Lecture Summary: Markov Jump Linear Systems

Vijay Gupta* and Richard M. Murray †

Definition: Consider a discrete time discrete state Markov process with state $r(k) \in \{1,2,\cdots,m\}$ at time k. Denote the transition probability $\text{Prob}(r(k+1)=j|r(k)=i)$ by q_{ij}, and the resultant transition probability matrix by Q. Also denote $\text{Prob}(r(k)=j)=\pi_j(k)$, with $\pi_j(0)$ as given. The evolution of a Markovian jump linear system (MJLS), denoted by $\mathcal{S}_1$ for future reference, can be described by the following equations

$$\begin{aligned} x(k+1) &= A_{r(k)}x(k)+B_{r(k)}u(k)+F_{r(k)}w(k) \\ y(k) &= C_{r(k)}x(k)+G_{r(k)}v(k), \end{aligned} \tag{1}$$

where $w(k)$ is zero mean white Gaussian noise with covariance R_w, $v(k)$ is zero mean white Gaussian noise with covariance R_v and the notation $X_{r(k)}$ implies that the matrix $X \in \{X_1,X_2,\cdots,X_m\}$ with the matrix X_i being chosen when $r(k)=i$. The initial state $x(0)$ is assumed to be a zero mean Gaussian random variable with variance $\Pi(0)$. For simplicity, we will consider $F_{r(k)}=G_{r(k)}\equiv I$ for all values of $r(k)$. We also assume that $x(0)$, $\{w(k)\}$, $\{v(k)\}$ and $\{r(k)\}$ are mutually independent.

LQ Control: The LQR problem aims at designing the control input $u(k)$ to minimize the finite horizon cost function

$$J_{LQR}=\sum_{k=1}^{K}\left(E_{\{r(j)\}_{j=k+1}^{K}}\left[x^T(k)Qx(k)+u^T(k)Ru(k)\right]\right)+x^T(K+1)P(K+1)x(K+1),$$

where the expectation at time k is taken with respect to the future values of the Markov state realization, $P(K+1)$, Q and R are all assumed to be positive definite, and the noises $w(k)$ and $v(k)$ are not present. The controller at time k has access to control inputs $\{u(j)\}_{j=0}^{k-1}$, state values $\{x(j)\}_{j=0}^{k}$ and the Markov state values $\{r(j)\}_{j=0}^{k}$. Moreover, the system is said to be stabilizable if the infinite horizon cost function $J_\infty \stackrel{def}{=} \lim_{K\to\infty}\frac{J_{LQR}}{K}$ is finite. The solution to this problem can readily be obtained through dynamic programming arguments.

- At time k, if $r(k)=i$, then the optimal control input is given by

  $$u(k)=-\left(R+B_i^TP_i(k+1)B_i\right)^{-1}B_i^TP_i(k+1)A_ix(k),$$

  where for $j=1,2,\cdots,m$,

  $$P_j(k)=\sum_{t=1}^{m}q_{tj}\left(Q+A_t^TP_t(k+1)A_t-A_t^TP_t(k+1)B_t\left(R+B_t^TP_t(k+1)B_t\right)^{-1}B_t^TP_t(k+1)A_t\right),$$

  and $P_j(K+1)=P(K+1),\forall j=1,2,\cdots,m$.
- Assume that the Markov states reach a stationary probability distribution. A sufficient condition for stabilizability of the system is that there exist m positive definite matrices X_1, X_2, $\cdots$, X_m and m^2 matrices $K_{1,1}$, $K_{1,2}$, $\cdots$, $K_{1,m}$, $K_{2,1}$, $\cdots$, $K_{m,m}$ such that for all $j=1,2,\cdots,m$,

  $$X_j>\sum_{i=1}^{m}q_{ij}\left((A_i^T+K_{i,j}B_i^T)X_i(A_i^T+K_{i,j}B_i^T)^T+Q+K_{ij}RK_{ij}^T\right).$$

*Department of Electrical Engineering, University of Notre Dame, vgupta2@nd.edu
†Control and Dynamical Systems, California Institute of Technology murray@caltech.edu

- A necessary condition for stabilizability is that

$$q_{i,i}\rho(A_i)^2 < 1, \qquad \forall i = 1,2,\cdots,m,$$

where $\rho(A_i)$ is the spectral radius of the matrix A_i that governs the dynamics of uncontrollable modes of the process in the i-th mode.

