COMMUTATION RELATIONS and STABILITY of LINEAR and NONLINEAR SWITCHED SYSTEMS

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SWITCHED vs. HYBRID SYSTEMS

Switched system:

\[ \dot{x} = f_\sigma(x) \]

- \( \dot{x} = f_p(x), \ p \in \mathcal{P} \) is a family of systems
- \( \sigma : [0, \infty) \rightarrow \mathcal{P} \) is a switching signal

Switching can be:
- State-dependent or time-dependent
- Autonomous or controlled

Details of discrete behavior are “abstracted away”

Further abstraction/relaxation:
- differential inclusion, measurable switching

Properties of the continuous state \( x \): stability
Asymptotic stability of each subsystem is not sufficient for stability
TWO BASIC PROBLEMS

• Stability for arbitrary switching

• Stability for constrained switching
TWO BASIC PROBLEMS

• Stability for arbitrary switching

• Stability for constrained switching
GLOBAL UNIFORM ASYMPTOTIC STABILITY

GUAS is Lyapunov stability

\[ \forall \varepsilon \exists \delta \ |x(0)| \leq \delta \Rightarrow |x(t)| \leq \varepsilon \ \forall t \geq 0, \forall \sigma \]

plus asymptotic convergence

\[ \forall \varepsilon, \delta \exists T \ |x(0)| \leq \delta \Rightarrow |x(t)| \leq \varepsilon \ \forall t \geq T, \forall \sigma \]

GUES: \[ |x(t)| \leq ce^{-\lambda t}|x(0)| \ \forall t \geq 0, \forall \sigma \]
COMMUTING STABLE MATRICES $\Rightarrow$ GUES

$\mathcal{P} = \{1, 2\}, \ A_1A_2 = A_2A_1$

(commuting Hurwitz matrices)

\[
\begin{array}{cccc}
\sigma = 1 & \sigma = 2 & \sigma = 1 & \sigma = 2 \\
 s_1 & t_1 & s_2 & t_2 \\
\end{array} \rightarrow t
\]

\[
x(t) = e^{A_2t_k}e^{A_1s_k} \cdots e^{A_2t_1}e^{A_1s_1}x(0)
\]

\[
= e^{A_2(t_k+\cdots+t_1)}e^{A_1(s_k+\cdots+s_1)}x(0) \rightarrow 0
\]

For $\geq 2$ subsystems – similarly
COMMUTING STABLE MATRICES => GUES

Alternative proof:

∃ quadratic common Lyapunov function
[Narendra–Balakrishnan ’94]

\[
P_1A_1 + A_1^TP_1 = -I
\]
\[
P_2A_2 + A_2^TP_2 = -P_1
\]
\[
\vdots
\]
\[
(P_mA_m + A_m^TP_m) = -P_{m-1}
\]

\[x^TP_mx\] is a common Lyapunov function
LIE ALGEBRAS and STABILITY

Lie algebra: \( \mathfrak{g} = \{ A_p : p \in \mathcal{P} \}_{\text{LA}} \)

Lie bracket: \( [A_1, A_2] = A_1A_2 - A_2A_1 \)

\( \mathfrak{g}^1 = \mathfrak{g}, \quad \mathfrak{g}^{k+1} = [\mathfrak{g}, \mathfrak{g}^k] \subset \mathfrak{g}^k \quad \mathfrak{g} \text{ is nilpotent if } \exists k \text{ s.t. } \mathfrak{g}^k = 0 \)

\( \mathfrak{g}^{(1)} = \mathfrak{g}, \quad \mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] \subset \mathfrak{g}^{(k)} \quad \mathfrak{g} \text{ is solvable if } \exists k \text{ s.t. } \mathfrak{g}^{(k)} = 0 \)

Nilpotent means suff. high-order Lie brackets are 0

e.g. \( [A_1, [A_1, A_2]] = [A_2, [A_1, A_2]] = 0 \)

Nilpotent \( \Rightarrow \) GUES [Gurvits '95]
SOLVABLE LIE ALGEBRA => GUES

Lie’s Theorem: \( \mathfrak{g} \) is solvable \( \Rightarrow \) triangular form

\[
A_p = \begin{pmatrix}
\lambda_1 & \cdots & * \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_n
\end{pmatrix}
\]

