A dissipativity-based approach to output regulation of non-minimum-phase systems

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ABSTRACT
The present paper presents a new contribution to the design of output regulators for a class of nonlinear systems characterized by a possibly unstable zero dynamics. It is shown that the problem in question can be reduced to a stabilization problem, with a supplementary "gain constraint", for a suitably defined reduced auxiliary plant.

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1. Introduction

In this paper, we address the problem of output regulation (sometimes also known as generalized tracking problem, or generalized servomechanism problem) for nonlinear systems whose zero dynamics is unstable. Formally, the problem of output regulation is cast in the following terms. The controlled plant is a finite-dimensional, time-invariant, nonlinear system modelled by equations of the form

\[ \dot{x} = F(w, x, u) \]
\[ e = H(w, x) \]
\[ y = K(w, x) \]

in which \( x \in \mathbb{R}^n \) is a vector of state variables, \( u \in \mathbb{R} \) is the input used for control purposes, \( w \in \mathbb{R}^s \) is a vector of inputs which cannot be controlled and include exogenous commands, exogenous disturbances and model uncertainties, \( e \in \mathbb{R} \) is a vector of regulated outputs which include tracking errors and any other variable that needs to be steered to 0, \( y \in \mathbb{R}^p \) is a vector of outputs that are available for measurement. The exogenous input \( w(t) \) is assumed to be a (undefined) member of the family of all solutions of a fixed ordinary differential equation of the form

\[ \dot{w} = s(w) \]

obtained when the initial condition \( w(0) \) is allowed to vary on a prescribed set \( W \). This system is usually referred to as the exosystem. The initial state of (1) and of (2) are assumed to range over fixed compact sets \( X \) and \( W \), the latter being invariant under the dynamics of (2). The problem of output regulation is to design a controller

\[ \dot{x} = \varphi(x, y) \]
\[ u = \gamma(x, y) \]

with initial state in a compact set \( \mathcal{S} \), yielding a closed-loop system in which

- the positive orbit of \( W \times X \times \mathcal{S} \) is bounded,
- \( \lim_{t \to \infty} e(t) = 0 \), uniformly in the initial condition (on \( W \times X \times \mathcal{S} \)).

The theory of output regulation of nonlinear systems, which uses a combination of geometry and nonlinear dynamical systems theory, was initiated by pioneering works of [1–3] who showed how to design a controller that provides a local solution near an equilibrium point, in the presence of exogenous signals which were produced by a neutrally stable system. Since these early contributions, the theory has experienced a tremendous growth, culminating in the recent development of design methods able to handle issues of global convergence (as in [4]), the case of parametric uncertainties affecting the autonomous (linear) system which generates the exogenous signals (such as in [5,6]), the case of nonlinear exogenous systems (such as in [7,8]), or a combination thereof (as in [9]). A thorough presentation of several recent advances in this area can also be found in the recent books [10–12].
However, most of the design methods proposed in these recent contributions still address a restricted class of systems, namely only systems in normal form with a (globally) stable zero dynamics. The solution of the problem for systems whose zero dynamics is unstable is still largely an open problem. The purpose of this paper is to present some further advances in this direction. We observe that, in the general setup presented above, the vector $w$ of exogenous inputs may well include (constant) uncertain parameters, which are hence assumed to range on a given compact set. Thus, if a controller solves the problem at issue, the goal of asymptotic regulation is achieved robustly with respect to (constant) parameter uncertainties.

2. Preliminaries

The design procedure presented in this paper is an enhancement of the procedure originally suggested in [13] and further explored in [14]. The underlying design philosophy of these two papers consisted in the reduction of the original problem of output regulation to a similar problem for a composite “auxiliary” plant, whose dynamics included the zero dynamics of the original plant and the dynamics of a suitably chosen internal model. The problem with these papers, though, was that no specific design procedure was suggested for the actual design of a controller that solves the problem for the auxiliary plant. This is because the conditions needed for the solution of the problem, reposing – among other things - on the property of uniform observability of a cascade of nonlinear systems, were difficult to check. In the present paper, instead, we show that if the design problem for the auxiliary plant is posed in slightly different terms, the design procedure can be carried out up to full completion. A few relevant examples are in fact carried out in detail, to this end.

