Port-Hamiltonian Systems:

From Geometric Network Modeling to Control

Module M13, Lecture 1

HYCON-EECI Graduate School on Control

April 07–10, 2009

Lecturers

- Arjan van der Schaft
  Institute of Mathematics and Computing Science
  University of Groningen
  The Netherlands
  http://www.math.rug.nl/∼arjan

- Dimitri Jeltsema
  Delft Institute of Applied Mathematics
  Delft University of Technology
  The Netherlands
  http://diam.ewi.tudelft.nl/∼jeltsema

Lecture Schedule

Tuesday April 07:

- Morning: Intro + port-based modelling (DJ)
- Afternoon: Port-Hamiltonian systems (AS)

Wednesday April 08:

- Morning: Analysis of port-Hamiltonian systems (AS)
- Afternoon: Control of port-Hamiltonian systems (DJ)

Morning: 09:00–10:30 and 11:00–12:30

Afternoon: 14:00–15:30 and 16:00–17:30

Lecture Schedule

Thursday April 09:

- Morning: Control of port-Hamiltonian systems (AS)
- Afternoon: Special subjects + case study (DJ)

Friday April 10:

- Morning: Distributed-parameter port-Hamiltonian systems (AS)

Course Material

- Copies of the slides
- Lecture notes

Available via: http://www.dcsc.tudelft.nl/∼djeltsema/EECI/

Examination

- Take home exam (not obligatory)

Objectives of the Course

- Present methods, techniques, and tools for modeling and control of complex dynamical systems
- Unified system approach allowing to deal with physical components stemming from different physical domains (electrical, mechanical, fluid, thermodynamic), both in the lumped-parameter and in the distributed parameter case
- Energy-based perspective using the port-Hamiltonian formalism.

Objectives of the Course

Why port-Hamiltonian modeling?

- Physical structure (energy and interconnection)
- Re-usability (‘libraries’)
- Multi-physics approach
- Modularity and flexibility
- Hierarchical analysis of complex systems
- ‘Modularity can beat complexity’
- Suited to design/control

This Morning

Brief introduction into port-based modeling issues

Objectives:

- Introduce some general modeling issues
- Define a dynamical system
- Define basic elements and their relationships
- Define energy storage and dissipation
- Towards a port-Hamiltonian system structure

Introduction into modeling issues

System: Object or set of objects from which we want to study the properties.

Model of a system: A tool we use to answer questions about the system without having to do an experiment.

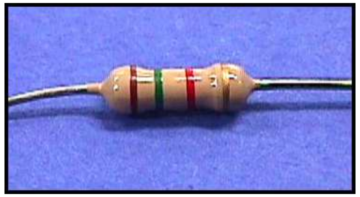
Types of models:

- mental: intuition and experience, verbal: if..., then...
- physical: scale models, laboratory set-ups
- **mathematical**: equations that describe relation between quantities that are important for behaviour of system, e.g., laws of nature. **Focus of this course!**

Introduction into modeling issues

A **model** depends on its **problem context**.

For example: a resistor



- Thermal characteristic
- Electromagnetic field
- Quantum effects
- Noise
- ...

Introduction into modeling issues

Simple and small models		Bad approximation of reality
	trade off	
Complex and large models		Good approximation of reality

It is **very important** to know for what purpose the model will be used!

Introduction into modeling issues

What **information** do we use to build models?

- Laws of physics, knowledge of the parameters by for example measurements
- Data obtained from the past
- Data obtained from experiments

Introduction into modeling issues

White box		based on underlying physics laws of nature and known parameters
- - - - -	**Grey box**	partly known, partly based on data
Black box		based on data e.g., i/o signals, no information on internal structure and relations ⇒**identification**

Example: Car test rig

Introduction into modeling issues

How well do you trust the prediction based on model?

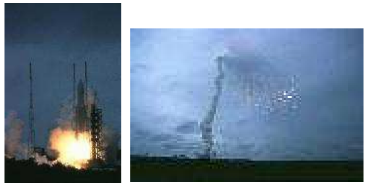
- compare behaviour model with behaviour system
- validation and verification

Be **always** aware of the assumptions and simplification that are made
⟶ **domain of validity** is limited!

Never use a model outside its domain of validity!

Ariane V

The rocket



The disaster launch in 1996

Control software of Ariane IV used in Ariane V, which caused the disaster launch. Outside domain of validity!

Different types of mathematical models

Continuous time	Discrete time
$t \in \mathbb{R}$	$k \in \mathbb{Z}$
relations with help of	relations with help of
differential equations	difference equations

Lumped	Distributed
Ordinary differential eq.^{s}	Partial differential eq.^{s}
finite # of variables,	infinite # of variables,
discrete set of variables	continuous set of variables
variables in time	variables in space and time

Different types of mathematical models

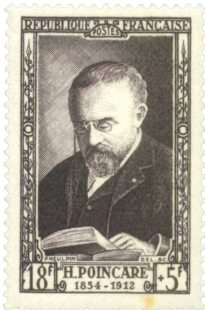
Change oriented	Hybrid	Discrete event
Difference or differential eq.^{s}	Mix	Event driven eq.^{s}

LET'S START MODELING...

