A method for robust regulation of non-minimum-phase linear systems

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Abstract: The paper discusses a new method for the design of output regulators for a class of nonlinear systems characterized by a possibly unstable zero dynamics. In the case of linear systems, the method is always applicable if controlled plant has a (multiple) zero at the origin and all other zeros with negative real part. If the plant has zeros with positive real part, the method is applicable if the frequencies which characterize the harmonic components of the exogenous input exceed a minimal value determined by the gain needed to solve an auxiliary stabilization problem.

Keywords: Output Regulation, Non-Minimum-Phase Systems, Robust Regulation, Tracking, Disturbance Rejection

1. INTRODUCTION

The theory of output regulation of nonlinear systems, which uses a combination of geometry and nonlinear dynamical systems theory, was initiated by pioneering works of [9, 8]. Since these early contributions, the theory has experienced a tremendous growth, culminating in the recent development of design methods able to handle issues of global convergence (as in [2]), the case of parametric uncertainties affecting the autonomous (linear) system which generates the exogenous signals (such as in [13]), the case of nonlinear exogenous systems (such as in [1, 3]), or a combination thereof (as in [10]). A thorough presentation of several recent advances in this area can also be found in the recent books [6, 12].

However, most of the design methods proposed in these recent contributions still address a restricted class of systems, namely systems in normal form with a (globally) asymptotically stable zero dynamics. The solution of the problem, in the presence of parametric uncertainties, for systems whose zero dynamics is unstable is still largely an open problem. Non-minimum-phase systems can be handled, in principle, by means of the “reduction procedure” suggested in [7] for nonlinear stabilization and later extended in [11] to cover the problem of output regulation. The problem with these papers, though, is that the suggested design procedure is based on hypotheses that are not readily checkable. The method in question has been revisited, from a different perspective, in the recent paper [4], where a more efficient design procedure has been proposed and few successful illustrative examples have been sketched. The purpose of this paper is to show in detail how the procedure of [4] can be used to address and solve, in general, the problem of robust regulation for a non-minimum phase linear system.

2. BACKGROUND MATERIAL

2.1 The setup

We begin with a summary of the setup and of the results of [4]. Consider a nonlinear system in normal form

\[
\begin{align*}
\dot{z}_0 &= f_0(w, z_0, \xi_1, \ldots, \xi_r) \\
\dot{\xi}_1 &= \xi_2 \\
& \quad \ldots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= q_0(w, z_0, \xi_1, \ldots, \xi_r) + u \\
e &= \xi_1
\end{align*}
\]
with control input \( u \in \mathbb{R} \), regulated output \( e \in \mathbb{R} \), in which \( w \in \mathbb{R}^r \) is a vector of exogenous inputs which cannot be controlled, solutions of a fixed ordinary differential equation of the form

\[
\dot{w} = s(w) .
\] (2)

In this setup, \( w \) can be viewed as a model of time-varying commands, external disturbances, and also uncertain constant plant parameters. The initial states of (1) and of (2) are assumed to range over a fixed compact set \( X \) and \( W \), with \( W \) invariant under the dynamics of (2). Motivated by well-known standard design procedures (see e.g. \([5]\)), we assume throughout that the measured output \( y \) coincides with the partial state \((\xi_1, \ldots, \xi_r)\). The states \( w \) and \( z_0 \) are, on the contrary, not available for measurement.

The problem of output regulation is to design a controller

\[
\dot{\xi} = \varphi(\xi, y) \\
u = \gamma(\xi, y)
\]

with initial state in a compact set \( \Xi \), yielding a closed-loop system in which

- the positive orbit of \( W \times X \times \Xi \) is bounded,
- \( \lim_{t \to \infty} e(t) = 0 \), uniformly in the initial condition
  
  (on \( W \times X \times \Xi \)).

The standard point of departure in the analysis of the problem of output regulation is the identification of a (smooth) controlled invariant manifold entirely contained in the set of all states at which \( e = 0 \) (see \([9]\)). In the present context, this can be specialized as follows. Let the aggregate of (1) and (2), be rewritten as

\[
\begin{align*}
\dot{w} &= s(w) \\
\dot{z} &= f(w, z, \zeta) \\
\dot{\zeta} &= q(w, z, \zeta) + u
\end{align*}
\] (3)

in which \( \zeta = \zeta_t = e^{r(1)} \) and \( z = \text{col}(z_0, \xi_1, \ldots, \xi_r) \).