We begin by reviewing the initial part of the approach of [13, 14]. In what follows, a nonlinear system having relative degree $r$ between control input $u \in \mathbb{R}$ and regulated output $e \in \mathbb{R}$ is considered, expressed in normal form as

$$
\dot{w} = s(w)
$$

$$
\dot{z}_0 = f_0(w, z_0, \xi_1, \ldots, \xi_r)
$$

$$
\dot{\xi}_1 = \xi_2
$$

$$
\ldots
$$

$$
\dot{\xi}_{r-1} = \xi_r,
$$

$$
\dot{\xi}_r = q_0(w, z_0, \xi_1, \ldots, \xi_r) + b_0(w, z_0, \xi_1, \ldots, \xi_r)u,
$$

where $e = \xi_1$.

The functions $s(\cdot), f_0(\cdot), q_0(\cdot)$ and $b_0(\cdot)$ are smooth functions of their arguments and $b_0(\cdot)$ is nowhere zero. Motivated by well-known standard design procedures (see e.g. [15]), we assume throughout that the entire partial state $(\xi_1, \ldots, \xi_r)$ and hence in particular $\xi_1$, is available for measurement. The states $w$ and $z_0$ are, on the contrary, not available for measurement. To simplify the exposition, we also assume that the so-called “high-frequency gain” coefficient $b_0(\cdot)$ is either constant, and accurately known, or depending on variables that are available for measurement. The general case in which this coefficient is a function of unmeasured state variables can be handled by means of minor adjustments of the procedure presented hereafter.

If the coefficient $b_0(\cdot)$ is assumed to be known, there is no loss of generality in addressing the case in which $b_0(\cdot) = 1$. In this setting, the Eq. (3) which describes the system can be rewritten in more concise form as

$$
\dot{w} = s(w)
$$

$$
\dot{z} = f(w, z, \theta + N(\varphi))
$$

$$
\dot{\varphi} = q(w, z, \xi) + u
$$

in which $\xi = \xi_1 = e^{(r-1)}$ and $z = \text{col}(z_0, \xi_1, \ldots, \xi_{r-1})$.

The point of departure for the solution of the problem is, as usual, the assumption of the existence of a (smooth) manifold which can be rendered invariant by feedback and on which the regulated output vanishes (see [1]). In the case of system (3), this amounts to the assumption of the existence of a smooth map $\pi_0 : W \rightarrow \mathbb{R}^{n-1}$ satisfying

$$
\frac{\partial \pi_0}{\partial w} s(w) = f_0(w, \pi_0(w), 0, \ldots, 0) \quad \forall w \in W.
$$

This being the case, it is readily seen that the map

$$
\pi : W \rightarrow \mathbb{R}^{n-1}
$$

$$
w \mapsto \text{col}(\pi_0(w), 0, \ldots, 0)
$$

satisfies (see system (4))

$$
\frac{\partial \pi}{\partial w} s(w) = f(w, \pi(w), 0) \quad \forall w \in W.
$$

The second step in the solution of the problem usually consists in making assumptions that make it possible to build an “internal” model. In a series of recent papers, it was shown how these assumptions could be progressively weakened, moving from the so-called assumption of “immersion into a linear observable system” (as in [3]) to “immersion into a nonlinear uniformly observable system (as in [7])” to the recent results of [9], in which it was shown that no assumption is in fact needed for the construction of an internal model if only continuous (thus possibly not locally Lipschitz) controllers are acceptable. Motivated by this, we assume, in what follows, the existence of a pair $F_0, G_0$, in which $F_0$ is a $d \times d$ Hurwitz matrix and $G_0$ is a $d \times 1$ column vector that makes the pair $F_0, G_0$ controllable, of a locally Lipschitz map $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}$ and a continuously differentiable map $\tau : W \rightarrow \mathbb{R}^d$ satisfying