Modeling

Restriction to **continuous-time** modeling based on our knowledge of the laws of nature in a structured way.

Deterministic dynamical **lumped-parameter** systems (ODE's) of the form



$$\dot{x}(t) = f\big(x(t), u(t), t\big), \quad \dot{x}(t) = \frac{dx(t)}{dt}$$
$$y(t) = h\big(x(t), u(t), t\big)$$

with $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $t \in \mathbb{R}$. Furthermore,

$$f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n, \quad h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p.$$

Such set of ODE's is called a **state space model** with state x.

Modeling

How to obtain such models? Consider following domains:

- **Electrical**
- **Mechanical translational**
- **Mechanical rotational**
- **Hydraulical**
- **Thermo-dynamical**

Modeling

In discussing a lumped-parameter system, we deal with:

- Sets of **elements** such as **ideal** springs, masses, dampers, inductors, capacitors, resistors, tubes, tanks, transformers, gyrators, etc..
- Sets of **variables** (signals) such as forces, velocities, voltages, currents, pressure, temperature, etc..
- Sets of **relationships** between variables.
- **Interactions** between elements.

Modeling

Two type of elements:

- **Dynamic** (mass, spring, inductor, capacitor, etc.)
  $\Rightarrow$ **energetic** (energy storage)
- **Static** (resistor, damper, transformer, etc.)
  $\Rightarrow$ **non-energetic** (dissipation, scaling)

Two type of element relationships:

- **Constitutive** relationships (all elements)
- **Dynamical** relationships (dynamic elements)

Modeling

Other types of relationships:

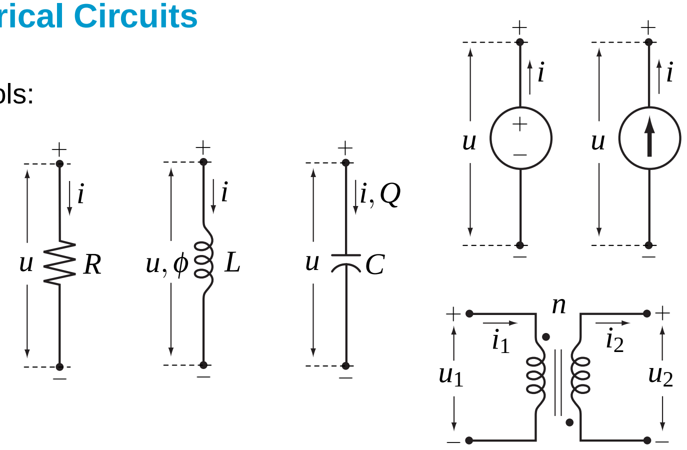
- **Interconnective** relationships. Interconnection of elements (Kirchhoff laws, D'Alembert, etc.)
- Combination of constitutive and dynamical relationships yields the **component** relationships.

These relationships define our

dynamical system model.

Electrical Circuits

Symbols:



Important physical variables:

- Charge $Q(t)$ [C]
- Voltage $u(t)$ [V]
- Flux-linkage $\phi(t)$ [Wb]
- Current $i(t)$ [A]

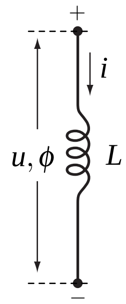
Electrical Circuits

Linear circuit theory and physics courses tell us that

- **Resistor:** $u = Ri$
- **Inductor:** $u = L\frac{di}{dt}$
- **Capacitor:** $i = C\frac{du}{dt}$

But what in case of time-varying resistors, inductors, or capacitors?

Electrical Circuits



Inductor L:

- Constitutive relationship:

$$\phi = \hat{\phi}(i) \quad [\text{linear: } \phi = Li].$$

- Dynamical relationship:

$$u = \frac{d\phi}{dt}, \text{ or } \phi = \phi(t_0) + \int_{t_0}^{t} u(\tau)d\tau.$$

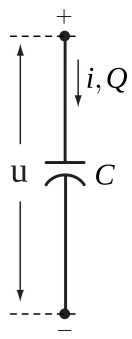
- Component relationship:

$$\frac{d\phi}{dt} = \frac{d\hat{\phi}(i)}{dt} = \frac{d\hat{\phi}(i)}{di}\frac{di}{dt} = L(i)\frac{di}{dt} \quad \left[\text{linear: } u = L\frac{di}{dt}\right].$$

(In NL case $L(i)$ is called the incremental inductance.)

Electrical Circuits

Capacitor C:



- Constitutive relationship:

$$Q = \hat{Q}(u) \quad [\text{linear: } Q = Cu].$$

- Dynamical relationship:

$$i = \frac{dQ}{dt}, \text{ or } Q = Q(t_0) + \int_{t_0}^{t} i(\tau)d\tau.$$

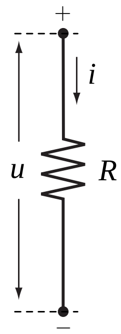
- Component relationship:

$$\frac{dQ}{dt} = \frac{d\hat{Q}(u)}{dt} = \frac{d\hat{Q}(u)}{du}\frac{du}{dt} = C(u)\frac{du}{dt} \quad \left[\text{linear: } i = C\frac{du}{dt}\right].$$

(In NL case $C(u)$ is called the incremental capacitance.)