Example:

\[
A_1 = \begin{pmatrix}
-a_1 & b_1 \\
0 & -c_1
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
-a_2 & b_2 \\
0 & -c_2
\end{pmatrix}
\]

\[
\dot{x}_2 = -c_\sigma x_2 \Rightarrow x_2 \to 0 \text{ exponentially fast}
\]

\[
\dot{x}_1 = -a_\sigma x_1 + b_\sigma x_2 \Rightarrow x_1 \to 0 \text{ exp fast}
\]

\( \exists \) quadratic common Lyap fcn \( x^TDx \), \( D \) diagonal

[Kutepov ’82, L–Hespanha–Morse ’99]
MORE GENERAL LIE ALGEBRAS

Levi decomposition: \[ g = r \oplus s \]

radical (max solvable ideal)

• \( s \) is compact (purely imaginary eigenvalues) \( \Rightarrow \) GUES, quadratic common Lyap fcn

• \( s \) is not compact \( \Rightarrow \) not enough info in Lie algebra:

There exists one set of stable generators for \( g \) which gives rise to a GUES switched system, and another which gives an unstable one

[Agrachev–L ’01]
SUMMARY: LINEAR CASE

Lie algebra \( \{A_p, p \in \mathcal{P}\}_{\text{LA}} \) w.r.t. \( [A_1, A_2] = A_1 A_2 - A_2 A_1 \)

Assuming GES of all modes, GUES is guaranteed for:

• commuting subsystems: \( [A_p, A_q] = 0 \ \forall p, q \in \mathcal{P} \)

• nilpotent Lie algebras (suff. high-order Lie brackets are 0)
  e.g. \( [A_1, [A_1, A_2]] = [A_2, [A_1, A_2]] = 0 \)

• solvable Lie algebras (triangular up to coord. transf.)

• solvable + compact (purely imaginary eigenvalues)

Quadratic common Lyapunov function exists in all these cases

Extension based only on the Lie algebra is not possible
SWITCHED NONLINEAR SYSTEMS

Lie bracket of nonlinear vector fields:

\[
[f_1, f_2] := \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2
\]

Reduces to earlier notion for linear vector fields (modulo the sign)
SWITCHED NONLINEAR SYSTEMS

• Commuting systems

\[[f_p, f_q] = 0 \Rightarrow \text{GUAS}\]

Can prove by trajectory analysis [Mancilla-Aguilar ’00]
or common Lyapunov function [Shim et al. ’98, Vu–L ’05]

• Linearization (Lyapunov’s indirect method)

\[A_p = \frac{\partial f_p}{\partial x}(0), \ p \in P\]

• Global results beyond commuting case – ?

[Unsolved Problems in Math. Systems and Control Theory]
SPECIAL CASE

$f_1, f_2$ globally asymptotically stable

\[ [f_1, [f_1, f_2]] = [f_2, [f_1, f_2]] = 0 \]

Want to show: \( \dot{x} = f_\sigma(x), \; \sigma \in \{1, 2\} \) is GUAS

Will show: differential inclusion

\[ \dot{x} \in \text{co}\{f_1(x), f_2(x)\} \]

is GAS
OPTIMAL CONTROL APPROACH

Associated control system:

\[ \dot{x} = f(x) + g(x)u \]

where \( f := f_1, \ g := f_2 - f_1, \ u \in [0, 1] \)

(original switched system \( \leftrightarrow u \in \{0, 1\} \))

Worst-case control law [Pyatnitskiy, Rapoport, Boscain, Margaliot]:

fix \( x_0 \) and small enough \( t_f \)

\[ |x(t_f)|^2 \to \max_u \]
MAXIMUM PRINCIPLE

\[ H(x, u, \lambda) = \lambda^T f(x) + \lambda^T g(x)u \]

Optimal control:

\[ u(t) = 0 \text{ if } \varphi(t) < 0, \quad u(t) = 1 \text{ if } \varphi(t) > 0 \]

\[ \dot{\varphi} = \lambda^T [f, g], \quad \ddot{\varphi} = \lambda^T [f, [f, g]] + \lambda^T [g, [f, g]]u = 0 \]

\[ \Downarrow \]

\[ \varphi \text{ is linear in } t \]

\[ \Downarrow \]

(\text{unless } \varphi \equiv 0) \]

at most 1 switch

\[ \Downarrow \]

GAS
SINGULARITY

Know: $\lambda$ nonzero on $L := \{f, g\}_{LA}$

$$\varphi = \lambda^T g, \quad \dot{\varphi} = \lambda^T [f, g]$$

Need: $\lambda$ nonzero on $L_0 := \text{ideal generated by } g$

(strong extremality)