$$
\frac{\partial \tau}{\partial w} s(w) = F_0 \tau(w) + G_0 \gamma(\tau(w)) \quad \forall w \in W
$$

$$
-q(w, \pi(w), 0) = \gamma(\tau(w)) \quad \forall w \in W.
$$

Consider now (see also [13, 14] in this respect), for the plant (4), a controller of the form

$$
u = N(\varphi) + \gamma(\eta) + v
$$

$$v = -k[\xi - N(\varphi)]
$$

$$\dot{\eta} = F_0(\eta - G_0[\xi - N(\varphi)]) + G_0[\gamma(\eta) + v]
$$

$$\dot{\varphi} = L(\varphi + M\theta) - Mu
$$

which is a dynamic controller, with internal state $(\eta, \varphi)$, “driven” only by the measured variable $\xi$. Changing $\xi$ into $\theta = \xi - N(\varphi)$ yields a system

$$
\dot{w} = s(w)
$$

$$\dot{z} = f(w, z, \theta + N(\varphi))
$$

$$\dot{\varphi} = q(w, z, \xi) + u
$$

in which $\xi = \xi_1 = e^{(r-1)}$ and $z = \text{col}(z_0, \xi_1, \ldots, \xi_{r-1})$. This system can be regarded as a system with input $v$ and output $\theta$, having relative degree 1, with input $v$ to be chosen as $v = -k\theta$, that is as a negative output feedback. To facilitate the analysis, we bring the system in normal form, changing variables as

$$
\chi = \varphi + M\theta
$$

$$x = \eta - G_0\theta
$$
which yields
\[
\dot{w} = s(w) \\
\dot{z} = f(w, z, \theta + N(\chi - M\theta)) \\
\dot{x} = L(\chi) + \gamma(x + G_0\theta) \\
\dot{\chi} = F_0x - G_0q(w, z, \theta + N(\chi - M\theta)) \\
\theta = q(w, z, \theta + N(\chi - M\theta)) + \gamma(x + G_0\theta) - k\theta.
\] (8)

The system in question can be seen as feedback interconnection of a system with input \(\theta\) and state \((w, z, x, \chi)\)
\[
w = s(w) \\
\dot{z} = f(w, z, \theta + N(\chi - M\theta)) \\
\dot{x} = L(\chi) + \gamma(x + G_0\theta) \\
\dot{\chi} = F_0x - G_0q(w, z, \theta + N(\chi - M\theta))
\] and of a system with input \((w, z, x, \chi)\) and state \(\theta\)
\[
\theta = q(w, z, \theta + N(\chi - M\theta)) + \gamma(x + G_0\theta) - k\theta.
\] (9)

Suppose that the latter possesses a compact invariant set \(A\) which is asymptotically stable, with a domain of attraction that contains the set of all admissible initial conditions. Then, it is known that, for every \(c > 0\), there is a number \(k\) such that, if \(k \geq k\), all trajectories of the composite system (8) remain bounded and there is a time \(T > 0\) such that
\[
|\theta(t)| \leq c \quad \forall t \geq T.
\]

If, in addition, the set \(A\) is locally exponentially stable and the map \(q(w, z, N(\chi)) + \gamma(x)\) vanishes on \(A\), one can invoke a local version of the small-gain theorem and claim the existence of a number \(k^*\) such that, if \(k \geq k^*\), all trajectories of the composite system (8) remain bounded and, moreover, \((w, z, x, \chi)\) converges to \(A\) while \(\theta\) converges to 0. If \(\xi_1\) vanishes on \(A\), then also \(e\) converges to 0 and the problem of output regulation is solved.

We have in this way identified an auxiliary problem which, if solved, makes the controller (7) solving the problem of output regulation for the original plant: find, if possible, a triplet \((L(\sigma), M, N(\psi))\) such that system (10) possesses a compact invariant set \(A\) which is locally exponentially stable and attracts all admissible initial conditions, and such that \(\xi_1\) and \(q(w, z, N(\chi)) + \gamma(x)\) vanish on this set.