Electrical Circuits

Resistor R:



Constitutive relationship:

- Current-controlled resistor

$$u = \hat{u}(i) \quad [\text{linear: } u = Ri].$$

- Voltage-controlled resistor

$$i = \hat{i}(u) \quad [\text{linear: } i = Gu, \; G = R^{-1}].$$

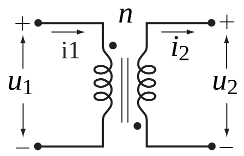
(**Note:** similar 'x-controlled' classifications holds also for inductors and capacitors.)

Electrical Circuits

Transformer Tr: (no power loss: $i_1 u_1 = i_2 u_2$)

- Constitutive relationships:

$$u_1 = n u_2$$
$$i_1 = \frac{i_2}{n}.$$

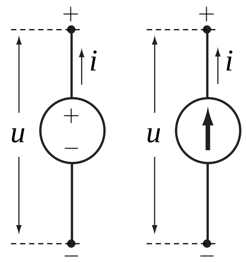


Sources S:

- Constitutive relationships:

voltage source: $u = u_S$

current source: $i = i_S$.



Electrical Circuits

Interconnective Relationships:

- Kirchhoff's Current Law (KCL):

$$\sum_k i_k = 0.$$

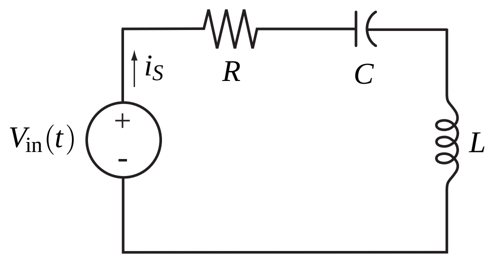
- Kirchhoff's Voltage Law (KVL):

$$\sum_k u_k = 0.$$

Example: Linear RLC circuit I

- Constitutive rel's:

$$u_R = R i_R$$
$$u_S = V_{in}$$
$$\phi_L = L i_L$$
$$Q_C = C u_C.$$



- Dynamical rel's:

$$u_L = \frac{d\phi_L}{dt}, \text{ or } \phi_L = \phi_L(t_0) + \int_{t_0}^{t} u_L(\tau)d\tau$$
$$i_C = \frac{dQ_C}{dt}, \text{ or } Q_C = Q_C(t_0) + \int_{t_0}^{t} i_C(\tau)d\tau.$$

Example: Linear RLC circuit II

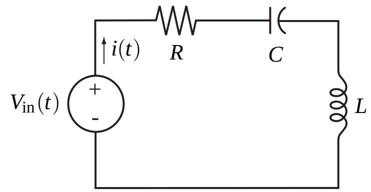
- Interconnective rel's:
  - KCL: $i_L = i_C = i_R = i_S$
  - KVL: $u_L + u_C + u_R = u_S$.
- Component rel's:

$$u_L = L\frac{di_L}{dt}$$
$$i_C = C\frac{du_C}{dt}.$$

State-Space Models

Combining dynamical and interconnective relationships, yields set of **first-order ODE's**:

$$\frac{dQ_C}{dt} = \frac{\phi_L}{L}$$
$$\frac{d\phi_L}{dt} = V_{\text{in}} - \frac{Q_C}{C} - R\frac{\phi_L}{L}.$$

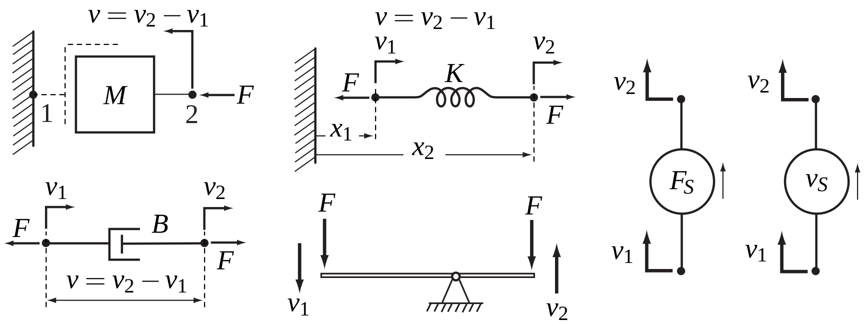


If initial charge $Q_C(t_0)$ and initial flux-linkage $\phi_L(t_0)$ are known, together with $V_{\text{in}}(t)$, $t > t_0$, then further evolution of $Q_C(t)$ and $\phi_L(t)$ is fully determined.

$Q_C, \phi_L \rightsquigarrow$ **state variables** $\rightsquigarrow$ **state-space system**.

