STABILITY of SWITCHED SYSTEMS under ARBITRARY SWITCHING

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SYSTEMS with SPECIAL STRUCTURE

• Triangular systems

• Feedback systems
  • passivity conditions
  • small-gain conditions

• 2-D systems
TRIANGULAR SYSTEMS

For linear systems, triangular form $\Rightarrow$ GUES

$$A_1 = \begin{pmatrix} -a_1 & b_1 \\ 0 & -c_1 \end{pmatrix}, \ A_2 = \begin{pmatrix} -a_2 & b_2 \\ 0 & -c_2 \end{pmatrix}$$

$$\dot{x}_2 = -c_\sigma x_2 \Rightarrow x_2 \to 0 \text{ exponentially fast}$$

$$\dot{x}_1 = -a_\sigma x_1 + b_\sigma x_2 \Rightarrow x_1 \to 0 \text{ exp fast}$$

$\exists$ quadratic common Lyap fcn $x^T D x$, $D$ diagonal

For nonlinear systems, not true in general

Need to know $x_2 \to 0 \Rightarrow x_1 \to 0$ (ISS)

[Angeli & L '00]
FEEDBACK SYSTEMS: ABSOLUTE STABILITY

\[ \dot{x} = Ax + bu \]
\[ y = c^T x \]

\( (A, b) \) controllable

\[ g(s) = c^T(sI - A)^{-1}b \]

\[ u = -\varphi_p(y) \]

\[ k_1 y^2 \leq y\varphi_p(y) \leq k_2 y^2 \ \forall p \]

Circle criterion: \( \exists \) quadratic common Lyapunov function \( \iff \)

\[ h(s) = \frac{1+k_2g(s)}{1+k_1g(s)} \text{ is strictly positive real (SPR): } \text{Re } h(i\omega) > 0 \]

For \( k_1 = 0, k_2 = \infty \) this reduces to \( g(s) \) SPR (passivity)

Popov criterion not suitable: \( V \) depends on \( \varphi_p \)
FEEDBACK SYSTEMS: SMALL-GAIN THEOREM

\[ \dot{x} = Ax + bu \]
\[ y = c^T x \]

\((A, b)\) controllable

\[ g(s) = c^T (sI - A)^{-1}b \]

\[ u = -\varphi_p(y) \]
\[ |\varphi_p(y)| \leq |y| \ \forall p \]
\[ (k_1 = -1, k_2 = 1) \]

Small-gain theorem:

\[ \exists \text{ quadratic common Lyapunov function} \]

\[ \|g\|_\infty = \max_\omega |g(i\omega)| < 1 \]
TWO-DIMENSIONAL SYSTEMS

Necessary and sufficient conditions for GUES known since 1970s

\[ \dot{x} = A_1 x, \quad \dot{x} = A_2 x, \quad x \in \mathbb{R}^2 \]

\[ \exists \text{ quadratic common Lyap fcn} \iff \exists \text{ convex combinations of } A_1, A_2, A_1^{-1}, A_2^{-1} \text{ Hurwitz} \]
OBSERVABILITY and ASYMPTOTIC STABILITY

Barbashin-Krasovskii-LaSalle theorem:

\( \dot{x} = f(x) \) is glob. asymp. stable (GAS) if \( \exists V \) s.t.

- \( \dot{V} := \frac{\partial V}{\partial x} f(x) \leq 0 \ \forall x \) (weak Lyapunov function)
- \( \dot{V} \) is not identically zero along any nonzero solution
  (observability with respect to \( \dot{V} \))

Example:

\[ \dot{x} = Ax, \quad V(x) = x^T Px \]

\[ A^T P + PA \leq -C^T C \]

\{ \text{observable} \} \Rightarrow \text{GAS}
SWITCHED LINEAR SYSTEMS  [Hespanha ’04]

\[ \dot{x} = A_\sigma x \]

**Theorem** (common weak Lyapunov function):

Switched linear system is GAS if

- \( \exists P > 0 \) s.t. \( A_p^T P + PA_p \leq -C_p^T C_p \) \( \forall p \)

- \((A_p, C_p)\) observable for each \( p \)

- \( \exists \) infinitely many switching intervals of length \( \geq \tau \)

Want to handle nonlinear switched systems and nonquadratic weak Lyapunov functions