Sussmann ’79:

constant control
(e.g., $u \equiv 0$)

strongly extremal
(time-optimal control for auxiliary system in $L_0$)

At most 2 switches $\Rightarrow$ GAS
GENERAL CASE

\[ \dot{x} = f(x) + \sum_{k=1}^{m} g_k(x) u_k \]

\[ \varphi_{ij} := \lambda^T (g_i(x) - g_j(x)) \]

**Want:** \( \varphi_{ij} \) polynomial of degree \(< r\)

\[ \Downarrow \ (\text{proof – by induction on } m) \]

bang-bang with \( (r + 1)^m - 1 \) switches

\[ \Downarrow \]

GAS
THEOREM

Suppose:

• $f_0, f_1, \ldots, f_m$ GAS, backward complete, analytic

• $\exists r > 0$ s.t.

\[
(ad f_0)^r f_i = 0 \quad \forall i > 0
\]

and

\[
[f_k - f_0, (ad f_0)^s(f_i - f_0)] = 0 \quad \forall i, k > 0, 0 \leq s \leq r - 1
\]

Then differential inclusion is GAS, and switched system is GUAS \[Margaliot–L ’06\]

Further work in \[Sharon–Margaliot ’07\]
REMARKS on LIE-ALGEBRAIC CRITERIA

• Checkable conditions
• In terms of the original data
• Independent of representation
• Not robust to small perturbations

In any neighborhood of any pair of $n \times n$ matrices there exists a pair of matrices generating the entire Lie algebra $gl(n, \mathbb{R})$ [Agrachev–L ’01]

How to measure closeness to a “nice” Lie algebra?
EXAMPLE

Switching between $x^+ = Ax$ and $x^+ = Bx$
(discrete time, or cont. time periodic switching: $A = e^L$, $B = e^M$)

Fact 1 Stable for dwell time $\geq 2$:
$AABBAAAAABB\ldots$

Fact 2 If $AB = BA$ then always stable:
$ABAB = AABBB$

When $AB \neq BA$ we have $ABAB = AABBB$ where

$$E = A^{-1}BAB^{-1}$$

Fact 3 Stable if $\|E\| \leq 1 + \varepsilon$ where $\varepsilon$ is small enough s.t.

$$\rho_A(1 + \varepsilon)\rho_B < 1$$

This generalizes Fact 1 (didn’t need $E$) and Fact 2 ($E = I$)
EXAMPLE

Switching between \( x^+ = Ax \) and \( x^+ = Bx \) (discrete time, or cont. time periodic switching: \( A = e^L, B = e^M \))

\[ A, B \text{ Schur} \Rightarrow \exists m: \|A^m\| \leq \rho_A < 1, \|B^m\| \leq \rho_B < 1 \]

Suppose \( m = 2 \)

Fact 1 Stable for dwell time \( \geq 2 \): \( AABBAABB \ldots \)

Fact 2 If \( AB = BA \) then always stable: \( ABAB = AABB \)

When \( AB \neq BA \) we have \( ABAB = AAEBB \) where

\[
E = A^{-1}BAB^{-1}
\]

\[
= e^{-L}e^M e^L e^{-M}
\]

\[
= e^{-[L,M] - \frac{1}{2}[L,[M,L]] - \frac{1}{2}[M,[M,L]] + \ldots}
\]
MORE GENERAL FORMULATION

\[ A = e^L, \quad B = e^M \quad \Rightarrow \|A^m\| \leq \rho_A < 1, \quad \|B^m\| \leq \rho_B < 1 \]

Assume switching period \( n \):

\[ \underbrace{A \cdots A}_n \underbrace{B \cdots B}_n \]
MORE GENERAL FORMULATION

\[ A = e^L, \quad B = e^M \quad \|A^m\| \leq \rho_A < 1, \quad \|B^m\| \leq \rho_B < 1 \]

Assume switching period \( n \)

Find smallest \( k \) s.t. \( kn \geq m \)

\[ \underbrace{A \cdots A}_{n} \underbrace{B \cdots B}_{n} \]
MORE GENERAL FORMULATION

\[ A = e^L, \quad B = e^M \quad \text{||} A^m \text{||} \leq \rho_A < 1, \quad \text{||} B^m \text{||} \leq \rho_B < 1 \]

Assume switching period \( n \)