The sufficient condition can be cast in alternate forms as linear matrix inequalities, that can be efficiently solved. A special case of Markov jump linear systems is when the discrete states are chosen independently from one time step to the next. For this case, consider system $\mathcal{S}_1$ with the additional assumption that the Markov transition probability matrix is such that for all states i and j, $q_{ij} = q_i$

- At time k, if $r(k) = i$, then the optimal control input is given by

$$u(k) = -\left(R + B_i^T P(k+1)B_i\right)^{-1} B_i^T P(k+1)A_i x(k),$$

where

$$P(k) = \sum_{t=1}^{m} q_t \left(Q + A_t^T P(k+1)A_t - A_t^T P(k+1)B_t \left(R + B_t^T P(k+1)B_t\right)^{-1} B_t^T P(k+1)A_t\right).$$

- Assume that the Markov states reach a stationary probability distribution. A sufficient condition for stabilizability of the system is that there exists a positive definite matrix X, and m matrices K_1, K_2, $\cdots$, K_m such that

$$X > \sum_{i=1}^{m} q_i \left((A_i^T + K_i B_i^T)X(A_i^T + K_i B_i^T)^T + Q + K_i R K_i^T\right).$$

- A necessary condition for stabilizability is that

$$q_i\rho(A_i)^2 < 1, \qquad \forall i = 1,2,\cdots,m,$$

where $\rho(A_i)$ is the spectral radius of the matrix A_i that governs the dynamics of uncontrollable modes of the process in the i-th mode.

MMSE Estimation The minimum mean squared error estimate problem for the system $\mathcal{S}_1$ is posed by assuming that the control $u_{r(k)}$ is identically zero. The objective is to identify at every time step k, an estimate $\hat{x}(k+1)$ of the state $x(k+1)$ that minimizes the mean squared error covariance

$$\Pi(k+1) = E_{\{w(j)\},\{v(j)\}}\left[(x(k+1) - \hat{x}(k+1))(x(k+1) - \hat{x}(k+1))^T\right],$$

where the expectation is taken with respect to the process and measurement noises (but not the Markov state realization). The estimator at time k has access to observations $\{y(j)\}_{j=0}^{k}$ and the Markov state values $\{r(j)\}_{j=0}^{k}$. Moreover, the error covariance is said to be stable if the expected steady state error covariance $\lim_{k\to\infty} E_{\{r(j)\}_{j=0}^{k-1}}[\Pi(k)]$ is bounded, where the expectation is taken with respect to the Markov process.

Since the estimator has access to the Markov state values till time k, the optimal estimate can be calculated through a time-varying Kalman filter. Thus, if at time k, $r_k = i$, the estimate evolves as

$$\hat{x}(k+1) = A_i\hat{x}(k) + K(k)\left(y(k) - C_i\hat{x}(k)\right),$$

where

$$K(k) = A_i\Pi(k)C_i^T \left(C_i\Pi(k)C_i^T + R_v\right)^{-1}$$
$$\Pi(k+1) = A_i\Pi(k)A_i^T + R_w - A_i\Pi(k)C_i^T \left(C_i\Pi(k)C_i^T + R_v\right)^{-1} C_i\Pi(k)A_i^T.$$