Mechanical Translation

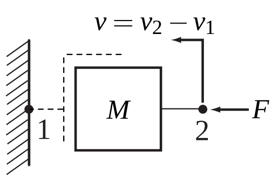
Symbols:



Important physical variables:

- Position $x(t)$ [m]
- Force $F(t)$ [N]
- Impulse momentum $p(t)$ [Ns]
- Velocity $v(t)$ [m/s]

Mechanical Translation



constant velocity reference v_1

Mass M: (non-relativistic)

- Constitutive relationship:

$$p = Mv.$$

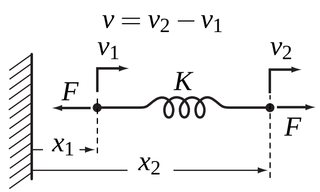
- Dynamical relationship:

$$F = \frac{dp}{dt}, \text{ or } p = p(t_0) + \int_{t_0}^{t} F(\tau)d\tau.$$

- Component relationship:

$$\frac{dp}{dt} = \frac{dMv}{dt} = M\frac{dv}{dt} = Ma.$$

Mechanical Translation



Translational spring K:

- Constitutive relationship:

$$x = \hat{x}(F) \quad [\text{linear: } x = K^{-1}F].$$

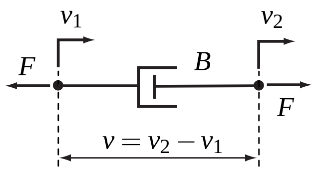
- Dynamical relationship:

$$v = \frac{dx}{dt}, \text{ or } x = x(t_0) + \int_{t_0}^{t} v(\tau)d\tau.$$

- Component relationship:

$$\frac{dx}{dt} = \frac{d\hat{x}(F)}{dt} = \frac{d\hat{x}(F)}{dF}\frac{dF}{dt} = K^{-1}(F)\frac{dF}{dt} \quad \left[\text{linear: } v = K^{-1}\frac{dF}{dt}\right].$$

Mechanical Translation



Translational damper B:

Constitutive relationship:

- Velocity-controlled damper

$$F = \hat{F}(v) \quad [\text{linear: } F = Bv].$$

- Force-controlled damper

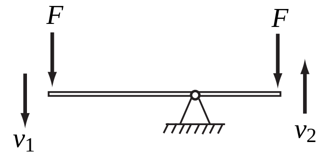
$$v = \hat{v}(F) \quad [\text{linear: } v = B^{-1}F].$$

Mechanical Translation

Transformer Tr: (no power loss: $v_1F_1 = v_2F_2$)

- Constitutive relationships:

$$F_1 = nF_2$$
$$v_1 = \frac{v_2}{n}.$$

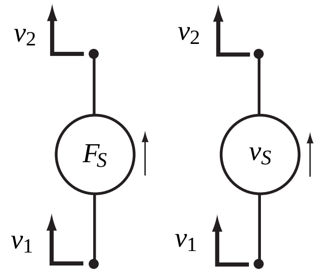


Sources S:

- Constitutive relationships:

force source: $F(t) = F_S(t)$

velocity source: $v(t) = v_S(t)$.



Mechanical Translation

Interconnective Relationships:

- Force balance:

$$\sum_k F_k = 0.$$

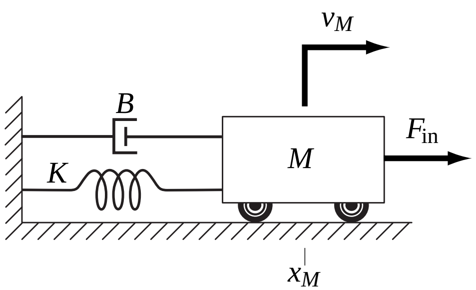
(d'Alembert's principle, law of conservation of impulse momenta
⇒ Translational mechanical "Kirchhoff" law.)

- Velocity balance not explicitly used.

Example: mass-spring-damper system I

- Constitutive rel's:

$$F_B = Bv_B$$
$$F_S = F_{in}$$
$$p_M = Mv_M$$
$$x_K = K^{-1}F_K.$$



- Dynamical rel's:

$$F_M = \frac{dp_M}{dt}, \text{ or } p_M = p_M(t_0) + \int_{t_0}^{t} F_M(\tau)d\tau$$
$$v_K = \frac{dx_K}{dt}, \text{ or } x_K = x_K(t_0) + \int_{t_0}^{t} v_K(\tau)d\tau.$$

Example: mass-spring-damper system II

- Interconnective rel.:
  - $F_M + F_K + F_B = F_S$
  - $(v_M = v_K = v_B = v_S)$
- Component rel's:

$$F_M = M\frac{dv_M}{dt}$$
$$v_K = K^{-1}\frac{dF_K}{dt}.$$

Electrical vs Mechanical

- State space model mechanical system:

$$\frac{dx_K}{dt} = \frac{p_M}{M}$$
$$\frac{dp_M}{dt} = F_{\text{in}} - Kx_K - B\frac{p_M}{M}.$$

- Recall electrical system:

$$\frac{dQ_C}{dt} = \frac{\phi_L}{L}$$
$$\frac{d\phi_L}{dt} = V_{\text{in}} - \frac{Q_C}{C} - R\frac{\phi_L}{L}.$$

Any resemblance? $L \rightleftharpoons M$, $C \rightleftharpoons K^{-1}$, $R \rightleftharpoons B$, $V_{\text{in}} \rightleftharpoons F_{\text{in}}$.