Need a suitable nonlinear observability notion
Several ways to define observability
(equivalent for linear systems)

Benchmarks:

• observer design or state norm estimation
• detectability vs. observability
• LaSalle’s stability theorem for switched systems

Joint work with Hespanha, Sontag, and Angeli

No inputs here, but can extend to systems with inputs
STATE NORM ESTIMATION

\[ \dot{x} = Ax, \quad y = Cx \]

\[ x(0) = W^{-1} \int_{0}^{\tau} e^{A^T t} C^T y(t) \, dt \quad \text{where} \]

\[ W = \int_{0}^{\tau} e^{A^T t} C^T C e^{At} \, dt \quad \text{(observability Gramian)} \]

\[ \dot{x} = f(x), \quad y = h(x) \]

Observability definition #1:

\[ |x(0)| \leq \gamma \left( \|y\|_{[0,\tau]} \right) \quad \text{where} \ \gamma \in \mathcal{K}_\infty \]

This is a robustified version of 0-distinguishability
OBSERVABILITY DEFINITION #1: A CLOSER LOOK

Small-time observability:

\[ \forall \tau > 0 \ \exists \gamma \in \mathcal{K}_\infty : \ |x(0)| \leq \gamma \left( \|y\|_{[0,\tau]} \right) \]

Large-time observability:

\[ \exists \tau > 0, \ \gamma \in \mathcal{K}_\infty : \ |x(0)| \leq \gamma \left( \|y\|_{[0,\tau]} \right) \]

Counterexample: \[ x = 1 \]

Initial-state observability:

\[ \forall \tau > 0 \ \exists \gamma \in \mathcal{K}_\infty : \ |x(0)| \leq \gamma \left( \|y\|_{[0,\tau]} \right) \]

Final-state observability:

\[ \forall \tau > 0 \ \exists \gamma \in \mathcal{K}_\infty : \ |x(\tau)| \leq \gamma \left( \|y\|_{[0,\tau]} \right) \]
DETECTABILITY vs. OBSERVABILITY

\[ \dot{x} = Ax, \quad y = Cx \]

Detectability \( \iff \exists L : A - LC \) is Hurwitz

\[ \dot{x} = (A - LC)x + Ly, \quad |x(t)| \leq ce^{-\lambda t}|x(0)| + d\|y\|_{[0,t]} \]

Observability \( \iff A - LC \) can have arbitrary eigenvalues

\[ \dot{x} = f(x), \quad y = h(x) \]

A natural detectability notion is output-to-state stability (OSS):

\[ |x(t)| \leq \beta(\|x(0)\|, t) + \gamma(\|y\|_{[0,t]}) \]

where \( \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty \) [Sontag-Wang]

Observability def'n #2: OSS, and \( \beta \) can decay arbitrarily fast
OBSERVABILITY DEFINITION #2: A CLOSER LOOK

Definition: \( \forall \varepsilon > 0, \nu \in \mathcal{K}_\infty \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty : \)

\[
|x(t)| \leq \beta(|x(0)|, t) + \gamma \left( \|y\|_{[0,t]} \right) \quad \forall t \geq 0
\]

and

\[
\beta(r, \varepsilon) \leq \nu(r) \quad \forall r \geq 0
\]

Theorem: This is equivalent to definition #1 (small-time obs.)

OSS admits equivalent Lyapunov characterization:

\[
|x| \geq \rho(|y|) \implies \dot{V} \leq -\alpha(|x|), \quad \alpha, \rho \in \mathcal{K}_\infty
\]

For observability, \( \alpha \) should have arbitrarily rapid growth
STABILITY of SWITCHED SYSTEMS

\[ \dot{x} = f_\sigma(x) \]

Theorem (common weak Lyapunov function):

Switched system is GAS if

- \( \exists V \) s.t. \( \frac{\partial V}{\partial x} f_p(x) \leq -W_p(x) \leq 0 \quad \forall x, \forall p \)
- \( \exists \) infinitely many switching intervals of length \( \geq \tau \)
- Each system
  \[ \dot{x} = f_p(x), \quad y = W_p(x) \]
  is observable:
  \[ \exists \gamma \in \mathcal{K}_\infty : \ |x(0)| \leq \gamma \left( \|y\|_{[0,\tau]} \right) \]