3. An alternative design procedure

It is from this point that the analysis of the present paper differs from the one described in [13]. Recall that, by assumption, there exists \(\sigma(w)\) and \(\tau(w)\) satisfying (5) and (6). Hence, it is readily seen that if \(L(0) = 0\) and \(N(0) = 0\), the set
\[
A = \{(w, z, \chi, x) : w \in W, z = \sigma(w), \chi = 0, x = \tau(w)\}
\]
is a compact invariant set of (10). Moreover, \(\xi_1\), and \(q(w, z, N(\chi)) + \gamma(x)\) vanish on this set. Thus, this is the set for which local exponential stability will be sought (with a domain of attraction that contains the compact set of all admissible initial conditions).

To determine whether this is achievable, it is convenient to perform another (last !) change of coordinates
\[
z_a = z - \sigma(w) \\
\tilde{x} = x - \tau(w)
\]
and define
\[
f_\delta(w, z_a, \chi) = f(w, z_a + \sigma(w), \chi) - f(w, \sigma(w), 0)
\]
\[
y_a = h_\delta(w, z_a, \chi) = \chi - h(w, \sigma(w), \chi)
\]
and define the system which, in view of a subsequent interpretation, we call a "weighting filter", modelled by equations of the form
\[
\dot{\tilde{x}} = F_0\tilde{x} - G_0h(w, z_a, N(\chi))
\]
\[
y_f = \gamma(\tilde{x} + \tau(w))
\]
and of a "controller" modelled by equations of the form
\[
\dot{x}_c = L(\chi) + Mu_c
\]
\[
y_c = N(\chi).
\]

As a matter of fact, system (10) can be obtained by setting (Fig. 1)
\[
u_c = y_a + v
\]
\[
u_a = y_c
\]
\[
y = y_f.
\]

The constraints (14) define – in particular - a system, with input \(v\) and output \(y_t\), modelled by equations of the form
\[
\dot{w} = s(w) \\
\dot{z}_a = f_\delta(w, z_a, N(\chi)) \\
\dot{\tilde{x}} = L(\chi) + M[h_\delta(w, z_a, N(\chi)) + v] \\
\dot{y}_f = (\tilde{x} + \tau(w)) - \gamma(\tilde{x} + \tau(w)).
\] (16)

The constraint (15) simply expresses the fact that the system thus defined is subject to unitary output feedback. Thus, the problem we face is to choose controller (13) in such a way that when system (16) is put in a unitary output feedback loop, the set \(A\) becomes locally exponentially stable, with a domain of attraction that contains the (compact) set of all admissible initial conditions. In this respect, the small-gain theorem is of help. System (16) being the cascade of two subsystems, one might be tempted to im-

Fig. 1. System (10).
pose that the "gain functions" of the two individual component subsystems be both strictly less than unitary gain functions. This does not help, though, because, as a simple example discussed later shows, such a condition cannot be fulfilled in general. It is important, therefore, to look at the cascade [16] as to a single system.

In the present context, the version of the small gain theorem that best suits the problem is the one for dissipative systems, which—within reference to the system at issue—can be briefly summarized as follows. Note (set, to this extent, $p = \text{col}(z_a, \chi, x)$) that (16) is a special case of a system of the form

$$\dot{w} = s(w)$$

$$\dot{p} = F(w, p, v)$$

$$y_1 = H(w, p, v)$$

in which $F(w, 0, 0) = 0$ and $H(w, 0, 0) = 0$. Retain the assumption that the state $w$ of $\dot{w} = s(w)$ evolves on a compact invariant set $W$. Let $V(p)$ be a positive definite and proper function with quadratic bounds for small $|p|$, namely satisfying

$$\alpha_1 |p|^2 \leq V(p) \leq \alpha_2 |p|^2,$$

for some $\alpha_1 > 0$, $\alpha_2 > 0$. Suppose that

$$\frac{\partial V}{\partial p} F(w, p, v) + g^2 |v|^2 + [H(w, p, v)]^2 \leq -\alpha |p|^2 \quad \forall(w, p, v)$$

