times V^* \):

\[ \langle (f^a, e^a), (f^b, e^b) \rangle := < e^a | f^b > + < e^b | f^a >, \]

\[ (f^a, e^a), (f^b, e^b) \in V \times V^*. \]
An $k$ dimensional storage element is determined by a $k$-dimensional state vector $x = (x_1, \cdots, x_k)$ and a Hamiltonian $H(x_1, \cdots, x_k)$ (energy storage), defining the lossless system

\[
\dot{x}_i = f_{Si}, \quad i = 1, \cdots, k \\
e_{Si} = \frac{\partial H}{\partial x_i}(x_1, \cdots, x_k) \\
\frac{d}{dt}H = \sum_{i=1}^{k} f_{Si}e_{Si}
\]

Such a $k$-dimensional storage component is written in vector notation as:

\[
\dot{x} = f_S \\
e_S = \frac{\partial H}{\partial x}(x)
\]

The elements of $x$ are called energy variables, those of $\frac{\partial H}{\partial x}(x)$ co-energy variables.

Basic property

\[
\frac{dH}{dt}(x(t)) = \frac{\partial H}{\partial x}(x(t))\dot{x}(t) = -e^T_x(t)f_x(t) = e^T_p(t)f^T_p(t)
\]

Example: The ubiquitous mass-spring-damper system:

Two storage elements:

- **Spring** Hamiltonian $H_s(q) = \frac{1}{2}kq^2$ (potential energy)

  \[
  \dot{q} = f_s, \quad = \text{velocity} \\
e_s = \frac{\partial H_s}{\partial q}(q) = kq \quad = \text{force}
  \]

- **Mass** Hamiltonian $H_m(p) = \frac{1}{2m}p^2$ (kinetic energy)

  \[
  \dot{p} = f_m, \quad = \text{force} \\
e_m = \frac{\partial H_m}{\partial p}(p) = \frac{p}{m} \quad = \text{velocity}
  \]

Interconnected by the Dirac structure

\[
f_s = e_m = y, \quad f_m = -e_s + u
\]

(power-conserving since $f_se_s + f_me_m = uy$) yields the port-Hamiltonian system

\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} 
\begin{bmatrix}
\frac{\partial H_s(q,p)}{\partial q} \\
\frac{\partial H_s(q,p)}{\partial p}
\end{bmatrix} + 
\begin{bmatrix}
0 \\
1
\end{bmatrix} u
\]

\[
y = 
\begin{bmatrix}
0 & 1
\end{bmatrix} 
\begin{bmatrix}
\frac{\partial H_s(q,p)}{\partial q} \\
\frac{\partial H_s(q,p)}{\partial p}
\end{bmatrix}
\]

with

\[
H(q,p) = H_s(q) + H_m(p)
\]
**Power-dissipation** is included by adding an **extra port** to the Dirac structure, terminated by power-conserving relations:

\[ R(f_R, e_R) = 0, \quad e_R f_R^T \geq 0 \]

**Example:** For the mass-spring system, the addition of the damper

\[ e_d = \frac{dR}{df_d} = c f_d, \quad R(f_d) = \frac{1}{2} c f_d^2 \] (Rayleigh function)

via the extended interconnection (Dirac structure)

\[ f_s = e_m = f_d = y, \quad f_m = e_s - e_d + u \]

leads to the **mass-damper-spring system**

\[
\begin{bmatrix}
\dot{q} \\
\dot{p} \\
y
\end{bmatrix} = 
\begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial q}(q, p) \\
\frac{\partial H}{\partial p}(q, p) \\
\frac{\partial H}{\partial \phi}(q, p)
\end{bmatrix} + 
\begin{bmatrix}
0 \\
1
\end{bmatrix} u
\]

Coupling electrical/mechanical domain via Hamiltonian \( H(q, p, \phi) \).

**Example: Electro-mechanical systems**

**Example: LC circuits**

Two inductors with magnetic energies \( H_1(\varphi_1), H_2(\varphi_2) \) (\( \varphi_1 \) and \( \varphi_2 \) magnetic flux linkages), and capacitor with electric energy \( H_3(Q) \) (\( Q \) charge).

\( V \) denotes the voltage of the source.

\[
\dot{\varphi}_1 = \frac{\partial H_1}{\partial \varphi_1}(\varphi_1, p) \quad (voltage) \\
\dot{\varphi}_1 = \frac{\partial H_1}{\partial \varphi_1}(\varphi_1, p) \quad (current) \\
\dot{Q} = \frac{\partial H_3}{\partial Q}(Q, p) \quad (voltage)
\]

All are port-Hamiltonian systems with \( J = 0 \) and \( g = 1 \).

If the elements are **linear** then the Hamiltonians are **quadratic**, e.g. \( H_1(\varphi) = \frac{1}{2L_1} \varphi_1^2 \), and \( \frac{\partial H_1}{\partial \varphi_1} = \frac{\varphi_1}{L_1} \) current, etc.
Kirchhoff’s interconnection laws in \( f_1, f_2, f_3, e_1, e_2, e_3, f = V, e = I \) are
\[
\begin{bmatrix}
-f_1 \\
-f_2 \\
-f_3 \\
e
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & -1 & e_1 \\
0 & 0 & -1 & 0 & e_2 \\
-1 & 1 & 0 & 0 & e_3 \\
1 & 0 & 0 & 0 & f
\end{bmatrix}
\]
Substitution of eqns. of components yields port-Hamiltonian system
\[
\begin{bmatrix}
\dot{\varphi}_1 \\
\dot{\varphi}_2 \\
\dot{Q}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & -1 & \frac{\partial H}{\partial \varphi_1} \\
0 & 0 & 1 & \frac{\partial H}{\partial \varphi_2} \\
1 & -1 & 0 & \frac{\partial H}{\partial Q}
\end{bmatrix} +
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} f
\]
with \( H(\varphi_1, \varphi_2, Q) := H_1(\varphi_1) + H_2(\varphi_2) + H_3(Q) \) total energy.