Port-Based Modeling of Physical Systems

Can we approach these domains in a similar way?

If so, why would we do that?

- Most **engineering applications** are mixtures of these domains! Treating the subsystems related to separate domains not simultaneously is time-consuming, and often yield surprises when connecting the subsystems.
- Main properties in common:

energy storage, dissipation, and transformation

Port-Based Modeling of Physical Systems

Motivations for adopting an **energy-based perspective** in modeling physical systems:

- Physical system can be viewed as a set of simpler subsystems that exchange energy (dynamics is exchange of energy!);
- Energy is neither allied to a particular physical domain nor restricted to linear elements and systems;
- Energy can serve as a *lingua franca* to facilitate communication among scientists and engineers from different fields;
- Role of energy and the interconnections between subsystems provide the basis for various control techniques.

Port-Based Modeling of Physical Systems

Each engineering domain consists of two sub-domains:

- Electrical $=$ electric $+$ magnetic
- Mechanical $=$ kinetic $+$ potential
- Hydraulic $=$ hydraulic kinetic $+$ hydraulic potential

Only thermal domain has no sub-domains.*

*Strongly related with irreversible transformation of energy).

Port-Based Modeling of Physical Systems

How to treat all domains on equal footing? $\Rightarrow$ Generalized Bond Graph (GBG) formalism [Breedveld 1982]. Main ideas:

- Decompose 'conventional' engineering domains, i.e., electrical, mechanical, hydraulical, into new domains.
- For each new domain introduce two variables, called **power conjugate variables**, whose product equals power (e.g., current $\times$ voltage, velocity $\times$ force, temperature $\times$ entropy flow).
- Label these variables as **efforts** $e \in \mathscr{E}$ and **flows** $f \in \mathscr{F}$.
- Each element defines a power port, with

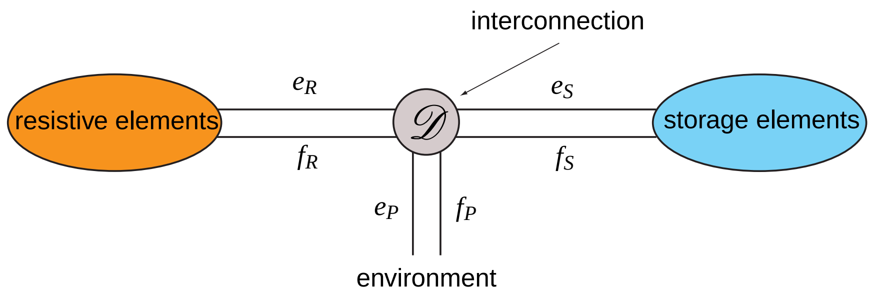
$$P = ef.$$

Port-Based Modeling of Physical Systems

physical domain	flow $f \in \mathscr{F}$	effort $e \in \mathscr{F}$
electric	current	voltage
magnetic	voltage	current
potential translation	velocity	force
kinetic translation	force	velocity
potential rotation	angular velocity	torque
kinetic rotation	torque	angular velocity
potential hydraulic	volume flow	pressure
kinetic hydraulic	pressure	volume flow
chemical	molar flow	chemical potential
thermal	entropy flow	temperature

Port-Based Modeling of Physical Systems

We thus view a physical system as interconnections between energy storage elements, resistive elements, and the environment:



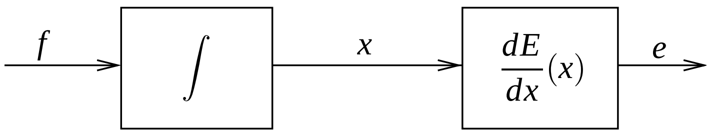
Natural partition $\mathscr{F} := \mathscr{F}_S \times \mathscr{F}_R \times \mathscr{F}_P$ and $\mathscr{E} := \mathscr{E}_S \times \mathscr{G}_R \times \mathscr{E}_P$.

Energy Storage

The structure of any **storage element** is the following:

$$\dot{x}(t) = f(t)$$
$$x(t) = x(0) + \int_0^t f(\tau)d\tau$$
$$e(t) = \frac{dE}{dx}(x(t)),$$

where $E(x)$ is the stored energy.