for some $\alpha > 0$. If this is the case, system (17) is said to be strictly dissipative, with respect to the supply rate $g(v, y_1) = g^2 v^2 - y_1^2$, with a locally quadratic storage function $V(p)$ (see e.g. [16, page 42]). If the above inequality holds for some $\alpha < 1$, it is readily seen that in the system obtained by closing the loop with $v = y_1$ (as (15) dictates), the compact invariant set $A = \{(w, p) : w \in W, |p| = 0\}$ is globally asymptotically and locally exponentially stable, actually with Lyapunov function $V(p)$. The dissipativity property (18) is precisely what will be sought for in the sequel.

Remark. It is interesting to see how the proposed procedure specializes in the case of a linear system. In this case, it is readily seen that the auxiliary plant is a linear system of the form

$$\dot{z}_2 = A_2 z_2 + B_2 u,$$

$$y_2 = C_2 z_2 + D_2 u,$$

and a linear controller

$$\dot{\chi} = L_2 x + M_2 u,$$

$$y_1 = N_2 \chi$$

can be chosen. The matrix $F_0$ being Hurwitz, system (16) is stable if and only if the "controller" (20) stabilizes the "auxiliary" plant (19). Let $P(s)$ denote the transfer function of system (19), let $G(s)$ denote the transfer function of the controller (20) and let $F(s)$ denote the transfer function of the filter (12). Set

$$T(s) = \frac{G(s)P(s)}{1 - G(s)P(s)}.$$  

Then, the transfer function of system (16), between input $v$ and output $y_1$, is simply

$$W(s) = F(s)T(s).$$

In view of the above, we can conclude that the design goal is achieved if the "controller" (20) stabilizes the "auxiliary" plant (19) and there is a number $0 < \gamma < 1$ such that

$$\|F(s)T(s)\|_{\infty} \leq \gamma.$$  

(22)

The result thus obtained shows that the problem of solving a (robust) problem of output regulation of a system of the form (4) can be reduced to the problem of robust stabilization— with a supplementary "filtered" gain constraint— of the auxiliary plant (11). We summarize the result as follows.

**Proposition 1.** Consider a problem of output regulation for a plant modelled by equations of the form (4). Let $(F_0, G_0)$ be a controllable pair, with $F_0$ a Hurwitz matrix, such that (6) hold for some locally Lipschitz $\gamma(\cdot)$ and some continuously differentiable $\tau(\cdot)$. Let $(L(t), M, N(\cdot))$ be such that the associated controller (13) renders system (16) strictly dissipative, with respect to the supply rate $q(v, y_1) = g^2 v^2 - y_1^2$, with a locally quadratic storage function and $g < 1$. Then, there exists a number $k^*$ such that, for all $k > k^*$, the controller (7) solves the problem of (semiglobal) output regulation.

We stress that the result summarized in this way may indeed cover cases in which the original plant (4) is non-minimum phase. In this case, in fact, the dynamics of the auxiliary plant (11) will be unstable, but indeed there might exist cases in which a controller of the form (13) exists, which stabilizes the auxiliary plant and meets the required "filtered" gain constraint. This is precisely what will be shown in the sequel.

4. Design for a non-minimum-phase plant

Of course, the approach described in the previous section is useful if the auxiliary stabilization problem, with a supplementary gain constraint, can be solved at all. We show that this is indeed the case, even if the original controlled plant (4) is non-minimum phase. In this case, an interesting trade-off between tracking specifications and "distance from stability" of the zero dynamics of the plant can also be established.