Find smallest \( k \) s.t. \( kn \geq m \), define \( E \) by

\[
\underbrace{A \cdots A}_{n} \underbrace{B \cdots B}_{n} \cdots \underbrace{A \cdots A}_{n} \underbrace{B \cdots B}_{n} = \underbrace{AA \cdots A}_{kn} \underbrace{E \cdots E}_{kn} \underbrace{BB \cdots B}_{kn}
\]

Intuitively, \( E \) captures:

- how far \( A \) and \( B \) are from commuting \( ([A, B] = 0 \Rightarrow E = I) \)
- how big \( m \) is compared to \( n \) \( (m = n \Rightarrow E = I) \)
MORE GENERAL FORMULATION

\[ A = e^L, \quad B = e^M \quad \|A^m\| \leq \rho_A < 1, \quad \|B^m\| \leq \rho_B < 1 \]

Assume switching period \( n \)

Find smallest \( k \) s.t. \( kn \geq m \), define \( E \) by

\[
\underbrace{A \cdot \cdots \cdot A}_{n} \underbrace{B \cdot \cdots \cdot B}_{n} \cdots \underbrace{A \cdot \cdots \cdot A}_{n} \underbrace{B \cdot \cdots \cdot B}_{n} = \underbrace{A A \cdots A}_{kn} \underbrace{E}_{kn} \underbrace{B B \cdots B}_{kn}
\]

\( k \) repetitions

Stability condition: \( \|E\| \leq 1 + \varepsilon \) where \( \rho_A(1 + \varepsilon)\rho_B < 1 \)

\( n = 1, \quad m = 2 \) already discussed: \( A B A B = A A E B B B \)

(“elementary shuffling”)

\( n = 2, \quad m = 4 \)

\[ A A B B A A B B B B = A A A A E B B B B B B \]

\[ E = A^{-1}A^{-1}BBAAB^{-1}B^{-1} = e^{-2L}e^{2M}e^{2L}e^{-2M} \]

\[ = e^{-[2L,2M]} - \frac{1}{2}[2L,[2M,2L]] - \frac{1}{2}[2M,[2M,2L]] + \cdots \]
SOME OPEN ISSUES

\[ A = e^L, \quad B = e^M \quad \|A^m\| \leq \rho_A < 1, \quad \|B^m\| \leq \rho_B < 1 \]

- Relation between \( E \) and Lie brackets of \( L \) and \( M \)
  general formula seems to be lacking
SOME OPEN ISSUES

\[ A = e^L, \quad B = e^M \quad \|A^m\| \leq \rho_A < 1, \quad \|B^m\| \leq \rho_B < 1 \]

• Relation between \( E \) and Lie brackets of \( L \) and \( M \)

• Smallness of higher-order Lie brackets
  Suppose \([L, M] \neq 0\) but \([L, [L, M]] = [M, [L, M]] = 0\)
  Elementary shuffling: \( BA = AEB \),
  \( E \) not nec. close to \( I \) but commutes with \( A \) and \( B \)

Example:

\[ A \quad B \quad A \quad B \quad A \quad B \]
SOME OPEN ISSUES

\[ A = e^L, \quad B = e^M \quad \|A^m\| \leq \rho_A < 1, \quad \|B^m\| \leq \rho_B < 1 \]

• Relation between \( E \) and Lie brackets of \( L \) and \( M \)

Elementary shuffling: \( BA = AEB \),
\( E \) not nec. close to \( I \) but commutes with \( A \) and \( B \)

Example:

\[
A \underbrace{BA}_n \underbrace{B}_n \underbrace{A}_n \underbrace{B}_n = A \underbrace{AEB}_n \underbrace{B}_n \underbrace{A}_n \underbrace{B}_n = A \underbrace{A}_n \underbrace{B}_n \underbrace{B}_n \underbrace{BA}_n
\]

This shows stability for \( m = 3 \) (Gurvits: true for any \( m \))

In general, for \( kn \geq m \) can define \( F \) by

\[
\underbrace{A \cdots A}_n \underbrace{B \cdots B}_n \cdots \underbrace{A \cdots A}_n \underbrace{B \cdots B}_n = \underbrace{AA \cdots A}_n \underbrace{F}_{kn-1} \underbrace{BB \cdots B}_n \underbrace{A}_{kn}
\]
SOME OPEN ISSUES

• Relation between $E$ and Lie brackets of $L$ and $M$

• Smallness of higher-order Lie brackets

• More than 2 subsystems
  can still define $E$ but relation with Lie brackets less clear

• Optimal way to shuffle
  especially when $m$ is not a multiple of $n$