The error covariance $\Pi(k)$ is available through the above calculations. The term $E_{\{r(j)\}_{j=0}^{k-1}}[\Pi(k)]$ obtained from the optimal estimator is upper bounded[1] by $M(k)=\sum_{j=1}^{m} M_j(k)$ where

$$M_j(k)=\sum_{t=1}^{m} q_{tj}\left(R_w+A_tM_t(k-1)A_t^T-A_tM_t(k-1)C_t^T\left(R_v+C_tM_t(k-1)C_t^T\right)^{-1}C_tM_t(k-1)A_t^T\right),$$

with $M_j(0)=\Pi(0)\ \forall j$. Moreover, if the Markov states reach a stationary probability distribution, then a sufficient condition for stabilizability of the system is that there exist m positive definite matrices X_1, X_2, $\cdots$, X_m and m^2 matrices $K_{1,1}$, $K_{1,2}$, $\cdots$, $K_{1,m}$, $K_{2,1}$, $\cdots$, $K_{m,m}$ such that for all $j=1,2,\cdots,m$,

$$X_j>\sum_{i=1}^{m} q_{ij}\left((A_i+K_{i,j}C_i)X_i(A_i+K_{i,j}C_i)^T+R_w+K_{ij}R_vK_{ij}^T\right).$$

A necessary condition for stabilizability is that

$$q_{i,i}\rho(A_i)^2<1,\qquad \forall i=1,2,\cdots,m,$$

where $\rho(A_i)$ is the spectral radius of the matrix A_i that governs the dynamics of unobservable modes of the process in the i-th mode.

For the case when the Markov transition probability matrix is such that for all states i and j, $q_{ij}=q_i$ (in other words, the states are chosen independently and identically distributed from one time step to the next). The term $E_{\{r(j)\}_{j=0}^{k-1}}[\Pi(k)]$ obtained from the optimal estimator is upper bounded by $M(k)$ where

$$M(k)=\sum_{t=1}^{m} q_t\left(R_w+A_tM(k-1)A_t^T-A_tM(k-1)C_t^T\left(R_v+C_tM(k-1)C_t^T\right)^{-1}C_tM(k-1)A_t^T\right),$$

with $M(0)=\Pi(0)$. Further, a sufficient condition for stabilizability of the system is that there exists a positive definite matrix X, and m matrices K_1, K_2, $\cdots$, K_m such that

$$X>\sum_{i=1}^{m} q_i\left((A_i+K_iC_i)X(A_i+K_iC_i)^T+R_w+K_iR_vK_i^T\right).$$

A necessary condition for stabilizability is that

$$q_i\rho(A_i)^2<1,\qquad \forall i=1,2,\cdots,m,$$

where $\rho(A_i)$ is the spectral radius of the matrix A_i that governs the dynamics of unobservable modes of the process in the i-th mode.

LQG Control The Linear Quadratic Gaussian (LQG) problem for the system $\mathcal{S}_1$ aims at designing the control input $u(k)$ to minimize the finite horizon cost function

$$J_{LQG}=E\left[\sum_{k=1}^{K}\left(x^T(k)Qx(k)+u^T(k)Ru(k)\right)+x^T(K+1)P(K+1)x(K+1)\right],$$

where the expectation at time k is taken with respect to the future values of the Markov state realization, and the measurement and process noises. Further, the matrices $P(K+1)$, Q and R are all assumed to be positive definite. The controller at time k has access to control inputs $\{u(j)\}_{j=0}^{k-1}$, measurements $\{y(j)\}_{j=0}^{k}$ and the Markov state values $\{r(j)\}_{j=0}^{k}$. The system is said to be stabilizable if the infinite horizon cost function $J_\infty \overset{def}{=} \lim_{K\to\infty}\frac{J_{LQG}}{K}$ is finite. The solution to this problem is provided by a separation principle and using the optimal LQ control in conjunction with the MMSE estimation. At time k, if $r(k)=i$, then the optimal control input is given by

$$u(k)=-\left(R+B_i^TP_i(k+1)B_i\right)^{-1}B_i^TP_i(k+1)A_i\hat{x}(k),$$

where $P_i(k)$ is calculated as in the optimal LQ control input and $\hat{x}(k)$ is calculated using a time-varying Kalman filter. Conditions for stabilizability of the system follow from stability conditions given for LQ control and MMSE estimation.

[1]We say that A is upperbounded by B if $B-A$ is positive semi-definite.