As a design example, we consider, in what follows, cases in which the internal model has a pair of purely imaginary eigenvalues at $\pm j\Omega$. This corresponds to a regulation problem in which the exogenous inputs (to be followed and/or rejected) are sinusoidal functions of time. Pick, without loss of generality, a controllable pair $F_0$, $G_0$

$$F_0 = \begin{pmatrix} 0 & 1 \\ -c_0 & -c_1 \end{pmatrix}, \quad G_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with $c_0 > 0$, $c_1 > 0$. In this case, one can take $\gamma(x) = \Psi x$ and $\Psi$, the unique vector which assigns to $F_0 + G_0 \Psi$ the characteristic polynomial $\lambda^2 + 2\Omega^2$, is necessarily

$$\Psi = (c_0 - \Omega^2)^{-1} c_1.$$  

The resulting transfer function $F(s)$ of the filter (12) has the expression

$$F(s) = \frac{-\Omega^2 + c_1 s + c_0}{s^2 + c_1 s + c_0}.$$  

In view of the fulfillment of the gain constraint (which, in the special case of a linear plant, takes the form of the constraint (22)), it is appropriate to choose the coefficients $c_0, c_1$, which are free parameters in the design, so as to reduce the amplitude of $|F(\omega)|$. Indeed, at $\omega = \Omega$ this amplitude is necessarily 1. But it can be lowered at different frequencies, for instance picking $c_0 = \Omega^2$ and $c_1 = 2 \Omega$, in which case

$$F(s) = \frac{-2 \Omega s}{(s + \Omega)^2},$$

with an amplitude which is maximal at $\omega = \Omega$.

With this choice of $F_0$, system (16) becomes

$$\dot{w} = s(w)$$

$$\dot{z}_2 = f_0(w, z_2, N(\chi))$$

$$\dot{\chi} = L(\chi) + M[h_1(w, z_2, N(\chi)) + v]$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\Omega^2 x_1 - 2 \Omega x_2 - h_2(w, z_2, N(\chi))$$

$$y_1 = 2 \Omega x_2$$.
Consider the (globally defined) change of variable
\[ \dot{x}_1 = -\Omega^2 x_1 - h_x(w, z, N(x)) \] (24)
which yields
\[ \dot{w} = s(w) \]
\[ \dot{z}_1 = f_a(w, z, N(x)) \]
\[ \dot{x} = L(x) + M[h_x(w, z, N(x)) + v] \] (25)
\[ \dot{y}_1 = \bar{h}_x(w, z, N(x), v) \]
\[ \dot{x}_2 = \dot{y}_1 - 2\Omega x_2 \]
\[ y_t = 2\Omega x_2 \]

where
\[ h_x(w, z, N(x), v) : = \frac{\partial h_x}{\partial w}s(w) + \frac{\partial h_x}{\partial z}f_a(w, z, N(x)) \]
\[ + \frac{\partial h_x}{\partial \chi}(L(x) + M[h_x(w, z, N(x)) + v]). \]

System (25) is simply the cascade connection of
\[ \dot{w} = s(w) \]
\[ \dot{z}_1 = f_a(w, z, N(x)) \]
\[ \dot{x} = L(x) + M[h_x(w, z, N(x)) + v] \]
\[ y = \bar{h}_x(w, z, N(x), v) \]
and
\[ \dot{x}_1 = -\Omega^2 x_1 - y \]
\[ \dot{x}_2 = \dot{x}_1 - 2\Omega x_2 \]
\[ y_t = 2\Omega x_2 \] (27)
The latter is a stable linear system, with transfer function
\[ \Phi(s) = -\frac{2\Omega}{(s + \Omega)^2} \]
whose \( H_\infty \) norm is equal to \( 2/\Omega \). Thus, by known facts, a sufficient condition for system (16) to be strictly dissipative, with respect to the supply rate \( q(v, y) = g^2 v^2 - y^2 \), with a locally quadratic storage function and \( g < 1 \), is that the “simpler” system (26) be strictly dissipative, with respect to the supply rate \( q(v, y) = g^2 v^2 - y^2 \), with a locally quadratic storage function and
\[ 2g < \Omega. \] (28)

Remark. The change of variable (24) simply “moves”, in the cascade of (11) and (12), a “derivative action” from the filter to the output of the auxiliary plant. This, as it will be shown in a moment, may simplify the task of meeting the gain constraint. Note also that this has reduced the problem of output regulation of (4) to the problem of globally asymptotically stabilizing, with a simple gain constraint, the closed loop system (26), which is a problem only dependent on the auxiliary plant (11) and not on the choice of the internal model.