Network modeling is prevailing in modeling and simulation of lumped-parameter physical systems (multi-body systems, electrical circuits, electro-mechanical systems, hydraulic systems, robotic systems, etc.), with many advantages:

- Modularity and flexibility. Re-usability (‘libraries’).
- Multi-physics approach.
- Suited to design/control.

Disadvantage of network modeling: it generally leads to a large set of DAEs, seemingly without any structure.

Port-based modeling and port-Hamiltonian system theory identifies the underlying structure of network models of physical systems, to be used for analysis, simulation and control.

However, this class of port-Hamiltonian systems is not closed under interconnection:

![Figure 2: Capacitors and inductors swapped](image)

Interconnection leads to algebraic constraints between the state variables \( Q_1 \) and \( Q_2 \).

For many systems, especially those with 3-D mechanical components, the interconnection structure will be modulated by the energy or geometric variables.

This leads to the notion of non-constant Dirac structures on manifolds.

**Definition 3** Consider a smooth manifold \( M \). A Dirac structure on \( M \) is a vector subbundle \( D \subset TM \oplus T^*M \) such that for every \( x \in M \) the vector space
\[
D(x) \subset T_xM \times T^*_xM
\]
is a Dirac structure as before.
Example: Mechanical systems with kinematic constraints

Constraints on the generalized velocities $\dot{q}$:

$$A^T(q)\dot{q} = 0.$$  

This leads to constrained Hamiltonian equations

$$\begin{align*}
\dot{q} &= \frac{\partial H}{\partial p}(q, p) \\
\dot{p} &= -\frac{\partial H}{\partial q}(q, p) + A(q)\lambda + B(q)f \\
0 &= A^T(q)\frac{\partial H}{\partial p}(q, p) \\
e &= B^T(q)\frac{\partial H}{\partial p}(q, p)
\end{align*}$$

with $H(q, p)$ total energy, and $\lambda$ the constraint forces.

Dirac structure is defined by the symplectic form on $T^*Q$ together with constraints $A^T(q)\dot{q} = 0$ and force matrix $B(q)$.

Can be systematically extended to general multi-body systems.

Example 4 (Rolling coin) Let $x, y$ be the Cartesian coordinates of the point of contact of the coin with the plane. Furthermore, $\varphi$ denotes the heading angle, and $\theta$ the angle of Queen Beatrix’ head (on the Dutch version of the euro).

The rolling constraints are

$$\begin{align*}
\dot{x} &= \dot{\theta} \cos \varphi, \\
\dot{y} &= \dot{\theta} \sin \varphi
\end{align*}$$

(rolling without slipping). The total energy is

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}p_\theta^2 + \frac{1}{2}p_\varphi^2,$$

and the constraints can be rewritten as

$$p_x = p_\theta \cos \varphi, \quad p_y = p_\theta \sin \varphi.$$  

Mathematical intermezzo: Jacobi identity and holonomic constraints

There is an important notion of integrability of a Dirac structure on a manifold.

**Definition 5** A Dirac structure $\mathcal{D}$ on a manifold $M$ is called integrable if

$$<L_{X_1}\alpha_2 | X_3> + <L_{X_2}\alpha_3 | X_1> + <L_{X_3}\alpha_1 | X_2> = 0$$

for all $(X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3) \in \mathcal{D}$.

For constant Dirac structures the integrability condition is automatically satisfied.

The Dirac structure $\mathcal{D}$ defined by the canonical symplectic structure and kinematic constraints $A^T(q)\dot{q} = 0$ satisfies the integrability condition if and only if the constraints are holonomic; that is, can be integrated to geometric constraints $\phi(q) = 0$.

Examples

(a) Let $J$ be a (pseudo-)Poisson structure on $M$, defining a skew-symmetric mapping $J : T^*M \to TM$. Then graph $J \subset T^*M \oplus TM$ is a Dirac structure.

Integrability is equivalent to the Jacobi-identity for the Poisson structure.

(b) Let $\omega$ be a (pre-)symplectic structure on $M$, defining a skew-symmetric mapping $\omega : TM \to T^*M$. Then graph $\omega \subset TM \oplus T^*M$ is a Dirac structure.

Integrability is equivalent to the closedness of the symplectic structure.

(c) Let $K$ be a constant-dimensional distribution on $M$, and let $annK$ be its annihilating co-distribution. Then $K \times annK \subset TM \oplus T^*M$ is a Dirac structure.

Integrability is equivalent to the involutivity of distribution $K$. 
Input-state-output port-Hamiltonian systems:

Particular case is a Dirac structure \( D(x) \subset T_x \mathcal{X} \times T^*_x \mathcal{X} \times \mathcal{F} \times \mathcal{F}^* \) given as the graph of the skew-symmetric map

\[
\begin{bmatrix}
  f_x \\
  e_P
\end{bmatrix} = \begin{bmatrix}
  -J(x) & -g(x) \\
  g^T(x) & 0
\end{bmatrix}
\begin{bmatrix}
  e_x \\
  f_P
\end{bmatrix},
\]

leading \( (f_x = -\dot{x}, e_x = \frac{\partial H}{\partial x}(x)) \) to a port-Hamiltonian system as before

\[
\begin{align*}
\dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)f_P, \quad x \in \mathcal{X}, f_P \in \mathbb{R}^m \\
e_P &= g^T(x)\frac{\partial H}{\partial x}(x), \quad e_P \in \mathbb{R}^m
\end{align*}
\]

Power-dissipation is included by terminating some of the ports by static resistive elements

\[
f_R = -F(e_R), \quad \text{where } e_H^T F(e_R) \geq 0, \quad \text{for all } e_R.
\]

This leads, e.g., for linear damping, to input-state-output port-Hamiltonian systems in the form

\[
\begin{align*}
\dot{x} &= [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)f_P \\
e_P &= g^T(x)\frac{\partial H}{\partial x}(x)
\end{align*}
\]

where \( J(x) = -J^T(x), R(x) = R^T(x) \geq 0 \) are the interconnection and damping matrices, respectively.