Assume the existence of a smooth map \( \pi_0 : W \to \mathbb{R}^{n-r} \) satisfying

\[
\frac{\partial \pi_0}{\partial w} s(w) = f_0(w, \pi_0(w), 0, \ldots, 0) \quad \forall w \in W .
\]

and note that the map \( \pi : W \to \mathbb{R}^{n-1} \) defined as

\[
\pi(w, \pi_0(w), 0, \ldots, 0) \]

satisfies

\[
\frac{\partial \pi}{\partial w} s(w) = f(w, \pi(w), 0) \quad \forall w \in W .
\]

Trivially, the smooth manifold

\[
\{(w, z, \zeta) : w \in W, z = \pi(w), \zeta = 0\},
\]

a subset of the set of all states at which \( e = \xi_t = 0 \), can be rendered invariant by feedback, actually by the control

\[
u = -q(w, \pi(w), 0).\] (5)

The second step in the solution of the problem usually consists in making assumptions that make it possible to generate the control (5) by means of an internal model. In a series of recent papers, it was shown how these assumptions could be progressively weakened, moving from the so-called assumption of “immersion into a linear observable system” (as in \([8]\)) to “immersion into a nonlinear uniformly observable system (as in \([1]\))” to the recent results of \([10]\), in which it was shown that no assumption is in fact needed for the construction of an internal model if only continuous (thus possibly not locally Lipschitz) controllers are acceptable. Motivated by this, we assume, in what follows, the existence of a pair \( F_0, G_0 \), in which \( F_0 \) is a \( d \times d \) Hurwitz matrix and \( G_0 \) is a \( d \times 1 \) column vector that makes the pair \( F_0, G_0 \) controllable, of a locally Lipschitz map \( \gamma : \mathbb{R}^d \to \mathbb{R} \) satisfying

\[
\frac{\partial \tau}{\partial w} s(w) = F_0 \tau(w) + G_0 \gamma(\tau(w)) \quad \forall w \in W .
\]

Properties (4) and (6) are instrumental in the design of a controller that solves the problem of output regulation.

### 2.2 The design method of \([4]\)

Consider, for the original plant, a controller of the form

\[
\begin{align*}
u &= \tilde{N}(\varphi) + \gamma(\eta) + v \\
v &= -k[\zeta - N(\varphi)] \\
\dot{\eta} &= F_0(\eta - G_0[\zeta - N(\varphi)]) + G_0[\gamma(\eta) + v] \\
\dot{\varphi} &= L(\varphi + M[\zeta - N(\varphi)]) - Mv
\end{align*}
\]

which is a dynamic controller, with internal state \((\eta, \varphi)\), “driven” only by the measured variable \( \zeta \). Change variables as

\[
\begin{align*}
\theta &= \zeta - N(\varphi) \\
\chi &= \varphi + M\theta \\
x &= \eta - G_0\theta
\end{align*}
\]

to obtain a system

\[
\begin{align*}
\dot{w} &= s(w) \\
\dot{z} &= f(w, z, \theta + N(x - M\theta)) \\
\dot{x} &= L(x) + M[q(w, z, \theta + N(x - M\theta)) + \gamma(x + G_0\theta)] + \gamma(x + G_0\theta) - k\theta.
\end{align*}
\] (8)
This system can be seen as feedback interconnection of a system with input \((w, z, \chi, x)\)
\[
\begin{align*}
\dot{w} &= s(w) \\
\dot{z} &= f(w, z, \theta + N(\chi - M\theta)) \\
\dot{\chi} &= L(\chi) + M[q(w, z, \theta + N(\chi - M\theta)) + \gamma(x + G_0\theta)] \\
\dot{x} &= F_0x - G_0q(w, z, \theta + N(\chi - M\theta))
\end{align*}
\]
and of a system with input \((w, z, \chi, x)\) and state \(\dot{\theta} = q(w, z, \theta + N(\chi - M\theta)) + \gamma(x + G_0\theta) - k\theta\).