We obtain, in this way, an alternative characterization of the result presented earlier, that can be summarized as follows.

**Proposition 2.** Consider a problem of output regulation for a plant modelled by equations of the form (4), and an internal model with a pair of imaginary eigenvalues at \( \pm i\Omega \). Let \( (L(\cdot), M, N(\cdot)) \) be such that the associated controller (13) renders system (26) strictly dissipative, with respect to the supply rate \( q(v, y) = g^2 v^2 - y^2 \), with a locally quadratic storage function and \( g < \Omega/2 \). Then, there exists a number \( k^* \) such that, for all \( k > k^* \), the controller (7) solves the problem of (semiglobal) output regulation.

As an example of application, we discuss now, in full, the case of a plant modelled by equations of the form
\[ \dot{w} = S(w) \]
\[ \dot{z}_1 = \bar{b}(z_1, z_2, w)z_2 \]
\[ \dot{e} = e \]
\[ y_t = z_1 + Q w + u \]
with
\[ S = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}, \quad Q = (1 \ 0) \]
in which we assume the existence of two numbers \( 0 < b_{\min} < b_{\max} \) such that
\[ b_{\min} \leq \bar{b}(z_1, z_2, w) \leq b_{\max} \forall (z_1, z_2, w). \]
To the best of our knowledge, this example cannot be treated by means of the currently known methods, because the zero dynamics of the controlled plant is unstable and only \( e \) is assumed to be available for measurement. One can immediately check that the map \( \pi_0(w) \) (which coincides with \( \pi(w) \) because the relative degree \( r = 1 \) is the trivial map \( \pi_0(w) = 0 \). One can also immediately check that a linear internal model of the form discussed above can be used.

In this specific example, the auxiliary plant (11) is a system of the form
\[ \dot{w} = S(w) \]
\[ \dot{z}_3 = \bar{b}(z_3, z_2, w)z_2 \]
\[ y = \bar{b}(z_3, z_2, w)z_2 \] (30)
and hence the output \( y \) of (26) is
\[ y = y_3 = \bar{b}(z_3, z_2, w)z_2. \]
We choose for (13) a linear controller
\[ \dot{\chi} = L\chi + Mu \]
\[ y_\chi = N\chi \]
in which \( L, M, N \) is a minimal realization of the transfer function
\[ G(s) = -\frac{\kappa (s + \varepsilon)}{(1 + T^2)^2} \] (31)
with \( T \) is a small number and \( \kappa > 0, \varepsilon > 0 \) are parameters to be determined so as to obtain the property indicated in Proposition 2.

To proceed with the design, we initially assume that \( T = 0 \), in which case the controller in question becomes
\[ y_\chi = -\kappa (u_\varepsilon + \varepsilon u_\varepsilon), \]
and system (26), accordingly, becomes
\[ \dot{w} = S(w) \]
\[ \dot{z}_3 = \bar{b}(z_3, z_2, w)z_2 \]
\[ y = \bar{b}(z_3, z_2, w)z_2. \] (33)
The change of variables
\[ x_1 = \frac{z_3}{k} \quad x_2 = \frac{z_2}{k} + v \]
yields a system of the form
\[ \dot{x} = \begin{pmatrix} 0 & b(x, v, v) \\ -k & -b(x, v, v) \end{pmatrix} x + \begin{pmatrix} -b(x, v, v) \\ b(x, v, v) - v \end{pmatrix} u \] (34)
with output
\[ y = b(x, w, v)x - v, \]  
(35)
in which
\[ b(x, w, v) = \tilde{b}(\kappa x_1, \kappa (x_2 - v), w). \]

For convenience, the exosystem has been omitted.