Example 6 (Boost converter) The circuit consists of an inductor \( L \) with magnetic flux linkage \( \phi_L \), a capacitor \( C \) with electric charge \( q_C \) and a resistance load \( R \), together with a diode and an ideal switch \( S \), with switch positions \( s = 1 \) (switch closed) and \( s = 0 \) (switch open).

The diode is modeled as an ideal diode:

\[
v_D i_D = 0, \quad v_D \leq 0, \quad i_D \geq 0.
\]

we Port-Hamiltonian model (with \( H = \frac{1}{2} q_C^2 + \frac{1}{2} \phi_L^2 \)):

\[
\begin{bmatrix}
  \dot{q}_C \\
  \dot{\phi}_L
\end{bmatrix} = \begin{bmatrix}
  -\frac{1}{R} & 1-s \\
  s-1 & 0
\end{bmatrix}\begin{bmatrix}
  \frac{\partial H}{\partial q_C} = \frac{q_C}{C} \\
  \frac{\partial H}{\partial \phi_L} = \frac{\phi_L}{L}
\end{bmatrix} + \begin{bmatrix}
  0 \\
  1
\end{bmatrix} E + \begin{bmatrix}
  s i_D \\
  (s-1) v_D
\end{bmatrix}
\]

\[
I = \frac{\phi_L}{L}
\]
Example 7 (Bouncing pogo-stick) Consider a vertically bouncing pogo-stick consisting of a mass \( m \) and a massless foot, interconnected by a linear spring (stiffness \( k \) and rest-length \( x_0 \)) and a linear damper \( d \).

The mass can move vertically under the influence of gravity \( g \) until the foot touches the ground. The states of the system are \( x \) (length of the spring), \( y \) (height of the bottom of the mass), and \( p \) (momentum of the mass, defined as \( p := m\dot{y} \)). Furthermore, the contact situation is described by a variable \( s \) with values \( s = 0 \) (no contact) and \( s = 1 \) (contact). The Hamiltonian of the system equals

\[
H(x, y, p) = \frac{1}{2}k(x - x_0)^2 + mg(y + y_0) + \frac{1}{2m}p^2 \tag{5}
\]

where \( y_0 \) is the distance from the bottom of the mass to its center of mass.

When the foot is not in contact with the ground total force on the foot is zero (since it is massless), which implies that the spring and damper force must be equal but opposite. When the foot is in contact with the ground, the variables \( x \) and \( y \) remain equal, and hence also \( \dot{x} = \dot{y} \).

For \( s = 0 \) (no contact) the system is described by the port-Hamiltonian system

\[
\frac{d}{dt} \begin{bmatrix} y \\ p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} mg \\ \frac{p}{m} \end{bmatrix}
\]

\[-d \dot{x} = k(x - x_0) \tag{6}\]

while for \( s = 1 \) the port-Hamiltonian description is

\[
\frac{d}{dt} \begin{bmatrix} x \\ y \\ p \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -d \end{bmatrix} \begin{bmatrix} k(x - x_0) \\ mg \\ \frac{p}{m} \end{bmatrix} \tag{7}\]
The two situations can be taken together into one port-Hamiltonian system with variable Dirac structure:

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
s & -1 & 0 & s \\
0 & 0 & 1 & 0 \\
-s & -1 & -sd & 0
\end{bmatrix}
\begin{bmatrix}
k(x - x_0) \\
mg \\
\frac{p_m}{m}
\end{bmatrix}
\]

The conditions for switching of the contact are functions of the states, namely as follows: contact is switched from off to on when \( y - x \) crosses zero in the negative direction, and contact is switched from on to off when the velocity \( \dot{y} - \dot{x} \) of the foot is positive in the no-contact situation, i.e. when \( \frac{p_m}{m} + \frac{k}{d}(x - x_0) > 0 \).

In both examples above we obtain a **switching** port-Hamiltonian system, specified by a Dirac structure \( D_s \) depending on the switch position \( s \in \{0, 1\} \) (here \( n \) denotes the number of independent switches), a Hamiltonian \( H : \mathcal{X} \to \mathbb{R} \) and a resistive structure \( \mathcal{R} \). Furthermore, every switching may be internally induced (like in the case of a diode in an electrical circuit or an impact in a mechanical system) or externally triggered (like an active switch in a circuit or mechanical system).

**Problems**

- Well-posedness questions: e.g., systems with reverse Coulomb friction may have multiple solutions.
- Computation of the next mode may be difficult.
- Collision rules.
- Investigation of limit cycles/periodic orbits.

**Representations and Transformations**

Dirac structures, and therefore port-Hamiltonian systems, admit different **representations**, with different properties for simulation and control.

Let \( \mathcal{D} \subset \mathcal{V} \times \mathcal{V}^* \), with \( \dim \mathcal{V} = n \), be a Dirac structure.