Set \(\theta = 0\) in the upper subsystem, to obtain
\[
\begin{align*}
\dot{w} &= s(w) \\
\dot{z} &= f(w, z, N(\chi)) \\
\dot{\chi} &= L(\chi) + M[q(w, z, N(\chi)) + \gamma(x)] \\
\dot{x} &= F_0x - G_0q(w, z, N(\chi)).
\end{align*}
\]
Suppose that the latter possesses a compact invariant set \(A\) which is asymptotically stable, with a domain of attraction that contains the set of all admissible initial conditions. Then, it is known that, for every \(\varepsilon > 0\), there is a number \(k^*\) such that, if \(k \geq k^*\), all trajectories of the composite system (8) remain bounded and there is a time \(T > 0\) such that
\[
\text{dist}\left(\{(w(t), z(t), \chi(t), x(t)): A\}\right) \leq \varepsilon \quad \text{and} \quad |\theta(t)| \leq \varepsilon
\]
for all \(t \geq T\). If, in addition, the set \(A\) is locally exponentially stable and the map \(q(w, z, N(\chi)) + \gamma(x)\) vanishes on \(A\), one can invoke a (local version of) the small-gain theorem and claim the existence of a number \(k^*\) such that, if \(k \geq k^*\), all trajectories of the composite system (8) remain bounded and, moreover, \((w, z, \chi, x)\) converges to \(A\) while \(\theta\) converges to 0. If \(\xi_1\) vanishes on \(A\), then also \(e\) converges to 0 and the problem of output regulation is solved.

We have in this way identified an auxiliary problem which, if solved, makes the controller (7) solving the problem of output regulation for the original plant: find, if possible, a triplet \(\{L(\varphi), N(\varphi)\}\) such that system (10) possesses a compact invariant set \(A\) which is locally exponentially stable and attracts all admissible initial conditions, and such that \(\xi_1, N(\chi)\) and \(q(w, z, N(\chi)) + \gamma(x)\) vanish on this set.

Recall that, by assumption, there exists \(\pi(w)\) and \(\tau(w)\) satisfying (4) and (6). Hence, it is readily seen that if \(L(0) = 0\) and \(N(0) = 0\), the set
\[
A = \{(w, z, \chi, x) : w \in W, z = \pi(w), \chi = 0, x = \tau(w)\}
\]
is a compact invariant set of (10). Moreover, \(\xi_1, N(\chi)\) and \(q(w, z, N(\chi)) + \gamma(x)\) vanish on this set. Thus, this is the set for which local exponential stability will be sought (with a domain of attraction that contains the compact set of all admissible initial conditions).

To determine whether this is achievable, it is convenient to change \(z\) into \(z = z - \pi(w)\), and to define
\[
\begin{align*}
f_a(w, z_a, \zeta) &= f(w, z_a + \pi(w), \zeta) - f(w, \pi(w), 0) \\
h_a(w, z_a, \zeta) &= q(w, z_a + \pi(w), \zeta) - q(w, \pi(w), 0).
\end{align*}
\]
Note that \(f_a(w, z_a, \zeta)\) and \(h_a(w, z_a, \zeta)\) both vanish at \((z_a, \zeta) = (0, 0)\). This being done, system (10) can be interpreted as interconnection of three subsystems: a system which we call the “auxiliary plant”, modelled by equations of the form
\[
\begin{align*}
\dot{w} &= s(w) \\
\dot{z}_a &= f_a(w, z_a, u_a) \\
y_a &= h_a(w, z_a, u_a),
\end{align*}
\]
a system which, in view of a subsequent interpretation, we call a “weighting filter”, modelled by equations of the form
\[
\begin{align*}
\dot{x} &= F_0x + G_0\gamma(\tau(w)) - G_0y_a \\
y &= \gamma(x) - \gamma(\tau(w)),
\end{align*}
\]
and of a “controller” modelled by equations of the form
\[
\begin{align*}
\dot{\chi} &= L(\chi) + M[y + \hat{u}] \\
u_a &= N(\chi).
\end{align*}
\]
As a matter of fact, system (10) can be obtained by closing a unitary feedback loop
\[
\hat{u} = \tilde{y},
\]
on the composite system
\[
\begin{align*}
\dot{w} &= s(w) \\
\dot{z}_a &= f_a(w, z_a, N(\chi)) \\
\dot{\chi} &= L(\chi) + M[h_a(w, z_a, N(\chi)) + \hat{u}] \\
\dot{x} &= F_0x + G_0\gamma(\tau(w)) - G_0h_a(w, z_a, N(\chi)) \\
y &= \gamma(x) - \gamma(\tau(w)).
\end{align*}
\]
In view of the above discussion, we seek a controller of the form (13) with the property that, when system (15) is controlled by \(\hat{u} = \tilde{y}\), the invariant set \(A\) has the required asymptotic properties.