Energy Storage

This completes the GBG table:

physical domain	flow $f \in \mathscr{F}$	effort $e \in \mathscr{F}$	state variable $x = \int f dt$
electric	current	voltage	charge
magnetic	voltage	current	flux linkage
potential translation	velocity	force	displacement
kinetic translation	force	velocity	momentum
potential rotation	angular velocity	torque	angular displacement
kinetic rotation	torque	angular velocity	angular momentum
potential hydraulic	volume flow	pressure	volume
kinetic hydraulic	pressure	volume flow	flow tube momentum
chemical	molar flow	chemical potential	number of moles
thermal	entropy flow	temperature	entropy

Energy Storage

In this setting, we have that the

- constitutive relationships are of the form $e = \hat{e}(x)$, and
- dynamical relationships are $\dot{x} = f$.

Furthermore, observe that the change in energy given by $\dot{E}(x) = \frac{d}{dt}E(x(t))$ is now **always** the external power flow of the storage element, i.e.,

$$\dot{E}(x) = \frac{dE}{dx}(x)\dot{x} = ef.$$

Thus, by construction: **product of effort and flow**!

Energy Storage

How do we obtain $E(x)$?

- Integral of power $P = ef$ w.r.t. time yields energy

$$E(t) = \int e(t)f(t)dt. \qquad (\star)$$

- Recall dynamic relationship: $\dot{x} = f$, or equivalently,

$$dx = fdt.$$

- Substitution of the latter into ($\star$), and using $e = \hat{e}(x)$, yields:

$$E(x) = \int \hat{e}(x)dx.$$

Energy Storage

Some examples:

- Electric energy stored in a capacitor: $e = u$ and $x = Q$. In the linear case, $u = \hat{u}(Q) = Q/C$, and thus

$$E(Q) = \int_0^Q \hat{u}(\bar{Q})d\bar{Q} = \int_0^Q \frac{\bar{Q}}{C}d\bar{Q} = \frac{1}{2C}Q^2.$$

- Potential energy stored in a spring: $e = F$. Suppose that $F = \hat{F}(x) = kx + kx^3$, and thus

$$E(x) = \int_0^x \hat{F}(\bar{x})d\bar{x} = \int_0^x (k\bar{x} + k\bar{x}^3)d\bar{x} = \frac{1}{2}kx^2 + \frac{1}{4}kx^4.$$

Energy Storage

Some examples (cont'd):

- Kinetic energy of a mass M: $e = v$ and $x = p$. We have $p = Mv$, and thus $v = p/M$ so that

$$E(p) = \int_0^p \hat{v}(\bar{p})d\bar{p} = \int_0^p \frac{\bar{p}}{M}d\bar{p} = \frac{1}{2M}p^2.$$

  Observe that the velocity can be obtained from

$$v = \frac{dE}{dp}(p) = \frac{p}{M}.$$

Energy versus Co-Energy

- Note that the energy $E(x)$ corresponds to the area underneath the e-x curve. The area above this curve is then defined by the **Legendre transformation**

$$E^*(e) = xe - E(x), \quad e = \frac{dE}{dx}(x),$$

  under the assumption that the constitutive rel. $e = \hat{e}(x)$ is invertible, that is, if $x = \hat{x}(e)$ exist.
- The quantity $E^*(e)$ is called the **co-energy**. In integral form

$$E^*(e) = \int \hat{x}(e)de.$$

- In the **linear case**: $E^*(v) \equiv E(p)$!

Energy versus Co-Energy

Some examples:

- Recall kinetic energy of a mass $E(p) = \frac{1}{2M}p^2$. Application of the Legendre transformation yields

$$E^*(v) = pv - \frac{1}{2M}p^2 = \frac{1}{2}Mv^2.$$

  with $p = Mv$. Note that in this (linear) case

$$\frac{1}{2}Mv^2 = \frac{1}{2}M\left(\frac{p}{M}\right)^2 = \frac{1}{2M}p^2 \;\Rightarrow\; E^*(v) \equiv E(p).$$

  $\Rightarrow$ For this reason $\frac{1}{2}Mv^2$ is commonly called the kinetic energy.
- How about a relativistic mass?

Energy versus Co-Energy

Some examples (cont'd):

- Of course, for a non-quadratic energy function, say

$$E(x) = \frac{1}{4}x^4$$

we have that $e = \hat{e}(x) = x^3$, or equivalently, $x = e^{1/3}$, for the corresponding co-energy reads

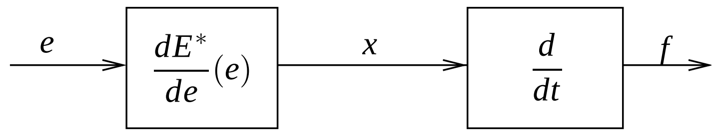
$$E^*(e) = \left[xe - \frac{1}{4}x^4\right]_{x=e^{1/3}} = e \cdot e^{1/3} - \frac{1}{4}e^{4/3} = \frac{3}{4}e^{4/3}.$$

Clearly

$$E(x) \neq E^*(e).$$

Energy versus Co-Energy

Note that while energy has an integrating causality, co-energy has a differentiating causality:



Furthermore, the rate of change of co-energy

$$\dot{E}^*(e) = \frac{dE^*}{de}(e)\dot{e} = x\dot{e} \neq ef \text{ (in general!)}$$

However, $\dot{E}^*(e)$ still has units of power, and in the linear case it does correspond to $\dot{E}(x) = ef$ (verify!).