1. **Kernel and Image representation**

\( \mathcal{D} = \{(f, e) \in \mathcal{V} \times \mathcal{V}^* \mid Ff + Ee = 0\} \), for \( n \times n \) matrices \( F \) and \( E \) (possibly depending on \( x \)) satisfying

\[
(i) \quad EF^T + FE^T = 0,
\]

\[
(ii) \quad \text{rank}[F; E] = n.
\]

It follows that \( \mathcal{D} \) can be also written in image representation as

\( \mathcal{D} = \{(f, e) \in \mathcal{V} \times \mathcal{V}^* \mid f = E^T\lambda, e = F^T\lambda, \lambda \in \mathbb{R}^n\} \).

2. **Constrained input-output representation**

Every Dirac structure \( \mathcal{D} \) can be written as

\( \mathcal{D} = \{(f, e) \in \mathcal{V} \times \mathcal{V}^* \mid f = Je + G\lambda, G^Te = 0\} \)

for a skew-symmetric matrix \( J \) and a matrix \( G \) such that

\( \text{im} \ G = \{f \mid (f, 0) \in \mathcal{D}\} \).

Furthermore, \( \ker J = \{e \mid (0, e) \in \mathcal{D}\} \).
3. Hybrid input-output representation

Let $\mathcal{D}$ be given by square matrices $E$ and $F$ as in 1. Suppose rank $F = m (\leq n)$. Select $m$ independent columns of $F$, and group them into a matrix $F_1$. Write (possibly after permutations) $F = [F_1; F_2]$, and correspondingly $E = [E_1; E_2]$, $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, $e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$.

Then the matrix $[F_1; E_2]$ is invertible, and

$$\mathcal{D} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = J \begin{bmatrix} e_1 \\ f_2 \end{bmatrix}$$

with $J := -[F_1; E_2]^{-1}[F_2; E_1]$ skew-symmetric.

4. Canonical coordinates

For simplicity take $\mathcal{F} \times \mathcal{F}^*$ to be void (no external ports). If the Dirac structure on $\mathcal{X}$ is integrable then there exist coordinates $(q, p, r, s)$ for $\mathcal{X}$ such that

$$\mathcal{D}(x) = \{(f_q, f_p, f_r, f_s, e_q, e_p, e_r, e_s) \in T_x \mathcal{X} \times T^*_x \mathcal{X}\}$$

$$\begin{cases} f_q = -e_p, \quad f_p = e_q \\ f_r = 0, \quad 0 = e_s \end{cases}$$

Hence the port-Hamiltonian system on $\mathcal{X}$ takes the form

$$\dot{q} = \frac{\partial H}{\partial p}(q, p, r, s)$$
$$\dot{p} = -\frac{\partial H}{\partial q}(q, p, r, s)$$
$$\dot{r} = 0$$
$$0 = \frac{\partial H}{\partial s}(q, p, r, s)$$

Mixture of constrained and hybrid input-output representation

By a hybrid input-output partition of the vector of port flows $(f, e) \in \mathcal{F} \times \mathcal{F}^*$ as $(u, y)$ we can represent any port-Hamiltonian system in constrained form as

$$\begin{cases} \dot{x} = J(x) \frac{\partial H}{\partial x}(x) + G(x) \lambda + g(x)u, \quad x \in \mathcal{X}, u \in \mathbb{R}^m \\ 0 = G^T(x) \frac{\partial H}{\partial x}(x) + D(x)u, \\ y = y^T(x) \frac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^m \end{cases}$$

where

$$J(x) = -J^T(x), \quad D(x) = -D^T(x)$$

This is the form as encountered before in the case of kinematic constraints.
Intermezzo: Relation with classical Hamiltonian equations

\[ \dot{x} = J(x) \frac{\partial H}{\partial x}(x) \]

with constant or 'integrable' $J$-matrix admits coordinates $x = (q, p, r)$ in which

\[
J = \begin{bmatrix}
0 & I & 0 \\
-I & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \dot{q} = \frac{\partial H}{\partial p}(q, p, r) \\
\dot{p} = -\frac{\partial H}{\partial q}(q, p, r) \\
\dot{r} = 0
\]

For constant or integrable Dirac structure one gets Hamiltonian DAEs

\[
\dot{q} = \frac{\partial H}{\partial p}(q, p, r, s) \\
\dot{p} = -\frac{\partial H}{\partial q}(q, p, r, s) \\
\dot{r} = 0 \\
0 = \frac{\partial H}{\partial s}(q, p, r, s)
\]

Recall of Hamiltonian dynamical systems from analytical mechanics

Historically, the Hamiltonian approach starts from the principle of least action, via the Euler-Lagrange equations and the Legendre transformation, towards the Hamiltonian equations of motion.

The standard Euler-Lagrange equations are given as

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right) - \frac{\partial L}{\partial q}(q, \dot{q}) = \tau,
\]

where $q = (q_1, \ldots, q_k)^T$ are generalized configuration coordinates for the system with $k$ degrees of freedom, the Lagrangian $L$ equals the difference $T - P$ between kinetic co-energy $T$ and potential energy $P$, and $\tau = (\tau_1, \ldots, \tau_k)^T$ is the vector of generalized forces acting on the system.

The vector of generalized momenta $p = (p_1, \ldots, p_k)^T$ is defined as

\[
p = \frac{\partial L}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{q}}
\]

By defining the state vector $(q_1, \ldots, q_k, p_1, \ldots, p_k)^T$ the $k$ second-order equations transform into $2k$ first-order equations

\[
\dot{q} = \frac{\partial H}{\partial p}(q, p) \\
\dot{p} = -\frac{\partial H}{\partial q}(q, p) + \tau
\]

where the Legendre transform

\[ H(q, p) = K(q, p) + P(q) \]

is the total energy of the system.
The above equations are called the Hamiltonian equations of motion, and \( H \) is called the Hamiltonian. The state space with local coordinates \((q, p)\) is called the phase space.