2.3 The case of a linear system

It is readily seen that – if controlled plant and exosystem are linear systems – the auxiliary plant is a linear system of the form
\[
\begin{align*}
\dot{z}_a &= A_\alpha z_a + B_\alpha u_a \\
y_a &= C_\alpha z_a + D_\alpha u_a,
\end{align*}
\]
the map \(\gamma(x)\) is a linear map, namely \(\gamma(x) = \Gamma x\), and a linear controller
\[
\begin{align*}
\dot{\chi} &= L \chi + M u_c \\
y_c &= N \chi
\end{align*}
\]
can be chosen. The matrices \(A_\alpha, B_\alpha, C_\alpha, D_\alpha\) may depend on a vector \(\mu\) of constant uncertain parameters.
If this is the case, it is assumed that the entries of these matrices are continuous functions of $\mu$ and that the latter ranges on a compact set. The matrix $F_0$ being Hurwitz, system (15) is stable if and only if the “controller” (17) stabilizes the “auxiliary” plant (16). Let $P_a(s)$ denote the transfer function of system (16), let $G(s)$ denote the transfer function of the controller (17) and let $\Phi(s)$ denote the transfer function of the filter (12), namely $\Phi(s) = -\Gamma(sI - F_0)^{-1}G_0$. Set

$$T(s) = \frac{G(s)P_a(s)}{1 - G(s)P_a(s)}$$

(18)

Then, the transfer function of system (15), between input $v$ and output $y$, is simply

$$W(s) = \Phi(s)T(s).$$

In view of the small gain theorem, we can conclude that the design goal is achieved if the “controller” (17) stabilizes the “auxiliary” plant (16) and there is a number $0 < \gamma < 1$ such that

$$\|\Phi(s)T(s)\|_{\infty} \leq \gamma.$$  

(19)

This design problem will be addressed in what follows.

3. THE BALANCED INTERNAL MODEL

We consider the case in which the internal model has purely imaginary nonzero eigenvalues, at $\pm i \Omega_k$, $k = 1, \ldots, p$. This corresponds to a regulation problem in which the exogenous inputs (to be followed and/or rejected) are sinusoidal functions of time. The fulfillment of the basic design goal, namely condition (19) for some $0 < \gamma < 1$, depends on the choice of the controller, but also on the choice of the pair $F_0, G_0$ which determines the transfer function $\Phi(s)$ of the filter (12). To this end, we observe (see [13]) that $\Gamma$ is necessarily the unique vector which assigns to $F_0 + \Gamma G_0$ the characteristic polynomial $q(s) = \prod_{k=1}^{p}(s^2 + \Omega_k^2)$. Since the characteristic polynomial of $F_0 + \Gamma G_0$ is precisely the numerator polynomial of the transfer function of

$$\dot{x} = F_0x + G_0u$$

$$y = \Gamma x + u$$

we see that

$$\prod_{k=1}^{p}(s^2 + \Omega_k^2) = 1 - \text{det}(sI - F_0) = 1 + \Phi(s).$$

Therefore, necessarily, $\Phi(\pm i \Omega_k) = -1$ regardless of how $F_0$ and $G_0$ are chosen. This may seem disappointing, because it implies $\|\Phi(s)\|_{\infty} \geq 1$. However, a clever choice of $F_0, G_0$ may still be sought to the purpose of lowering the magnitude of $\Phi(i\omega)$ at frequencies other than $\Omega_k$, in view of the fact that – after all – it is the $H_{\infty}$ norm of the product $\Phi(s)T(s)$ that matters in the basic condition (19).