Energy Dissipation

Dissipation of energy is expressed by direct (static) relation between effort and flow variables. In the linear case

$$e = Rf \Rightarrow P_{\text{diss}} = ef = Rf^2 \geq 0, \quad R \geq 0.$$

Electrical dissipation: $P_{\text{diss}} = Ri^2 = \frac{1}{R}u^2$.

In general: $P_{\text{diss}} \geq 0$, otherwise the element would generate energy instead of dissipating it!

Remark: for the thermo-dynamical domain it is no dissipation anymore (but rather storage of heat!).

Energy Dissipation

In the general nonlinear case, we have

- **Impedance** or **flow-controlled** form

$$e = \hat{e}(f),$$

- **Admittance** or **effort-controlled** form

$$f = \hat{f}(e).$$

- For dissipating (**passive**) resistors

$$\hat{e}(f)f \geq 0, \quad \hat{f}(e)e \geq 0.$$

Energy Dissipation

Energy dissipation can be associated with two state functions:

- The area underneath the e-f curve

$$D(f) = \int \hat{e}(f)df$$

is defined as the generalized **content**, whereas

- The area above the e-f curve

$$D^*(e) = \int \hat{f}(e)de$$

is defined as the generalized **co-content**.

Note: $D(f) + D^*(e) = ef \Rightarrow$ element power!

Energy Dissipation

Some examples:

- Electrical resistor: $e = u$ and $f = i$. If $u = Ri$, then

$$D(i) = \int_0^i R\bar{i}d\bar{i} = \frac{1}{2}Ri^2,$$

whereas

$$D^*(u) = \int_0^u \frac{\bar{u}}{R}d\bar{u} = \frac{1}{2R}u^2.$$

In this case,

$$\frac{1}{2}Ri^2 = \frac{1}{2}R\left(\frac{u}{R}\right)^2 = \frac{1}{2R}u^2 \Rightarrow D(i) \equiv D^*(u).$$

Energy Dissipation

Some examples (cont'd):

- Fluid orifice dissipator: $p = \hat{p}(Q) = rQ^2$, with pressure drop $e = p$ and flow rate $f = Q$. Hence

$$D(Q) = \int_0^Q r\bar{Q}^2 d\bar{Q} = \frac{1}{3}rQ^3.$$

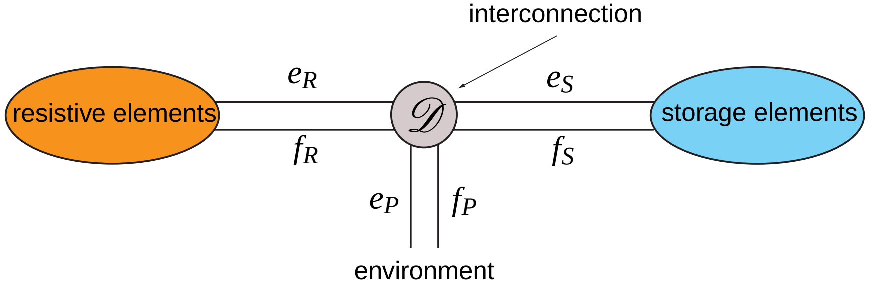
Since $p = rQ^2$ is invertible, that is, $Q = \sqrt{\frac{p}{r}}$, we have

$$D^*(p) = \int_0^p \sqrt{\frac{\bar{p}}{r}}d\bar{p} = \frac{2}{3}\left(\frac{p}{r}\right)^{3/2}.$$

Clearly, $D(Q) \neq D^*(p)$.

Port-Based Modeling of Physical Systems

Recall our perspective of a physical system as interconnections between energy storage elements, resistive elements, and the environment:



with natural partition $\mathscr{F} := \mathscr{F}_S \times \mathscr{F}_R \times \mathscr{F}_P$ and $\mathscr{E} := \mathscr{E}_S \times \mathscr{G}_R \times \mathscr{E}_P$.

Port-Based Modeling of Physical Systems

- The **energy storage port** containing n_S energy storage elements can now be characterized by

$$f_S = -\dot{x}, \quad e_S = \frac{\partial E}{\partial x}(x),$$

with total energy

$$E(x) = \sum_{k=1}^{n_S} \int_0^{x_k} \hat{e}_k(\ldots,\bar{x}_k,\ldots)d\bar{x}_k = \int_0^x \hat{e}^T(\bar{x})d\bar{x}.$$

Power at this port can be written as

$$\dot{E}(x) = \frac{\partial^T E}{\partial x}(x)\dot{x} = -e_S^T f_S.$$

Port-Based Modeling of Physical Systems

- The **resistive port** containing n_R resistive elements can be characterized by either

$$e_R = -G_R(f_R),$$

or

$$f_R = -F_R(e_R),$$

where G_R and F_R can be derived from the **total content** and **total co-content** functions

$$G_R(f_R) = \frac{\partial D}{\partial f_R}(f_R), \text{ and } F_R(e_R) = \frac{\partial D^*}{\partial e_R}(e_R),$$

respectively.