The following energy balance immediately follows:

\[
\frac{d}{dt} H = \frac{\partial T}{\partial q}(q, p) \dot{q} + \frac{\partial T}{\partial p}(q, p) \dot{p} = \frac{\partial T}{\partial q}(q, p) \tau = \dot{q}^T \tau,
\]

expressing that the increase in energy of the system is equal to the supplied work (conservation of energy).

A major generalization of the class of Hamiltonian systems consists in considering systems which are described in local coordinates as

\[
\dot{x} = J(x) \frac{\partial H}{\partial x}(x) + g(x)u, \quad x \in X, u \in \mathbb{R}^m
\]

\[
y = g^T(x) \frac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^m
\]

Here \( J(x) \) is an \( n \times n \) matrix which is skew-symmetric:

\[
J(x) = -J^T(x),
\]

and \( x = (x_1, \ldots, x_n) \) are local coordinates for an \( n \)-dimensional state space manifold \( X \). We recover the energy-balance

\[
\frac{d}{dt} H(x(t)) = u^T(t)y(t). \quad \text{In the previous case we had } J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

A Hamiltonian system with collocated inputs and outputs is more generally given in the following form

\[
\dot{q} = \frac{\partial H}{\partial p}(q, p), \quad (q, p) = (q_1, \ldots, q_k, p_1, \ldots, p_k)
\]

\[
\dot{p} = -\frac{\partial H}{\partial q}(q, p) + B(q)u, \quad u \in \mathbb{R}^m
\]

\[
y = B^T(q) \frac{\partial H}{\partial p}(q, p) \quad (= B^T(q) \dot{q}), \quad y \in \mathbb{R}^m,
\]

Here \( B(q) \) is the input force matrix. In case \( m < k \) we speak of an underactuated system.

By definition of the output \( y = B^T(q) \dot{q} \) we again obtain

\[
\frac{d}{dt} H(q(t), p(t)) = u^T(t)y(t).
\]

**Example 8** Consider a rigid body spinning around its center of mass in the absence of gravity. The energy variables are the three components of the body angular momentum \( p \) along the three principal axes: \( p = (p_x, p_y, p_z) \), and the energy is the kinetic energy

\[
H(p) = \frac{1}{2} \left( \frac{p_x^2}{I_x} + \frac{p_y^2}{I_y} + \frac{p_z^2}{I_z} \right),
\]

where \( I_x, I_y, I_z \) are the principal moments of inertia. Euler’s equations are

\[
\begin{bmatrix}
\dot{p}_x \\
\dot{p}_y \\
\dot{p}_z
\end{bmatrix} =
\begin{bmatrix}
0 & -p_z & p_y \\
p_z & 0 & -p_x \\
-p_y & p_x & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial p_x} \\
\frac{\partial H}{\partial p_y} \\
\frac{\partial H}{\partial p_z}
\end{bmatrix}
+ g(p)u,
\quad y = g^T(p) \frac{\partial H}{\partial p}
\]
$J(p)$ is the canonical Lie-Poisson structure matrix on the dual of the Lie algebra $so(3)$ corresponding to the configuration space $SO(3)$ of the rigid body.

Equations arise from the standard (6-dimensional) Hamiltonian equations by reduction ("symmetry").

Thus

$D_A || D_B := \{(f_1, e_1, f_3, e_3) \in F_1 \times F_1^* \times F_3 \times F_3^* \mid \exists (f_2, e_2) \in F_2 \times F_2^* \text{ s.t.} (f_1, e_1, f_2, e_2) \in D_A \text{ and } (-f_2, e_2, f_3, e_3) \in D_B\}$

**Theorem 9** Let $D_A$, $D_B$ be Dirac structures (defined with respect to $F_1 \times F_1^* \times F_2 \times F_2^*$, respectively $F_2 \times F_2^* \times F_3 \times F_3^*$ and their bilinear forms). Then $D_A || D_B$ is a Dirac structure with respect to the bilinear form on $F_1 \times F_1^* \times F_3 \times F_3^*$.
Proof

Consider $D_A, D_B$ defined in matrix kernel representation by

$$D_A = \{(f_1, e_1, f_A, e_A) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^* \mid F_1 f_1 + E_1 e_1 + F_2 A f_A + E_2 A e_A = 0\}$$

$$D_B = \{(f_B, e_B, f_3, e_3) \in \mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^* \mid F_2 B f_B + E_2 B e_B + F_3 f_3 + E_3 e_3 = 0\}$$

Make use of the following basic fact from linear algebra:

$$(\exists \lambda \text{ s.t. } A \lambda = b) \Leftrightarrow [\forall \alpha \text{ s.t. } \alpha^T A = 0 \Rightarrow \alpha^T b = 0]$$

Note that $D_A, D_B$ are alternatively given in matrix image representation as

$$D_A = \text{im} \begin{bmatrix} E_1^T & 0 \\ F_1^T & 0 \\ E_{2A}^T & E_{2B}^T \\ F_{2A}^T & -F_{2B}^T \\ 0 & F_3^T \\ 0 & E_3^T \end{bmatrix} \quad D_B = \text{im} \begin{bmatrix} 0 \\ 0 \\ E_{2B}^T \\ F_{2B}^T \\ E_3^T \\ F_3^T \end{bmatrix}$$