Choose the characteristic polynomial of $F_0$ as $d_0(s) = \prod_{k=1}^{p}(s + \Omega_k)^2$, in which case (see above)

$$\Phi(s) = \prod_{k=1}^{p}(s^2 + \Omega_k^2) - 1.$$  

Set

$$\Phi_k(s) = T_1(s) \cdots T_k(s) - 1,\quad T_k(s) = \frac{(s^2 + \Omega_k^2)}{(s + \Omega_k)^2}.$$  

Since $\Phi_k(0) = 0$ write $\Phi_k(s) = sR_k(s)$. Then,

$$sR_k(s) = (sR_{k-1}(s) + 1)T_k(s) - 1 = \frac{2\Omega_k}{(s + \Omega_k)^2}.$$  

It is easily checked that

$$\|T_k(i\omega)\|_{\infty} \leq 1,\quad \frac{2\Omega_k}{|1 + \Omega_k^2|} \leq \frac{2}{\Omega_k}.$$  

Hence

$$\|R_k(i\omega)\|_{\infty} \leq \frac{2}{\Omega_k} \leq \frac{2}{\Omega_k}.\quad (20)$$

In summary, we have shown that the choice of $d_0(s)$ as characteristic polynomial for $F_0$ entails a choice of a transfer function $\Phi(s)$ for the filter (12) which can be expressed as

$$\Phi(s) = sR(s), \quad \|R(s)\|_{\infty} \leq \frac{2}{\Omega_k}.\quad (20)$$

With this in mind, we see that the basic problem to fulfill the requirement (19) with $\gamma < 1$ can be (trivially) recast as the problem of rendering $\|sR(s)T(s)\|_{\infty} < 1$. Since a bound for the $\|R(s)\|_{\infty}$ is known, the problem is reduced to a problem in which $\|sT(s)\|_{\infty}$ is constrained. In summary, the proposed design scheme works if it is possible to choose the controller (17) in such a way that

$$\|s\frac{G(s)P_a(s)}{1 - G(s)P_a(s)}\|_{\infty} \leq \gamma$$  

(21)

for some $\gamma$ satisfying

$$\gamma \left(\sum_{k=1}^{p} \frac{2}{\Omega_k}\right) < 1.$$  

(22)
4. ROBUST DESIGN FOR NON-MINIMUM PHASE LINEAR SYSTEMS

Case 1: the controlled plant has a multiple zero at the origin. In this case the problem of (robust) output regulation can always be solved. The following result is instrumental in this respect.

Proposition 1. Let two polynomials \( N_0(s) \) and \( D_0(s) \) be given, with \( D_0(s) \) monic and Hurwitz, and \( \deg[D_0] \geq \deg[N_0] \). Set, for \( k = 1, \ldots \)

\[
N_1(s) = gN_0(s) \quad N_k(s) = (s + z_{k-1})N_{k-1}(s) \\
D_1(s) = sD_0(s) \quad D_k(s) = sD_{k-1}(s) \\
P_k(s) = D_k(s) + N_k(s) \quad T_k(s) = \frac{N_k(s)}{D_k(s) + N_k(s)}.
\]

Let \( \gamma > 0 \) be fixed. For any choice of \( a > 1 \) there is a choice of real numbers \( g, z_1, z_2, \ldots, z_k \) such that \( P_{k+1}(s) \) is Hurwitz, \( T_{k+1}(0) = 1 \) and

\[
\|sT_{k+1}(s)\|_\infty \leq a^k \gamma.
\]

Proof. Let \( k = 1 \). Indeed, \( P_1(s) = D_0(s) + gN_0(s) \). Let \( c_0 \) denote the leading coefficient of \( N_0(s) \) and set \( g = \sgn(c_0)|g| \). Since \( D_0(s) \) is Hurwitz, standard arguments prove that there is a number \( g^* > 0 \) such that, if \( 0 < |g| < g^* \), \( P_1(s) \) is Hurwitz. Since

\[
T_1(s) = \frac{gN_0(s)}{sD_0(s) + gN_0(s)}
\]

we see that \( T_1(0) = 1 \). Finally, consider

\[
sT_1(s) = \frac{gsN_0(s)}{sD_0(s) + gN_0(s)}.
\]