Port-Based Modeling of Physical Systems

- Hence the power at the resistive port is given by either

$$\frac{\partial^T D}{\partial f_R}(f_R)f_R \geq 0,$$

or

$$\frac{\partial^T D^*}{\partial e_R}(e_R)e_R \geq 0,$$

or, equivalently,

$$-G_R(f_R)f_R = e_R^T f_R \leq 0, \quad -F_R(e_R)e_R = f_R^T e_R \leq 0.$$

Dynamics = Exchange of Energy

Summarizing:

- Storage of energy corresponds to a state.
- The most **natural physical states** are in each engineering domain given by the **integrated flow variables** x.
- Hence we refer to x as the **energy variables**, whereas e are the **co-energy variables**.
- Dynamics iff there is **exchange of energy** among the elements.

Next question:

Can we also generalize the **interconnection structure**?

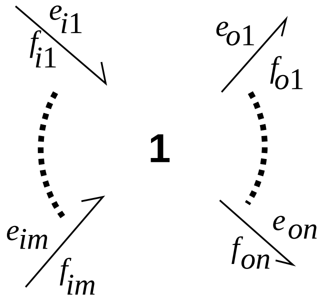
Junction Structure

Interconnection in bond graphs $\Rightarrow$ **Junction structure**

1-Junction (or flow junction):

$$\sum_{k=1}^{m} e_{ik} = \sum_{k=1}^{n} e_{ok}$$

$$f_{i1} = \cdots = f_{im} = f_{o1} \cdots = f_{on}$$



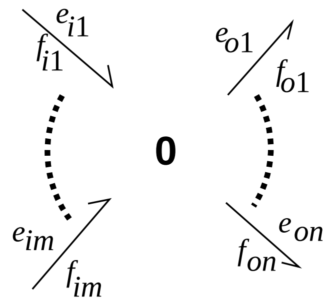
$\Rightarrow$ e.g., Kirchhoff's voltage law (KVL)

Junction Structure

0-Junction (or effort junction):

$$\sum_{k=1}^{m} f_{ik} = \sum_{k=1}^{n} f_{ok}$$

$$e_{i1} = \cdots = e_{im} = e_{o1} \cdots = e_{on}$$



$\Rightarrow$ e.g., Kirchhoff's current law (KCL)

Note: **Power continuity** for both junction structures (verify!)

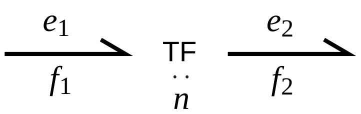
Power conserving Two-Ports

Ideal transformer "TF":

$$\begin{bmatrix} e_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 0 & n \\ n & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ e_2 \end{bmatrix},$$

with transformation ratio n.

In bond graph representation



with $e_1 f_1 = e_2 f_2$.

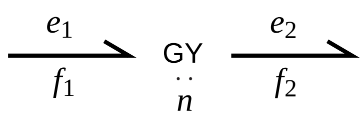
Power conserving Two-Ports

Ideal gyrator "GY":

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & n \\ n & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

with gyration ratio n.

In bond graph representation



with $e_1 f_1 = e_2 f_2$.

Multi-Port Gyrator

Multi-port version

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & N^T \\ N & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

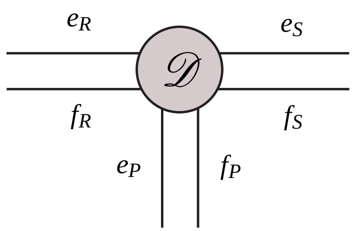
with gyration matrix N. To underscore power continuity, let $e = (e_1^T \; e_2^T)^T$ and $f = (f_1^T \; - f_2^T)^T$, then

$$e = Jf, \quad \text{with } J = \begin{bmatrix} 0 & -N^T \\ N & 0 \end{bmatrix} = -J^T$$

Hence $f^T e = f^T J f = 0$.

Power and Energy Balance

Interconnection structure satisfies the **power-conservation** property



$$e_S^T f_S + e_R^T f_R + e_P^T f_P = 0,$$

or, in terms of energy storage

$$\dot{E}(x) = -e_S^T f_S = e_R^T f_R + e_P^T f_P.$$

Integrating from t_0 to t yields the **energy balance**

$$E[x(t)] - E[x(t_0)] = \int_{t_0}^{t} e_R^T(\tau) f_R(\tau) \mathrm{d}\tau + \int_{t_0}^{t} e_P^T(\tau) f_P(\tau) \mathrm{d}\tau$$

Mathematical Formalization

- Power-conservation property

$$e_S^T f_S + e_R^T f_R + e_P^T f_P = 0$$

  can be formalized by the geometric notion of a **Dirac structure**.
- 1- and 0-junctions, transformers, and gyrators are examples of a Dirac structure.
- Naturally leads to **Port-Hamiltonian systems** $\Rightarrow$ this afternoon.