Hence, $(f_1, e_1, f_3, e_3) \in D_A \parallel D_B \Leftrightarrow \exists \lambda_A, \lambda_B$ such that

$$\begin{bmatrix} f_1 \\ e_1 \\ 0 \\ 0 \\ f_3 \\ e_3 \end{bmatrix} = \begin{bmatrix} E_1^T & 0 \\ F_1^T & 0 \\ E_{2A}^T & E_{2B}^T \\ F_{2A}^T & -F_{2B}^T \\ 0 & F_3^T \\ 0 & E_3^T \end{bmatrix} \begin{bmatrix} \lambda_A \\ \lambda_B \end{bmatrix} \Leftrightarrow [\forall (\beta_1, \alpha_1, \beta_2, \alpha_2, \beta_3, \alpha_3) \text{ s.t.}]

$$

$$\begin{bmatrix} E_{2A} & E_{2B} \\ F_{2A} & -F_{2B} \\ 0 & F_3 \\ 0 & E_3 \end{bmatrix} = 0,$$
\( \Rightarrow (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) \) s.t. 
\[
\begin{bmatrix}
F_1 & E_1 & F_{2A} & E_{2A} & 0 & 0 \\
0 & 0 & -F_{2B} & E_{2B} & F_3 & E_3
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\beta_1 \\
\alpha_2 \\
\beta_2 \\
\alpha_3 \\
\beta_3
\end{bmatrix} = 0,
\]

Thus \( \mathcal{D}_A \parallel \mathcal{D}_B = (\mathcal{D}_A \parallel \mathcal{D}_B)^\perp \), and so it is a Dirac structure.

This relaxed kernel/image representation can be readily understood by premultiplying the equations characterizing the composition of \( \mathcal{D}_A \) with \( \mathcal{D}_B \)

\[
\begin{bmatrix}
F_1 & E_1 & F_{2A} & E_{2A} & 0 & 0 \\
0 & 0 & -F_{2B} & E_{2B} & F_3 & E_3
\end{bmatrix}
\begin{bmatrix}
f_1 \\
e_1 \\
f_2 \\
e_2 \\
f_3 \\
e_3
\end{bmatrix} = 0,
\]

by the matrix \( L := [L_A|L_B] \). Since \( LM = 0 \) this results indeed in the relaxed kernel representation

\[
L_A F_1 f_1 + L_A E_1 e_1 + L_B F_3 f_3 + L_B E_3 e_3 = 0
\]
Example 10 (1-D mechanical system) Consider a spring with elongation $q$ and energy function $H_s(q) = \frac{1}{2}kq^2$. Let $(v_s, F_s)$ represent the external port through which energy can be exchanged with the spring, where $v_s$ is equal to the rate of elongation (velocity) and $F_s$ is equal to the elastic force. This port-Hamiltonian system can be written in kernel representation as

$$
\begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
-\dot{q} \\
v_s
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
kq \\
F_s
\end{bmatrix} = 0
$$

Similarly we model a moving mass $m$ with scalar momentum $p$ and kinetic energy $H_m(p) = \frac{1}{2}p^2$ as the port-Hamiltonian system

$$
\begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
-\dot{p} \\
F_m
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
p \\
v_m
\end{bmatrix} = 0
$$

where $(F_m, v_m)$ are respectively the external force exerted on the mass and the velocity of the mass.

The mass and the spring can be interconnected to each other using the symplectic gyrator

$$
\begin{bmatrix}
v_s \\
F_m
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{F}_s \\
v_m
\end{bmatrix}
$$

Collecting all equations we have obtained a port-Hamiltonian system with energy variables $x = (q, p)$, total energy $H(q, p) = H_s(q) + H_m(p)$ and with interconnection port variables $(v_s, F_s, F_m, v_m)$. After elimination of the interconnection variables $(v_s, F_s, F_m, v_m)$ one obtains the port-Hamiltonian system

$$
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
-\dot{q} \\
-\dot{p}
\end{bmatrix}
+ \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
kq \\
\frac{p}{m}
\end{bmatrix} = 0
$$

which is the ubiquitous mass-spring system.

Example: Coupled masses Consider two point masses $m_1$ and $m_2$ which are rigidly linked to each other. When decoupled the masses are described by the port-Hamiltonian systems

$$
\begin{align*}
\dot{p}_i &= F_i, \\
v_i &= \frac{p_i}{m_i}, \\
i &= 1, 2
\end{align*}
$$

with $F_i$ the force exerted on mass $m_i$. Rigid coupling amounts to

$$
F_1 = -F_2, \quad v_1 = v_2
$$

This leads to the port-Hamiltonian system

$$
\begin{bmatrix}
p_1 \\
p_2
\end{bmatrix} = \begin{bmatrix}
1 \\
-1
\end{bmatrix}\lambda
$$

where $\lambda = F_1 = -F_2$ now denotes the internal constraint force.
The resulting interconnected system does not have external ports anymore. On the other hand, external ports for the interconnected system can be included by either extending (16) to
\[
\dot{p}_i = F_i + F_i^{\text{ext}} \\
v_i = \frac{p_i}{m_i} \\
v_i^{\text{ext}} = \frac{p_i^{\text{ext}}}{m_i}
\]
with $F_i^{\text{ext}}$ and $v_i^{\text{ext}}$ denoting the external forces and velocities, or by modifying the interconnection constraints (17) to e.g.
\[
F_1 + F_2 + F^{\text{ext}} = 0, \quad v_1 = v_2 = v^{\text{ext}},
\]
with $F^{\text{ext}}$ and $v^{\text{ext}}$ denoting the external force exerted on the coupled masses, respectively the velocity of the coupled masses.

Consider the port-Hamiltonian system
\[
\dot{x} = J(x) \frac{\partial H}{\partial x}(x) + g(x)u + b(x)\lambda \\
\Sigma : \quad y = g^T(x) \frac{\partial H}{\partial x}(x) \\
\quad 0 = b^T(x) \frac{\partial H}{\partial x}(x)
\]
The Lagrange multipliers $\lambda$ can be eliminated by constructing a matrix $b^\perp(x)$ of maximal rank such that
\[
b^\perp(x)b(x) = 0
\]
By premultiplication with $b^\perp(x)$ one obtains
\[
b^\perp(x)\dot{x} = b^\perp(x)J(x) \frac{\partial H}{\partial x}(x) + b^\perp(x)g(x)u
\]
\[
\Sigma : \quad y = g^T(x) \frac{\partial H}{\partial x}(x) \\
\quad 0 = b^T(x) \frac{\partial H}{\partial x}(x)
\]
This is a kernel representation of the port-Hamiltonian system.