Let \( d \) be the degree of \( D_0(s) \). From root locus, we know that if \( |g| \) is small, \( d \) roots of \( P_1(s) \), which we will denote as \( -p_i \) with \( i = 1, \ldots, k \), are close to the \( d \) roots, denoted by \( -p_i \) with \( i = 1, \ldots, k \), of \( D_0(s) \), while the extra \( (p + 1) \)-th root, denoted by \( -p_{k+1} \) is close to the origin. Thus, we write

\[
sT_1(s) = \frac{gsN_0(s)}{(s + p_{d+1})(s + p_d) \cdots (s + p_1)} = \frac{gs}{(s + p_{d+1})} \frac{N_0(s)}{D_0(s)} \frac{(s + p_d) \cdots (s + p_1)}{(s + p_d) \cdots (s + p_1)}.
\]

Claim 1. There is \( g^{**} > 0 \) such that, if \( 0 < |g| < g^{**} \),

\[
\|\frac{(s + p_d) \cdots (s + p_1)}{(s + p_d) \cdots (s + p_1)}\|_\infty \leq 2.
\]

Proof of Claim 1. We know that, given any \( \varepsilon \) there is a \( \delta \) such that if \( 0 < |g| < \delta \), then \( |p_i - p_i| \leq \varepsilon \) for all \( i = 1, \ldots, p \). From this, the result follows by continuity.

Let \( \gamma_0 := \|N_0(s)/D_0(s)\|_\infty \) and observe that

\[
\|\frac{gs}{(s + p_{d+1})}\|_\infty = |g|.
\]

In this way we have shown the existence of a number \( g^{**} > 0 \) such that, for all \( 0 < |g| < g^{**} \),

\[
\|sT_1(s)\|_\infty \leq |g| 2 \gamma_0.
\]

For any given \( \gamma \), pick \( |g| \leq \min\{g^{**}, \gamma/2\gamma_0\} \), so that

\[
\|sT_1(s)\|_\infty \leq \gamma.
\]

For higher values of \( k \), we proceed by induction. Assume that \( P_k(s) \) is Hurwitz, \( T_k(0) = 1 \) and that \( \|sT_k(s)\|_\infty \leq a^{k-1} \gamma \). Note that

\[
P_{k+1}(s) = sD_k(s) + (s + z_k)N_k(s) = s[D_k(s) + N_k(s)] + z_kN_k(s) = sP_k(s) + z_kN_k(s).
\]

Let \( c_0 \) denote the leading coefficient of \( N_k(s) \) and set \( z_k = \sgn(c_0)|z_k| \). By assumption, \( P_k(s) \) is Hurwitz. Hence, by standard (root-locus or passivity-based) arguments, we see that there is a number \( \delta_1 > 0 \) such that for all \( 0 < |z_k| < \delta_1 \), the polynomial \( P_{k+1}(s) \) is Hurwitz. Note also that \( T_{k+1}(0) = 1 \).

From root locus, we know that if \( \delta_1 \) is small, \( k + d \) roots of \( P_{k+1}(s) \), which we will denote as \( -\hat{p}_i \) with \( i = 1, \ldots, k + d \), are close to the \( k \) roots, denoted by \( -p_i \) with \( i = 1, \ldots, k + d \), of \( P_k(s) \), while the extra \( (k + d + 1) \)-th root, denoted by \( -p_{k+d+1} \) is close to the origin. Thus, we can write

\[
T_{k+1}(s) = \frac{(s + z_k)N_k(s)}{P_{k+1}(s)} = \frac{(s + z_k)N_k(s)}{(s + \hat{p}_{d+1}) \cdots (s + \hat{p}_1)}
\]

\[
= \frac{(s + z_k)N_k(s)}{(s + \hat{p}_{d+1}) \cdots (s + \hat{p}_1)} \frac{(s + p_{d+1}) \cdots (s + p_1)}{(s + p_{d+1}) \cdots (s + p_1)}.
\]

Claim 2. Given any \( a > 1 \), there is \( \delta_2 > 0 \) such that, if \( 0 < |z_k| < \delta_2 \),

\[
\|\frac{(s + p_{d+1}) \cdots (s + p_1)}{(s + p_{d+1}) \cdots (s + p_1)}\|_\infty \leq \sqrt{a}.
\]

The proof of Claim 2 is exactly the same as the proof of Claim 1.