**Example 11 (Coupled masses continued)** Consider the system of two coupled masses. Premultiplication of the dynamic equations by the row vector $[1 \ 1]$ yields the equations
\[
\dot{p}_1 + \dot{p}_2 = 0, \quad p_1 \frac{1}{m_1} - p_2 \frac{1}{m_2} = 0,
\]
which constitutes a kernel representation of the port-Hamiltonian system.
A more difficult question concerns the possibility to solve for the algebraic constraints of a port-Hamiltonian system:

$$0 = b^T(x) \frac{\partial H}{\partial x}(x)$$  (23)

Under constant rank assumptions the set

$$\mathcal{X}_c := \{ x \in \mathcal{X} \mid b^T(x) \frac{\partial H}{\partial x}(x) = 0 \}$$

defines a submanifold of the total state space $\mathcal{X}$; the constrained state space. In order that this constrained state space qualifies as the state space for a port-Hamiltonian system without further algebraic constraints one needs to be able to restrict the dynamics of the port-Hamiltonian system to the constrained state space. This is always possible under the condition that the matrix

$$b^T(x) \frac{\partial^2 H}{\partial x^2}(x)b(x)$$  (24)

has full rank, since in this case the differentiated constraint equations

$$0 = \frac{d}{dt} (b^T(x) \frac{\partial H}{\partial x}(x)) = \ast + b^T(x) \frac{\partial^2 H}{\partial x^2}(x)b(x)\lambda$$  (25)

can be always uniquely solved for $\lambda$.

**Example 12 (Coupled masses continued)**

Differentiating the constraint equation $\frac{p_1}{m_1} - \frac{p_2}{m_2} = 0$ and using $\dot{p}_1 = \lambda$ and $\dot{p}_2 = -\lambda$ one obtains

$$\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\lambda = 0$$  (26)

which determines the constraint force $\lambda$ (to be equal to 0).

Defining the total momentum $p = p_1 + p_2$ one obtains the reduced system $\dot{p} = \dot{p}_1 + \dot{p}_2 = 0$.

On the other hand, suppose that the mass $m_1$ is connected to a linear spring with spring constant $k_1$ and elongation $q_1$ and that the mass $m_2$ is connected to a linear spring with spring constant $k_2$ and elongation $q_2$. Then the dynamical equations change into $\dot{p}_1 = -k_1 q_1 + \lambda$ and $\dot{p}_2 = -k_2 q_2 - \lambda$, and differentiation of the constraint $\frac{p_1}{m_1} - \frac{p_2}{m_2} = 0$ leads to

$$-\frac{k_1}{m_1}q_1 + \frac{k_2}{m_2}q_2 + \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\lambda = 0$$  (27)

which determines the constraint force $\lambda$ as $\lambda = \frac{m_1 m_2}{m_1 + m_2} (\frac{k_1}{m_1}q_1 - \frac{k_2}{m_2}q_2)$, and results in the dynamical equation for the total momentum $p$ given by

$$\dot{p} = -k_1 q_1 - k_2 q_2$$  (28)
Consider the equations of a general mechanical system subject to kinematic constraints. The constrained Hamiltonian equations define a port-Hamiltonian system, with respect to the Dirac structure \( \mathcal{D} \) (in constrained input-output representation)

\[
\mathcal{D} = \{ (f_s, e_s, f_c, e_C) \mid 0 = \begin{bmatrix} 0 & A^T(q) \\ B(q) \end{bmatrix} \begin{bmatrix} e_s \\ e_C \end{bmatrix}, \quad e_s = [0 \ B^T(q)] e_s, \quad -f_s = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} e_s + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} f_c, \quad \lambda, f_c, e_C, e_s \in \mathbb{R}^k \}
\]

The algebraic constraints on the state variables \((q, p)\) are

\[
0 = A^T(q) \frac{\partial H}{\partial p}(q, p)
\]

and the constrained state space is \( \mathcal{X}_c = \{ (q, p) \mid A^T(q) \frac{\partial H}{\partial p}(q, p) = 0 \} \).

We may solve for the algebraic constraints and at the same time eliminate the constraint forces \( A(q) \lambda \) in the following way. Since rank \( A(q) = k \), there exists locally an \( n \times (n - k) \) matrix \( S(q) \) of rank \( n - k \) such that

\[
A^T(q) S(q) = 0
\]

Now define \( \tilde{p} = (\tilde{p}^1, \tilde{p}^2) = (\tilde{p}_1, \ldots, \tilde{p}_{n-k}, \tilde{p}_{n-k+1}, \ldots, \tilde{p}_n) \) as

\[
\tilde{p}^1 := S^T(q)p, \quad \tilde{p}^1 \in \mathbb{R}^{n-k}, \\
\tilde{p}^2 := A^T(q)p, \quad \tilde{p}^2 \in \mathbb{R}^k
\]

The map \((q, p) \mapsto (q, \tilde{p}^1, \tilde{p}^2)\) is a coordinate transformation. Indeed, the rows of \( S^T(q) \) are orthogonal to the rows of \( A^T(q) \).