Claim 3. Given any \( a > 1 \), there is \( \delta_3 > 0 \) such that, if \( 0 < |z_k| < \delta_3 \),

\[
\|\frac{(s + \hat{p}_{d+1}) \cdots (s + \hat{p}_1)}{(s + \hat{p}_{d+1}) \cdots (s + \hat{p}_1)}\|_\infty \leq \sqrt{a}.
\]
Proof of Claim 3. Note that
\[ sP_k(s) + z_k N_k(s) = (s + p_{k+1}) \cdots (s + p_1) . \]
Hence
\[ -\bar{p}_{k+1} P_k(-\bar{p}_{k+1}) + z_k N_k(-\bar{p}_{k+1}) = 0 \]
i.e.
\[ \frac{\bar{p}_{k+1}}{z_k} = \frac{N_k(-\bar{p}_{k+1})}{P_k(-\bar{p}_{k+1})} = T_k(-\bar{p}_{k+1}) . \]
From root locus, we know that, given any \( \varepsilon \), there is a minimal value, which is determined by an auxiliary \( \delta \) such that, if \( 0 < |z_k| < \delta , \bar{p}_{k+1} < \varepsilon . \) Thus, since \( T_k(0) = 1 \), using continuity we know that for any \( \varepsilon ' \) there is \( \delta ' \) such that, if \( 0 < |z_k| < \delta ' , 1 - \varepsilon ' \leq T(-\bar{p}_{k+1}) \leq 1 + \varepsilon ' . \)

Then, the result follows from
\[ \| (s + z_k) \|_{\infty} = \max \left\{ 1, \frac{z_k}{\bar{p}_{k+1}} \right\} = \max \left\{ 1, \frac{1}{T_k(-\bar{p}_{k+1})} \right\} . \]

Using Claims 2 and 3, we see that for any choice of \( \alpha > 1 \) there is a number \( \delta \) such that, if \( 0 < |z_k| < \delta \),
\[ \| sT_k(s) \|_{\infty} \leq a \| sT_k(s) \| \]
and this, by induction, completes the proof. \( \square \)

Let us now return of the basic problem, which is to make (21) fulfilled for a \( \gamma \) satisfying (22). Let \( P_a(s) := N_a(s)/s^n D_a(s) \) with \( D_a(s) \) a Hurwitz polynomial. Pick any monic Hurwitz polynomial \( D^*(s) \) of degree \( n \). Then, from the previous Proposition (with \( \gamma = \gamma_1/2 \) and \( a = (\gamma_1/2)^{1/(n-1)} \)), we conclude that the controller
\[ G(s) = \frac{(s + z_{n-1})(s + z_{n-2}) \cdots (s + z_1)}{D^*(s)} \]
solves the problem.

**Case II:** the controlled plant has a zero with positive real part. In this case, the auxiliary plant (16) has a pole with positive real part, and the inequality (21) can only be fulfilled if \( \gamma \) exceeds a minimal value. Nevertheless, the approach presented above is still applicable, if the frequencies which characterize the harmonic components of the exogenous input exceed a minimal value, which is determined by an auxiliary \( H_{2\infty} \) minimization problem. In fact, let \( \gamma^* \) be defined as
\[ \gamma^* = \min_{\gamma \in (17) \text{ stabilizes (16)}} \left\{ \| s \frac{G(s)P_a(s)}{1 - G(s)P_a(s)} \|_{\infty} \right\} . \]

Then, from our previous analysis, it is clear that the problem output regulation is solvable by means of the proposed approach if frequencies \( \Omega_1, \ldots, \Omega_p \) which characterize the harmonic components of the exogenous input satisfy
\[ \frac{1}{\gamma^*} > \left( \sum_{k=1}^{p} \frac{2}{\Omega_k} \right) . \]

**REFERENCES**