In the new coordinates the constrained Hamiltonian system becomes

\[
\begin{bmatrix}
\dot{q} \\
\dot{\tilde{p}}^1 \\
\dot{\tilde{p}}^2
\end{bmatrix}
= 
\begin{bmatrix}
0_n & S(q) & * \\
-S^T(q) & -p^T [S_i, S_j](q)_{i,j} & * \\
* & * & *
\end{bmatrix}
+ 
\begin{bmatrix}
\frac{\partial \tilde{H}}{\partial q} \\
\frac{\partial \tilde{H}}{\partial p^1} \\
\frac{\partial \tilde{H}}{\partial p^2}
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
\lambda + B_c(q)
\end{bmatrix} u
\]

\[
A^T(q) \frac{\partial H}{\partial q} = A^T(q) A(q) \frac{\partial H}{\partial p^2} = 0
\]

Here \( S_i \) denotes the \( i \)-th column of \( S(q) \), \( i = 1, \ldots, n - k \), and \([S_i, S_j]\) is the Lie bracket of \( S_i \) and \( S_j \), in local coordinates given as:

\[
[S_i, S_j](q) = \frac{\partial S_j}{\partial q}(q) S_i(q) - \frac{\partial S_i}{\partial q} S_j(q)
\]
Since $\lambda$ only influences the $\tilde{p}^2$-dynamics, and the constraints $A^T(q)\frac{\partial H}{\partial p}(q,\tilde{p}) = 0$ are equivalently given by $\frac{\partial \tilde{H}}{\partial p}(q,\tilde{p}) = 0$, the constrained dynamics is determined by the dynamics of $q$ and $\tilde{p}^1$ (coordinates for the constrained state space $\mathcal{X}_c$)

$$
\begin{bmatrix}
\dot{\tilde{p}} \\
\dot{p}_1 \\
\dot{p}_2 \\
\dot{p}_3 \\
\dot{p}_4
\end{bmatrix} = J_c(q, \tilde{p}^1) \begin{bmatrix}
\frac{\partial H}{\partial q(q,\tilde{p}^1)} \\
\frac{\partial H}{\partial p_1(q,\tilde{p}^1)} \\
\frac{\partial H}{\partial p_2(q,\tilde{p}^1)} \\
\frac{\partial H}{\partial p_3(q,\tilde{p}^1)} \\
\frac{\partial H}{\partial p_4(q,\tilde{p}^1)}
\end{bmatrix} + \begin{bmatrix} 0 \\
B_c(q) \end{bmatrix} u,
$$

where $H_c(q, \tilde{p}^1)$ equals $\tilde{H}(q, \tilde{p})$ with $\tilde{p}^2$ satisfying $\frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$, and where the skew-symmetric matrix $J_c(q, \tilde{p}^1)$ is given as the left-upper part of the structure matrix, that is

$$J_c(q, \tilde{p}^1) = \begin{bmatrix}
O_n & S(q) \\
-S^T(q) & -\tilde{p}^T[S_i, S_j]|(q)\end{bmatrix}_{i,j},$$

where $p$ is expressed as function of $q, \tilde{p}$, with $\tilde{p}^2$ eliminated from $\frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$.

Furthermore, in the coordinates $q, \tilde{p}$, the output map is given in the form

$$y = \begin{bmatrix} B_c^T(q) \end{bmatrix} \begin{bmatrix}
\frac{\partial \tilde{H}}{\partial \tilde{p}^1}(q, \tilde{p}^1) \\
\frac{\partial \tilde{H}}{\partial \tilde{p}^2}(q, \tilde{p}^1)
\end{bmatrix},$$

which reduces on the constrained state space $\mathcal{X}_c$ to

$$y = B_c^T(q) \frac{\partial \tilde{H}}{\partial \tilde{p}^1}(q, \tilde{p}^1)$$

These equations define a port-Hamiltonian system on $\mathcal{X}_c$, with Hamiltonian $H_c$ given by the constrained total energy, and with structure matrix $J_c$.

**Example 13 (Example 4 continued: The rolling euro)** Define the new $p$-coordinates

- $p_1 = p_\varphi$
- $p_2 = p_\theta + p_x \cos \varphi + p_y \sin \varphi$
- $p_3 = p_x - p_\theta \cos \varphi$
- $p_4 = p_y - p_\theta \sin \varphi$

The constrained state space $\mathcal{X}_c$ is given by $p_3 = p_4 = 0$, and the dynamics on $\mathcal{X}_c$ is computed as

$$
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta} \\
\dot{\varphi} \\
\dot{p}_1 \\
\dot{p}_2 \\
\dot{Y}_1 \\
\dot{Y}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & \cos \varphi & 0 & 0 \\
0 & 0 & -\sin \varphi & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & -\cos \varphi & -1 & 0 \\
0 & 0 & -\sin \varphi & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H_c}{\partial x} \\
\frac{\partial H_c}{\partial y} \\
\frac{\partial H_c}{\partial \theta} \\
\frac{\partial H_c}{\partial \varphi} \\
\frac{\partial H_c}{\partial p_1} \\
\frac{\partial H_c}{\partial p_2} \\
\frac{\partial H_c}{\partial Y_1} \\
\frac{\partial H_c}{\partial Y_2}
\end{bmatrix}
$$

where $H_c(x, y, \theta, \varphi, p_1, p_2) = \frac{1}{2} p_1^2 + \frac{1}{4} p_2^2$. 

Since $\lambda$ only influences the $\tilde{p}^2$-dynamics, and the constraints $A^T(q)\frac{\partial H}{\partial p}(q, \tilde{p}) = 0$ are equivalently given by $\frac{\partial \tilde{H}}{\partial p}(q, \tilde{p}) = 0$, the constrained dynamics is determined by the dynamics of $q$ and $\tilde{p}^1$ (coordinates for the constrained state space $\mathcal{X}_c$)