In view of all of the above, and of the fact that \( b(x, w, v) \leq b_{\text{max}} \), the task is achieved if system (34), with output
\[ \dot{y} = (0 \ 1) x - v, \]  
(36)
is strictly dissipative, with respect to the supply rate \( q(v, y) = g^2v^2 - y^2 \), with a locally quadratic storage function, and \( g \) satisfies
\[ 2g b_{\text{max}} < \Omega. \]  
(37)

System (34) with output (36) can be formally written as a linear system
\[ \dot{x} = A(x, w, v)x + B(x, w, v)v \]
\[ \dot{y} = Cx - v \]
with obvious meaning of \( A(x, w, v), B(x, w, v), C \). To check the property of strict dissipativity, we appeal to the existence of a quadratic storage function \( V(x) = x^TPx \). In this case, the dissipation inequality (18) reduces to
\[ 2x^TP[A(x, w, v)x + B(x, w, v)v] - g^2v^2 + [Cx - v]^2 \leq -\alpha x^2 \ \forall (x, w, v). \]

By standard completion of the squares, and using the fact that the coefficient \( b(x, w, v) \) ranges on a compact set, it is readily seen that this inequality holds if \( g > 1 \) and the symmetric matrix
\[ PA(x, w, v) + A(x, w, v)^TP + C^T \]
\[ + \frac{1}{g^2 - 1}(PB(x, w, v) - C^T)(PB(x, w, v) - C^T)^T \]  
(38)
is negative definite, for all \( (x, w, v) \) (see e.g. [16, page 64]).

Choose now
\[ P = \alpha \begin{pmatrix} \kappa & 1 \\ 1 & \kappa \end{pmatrix} \]
in which \( \alpha > 0 \). This matrix is positive definite. Appropriate calculations show that (38) is a matrix \( Q \) in which
\[ q_{11} = -2\alpha \epsilon + \frac{g^2\epsilon^2}{g^2 - 1} \]
\[ q_{22} = -2\alpha \epsilon + \frac{1}{g^2 - 1} \left( \frac{ab}{\kappa} - 2\alpha \epsilon \right)^2 + 1 \]
\[ q_{12} = -2\alpha \epsilon + \frac{1}{g^2 - 1} \left( \alpha \epsilon - \frac{a^2b}{\kappa} + \frac{2\alpha \epsilon^2}{\kappa} \right). \]

In these expressions, we have replaced \( b(x, w, v) \) simply by \( b \). The matrix \( Q \) is negative definite if and only if \( q_{11} < 0 \) and \( \text{det}(Q) > 0 \). The first condition is indeed achieved if \( \epsilon \) is small. As for the second condition, note that \( \text{det}(Q) \) is a polynomial in \( \epsilon \)
\[ \text{det}(Q) = \epsilon (a_0 + a_1 \epsilon + a_2 \epsilon^2 + a_3 \epsilon^3) \]
in which
\[ a_0 = -2\alpha \epsilon \left( -2ab + \frac{1}{g^2 - 1} (ab - 1)^2 + 1 \right). \]
Seeking a small value of \( \epsilon \), we try to make \( a_0 > 0 \), which is the case if
\[ (ab)^2 - 2g^2(ab) + g^2 < 0. \]  
(39)

This is a second degree inequality in \( (ab) \), which holds if and only if
\[ g^2 - \sqrt{g^4 - g^2} < \alpha b(x, w, v) < g^2 + \sqrt{g^4 - g^2}. \]  
(40)

Recall that \( 0 < b_{\text{min}} \leq b(x, w, v) \leq b_{\text{max}} \), and define
\[ b_0 = \frac{1}{2} (b_{\text{max}} + b_{\text{min}}) \]
in which \( 0 < c < b_{\text{min}} \). Note that \( 0 < \delta < 1 \). Trivially
\[ 0 < b_0(1 - \delta) < b(x, w, v) < b_0(1 + \delta). \]
Choosing
\[ \bar{g} = \frac{1}{1 - \delta^2} \]
\[ a = \frac{1}{b_0(1 - \delta^2)} \]
it is seen that the inequality (40) holds, and therefore the left-hand side of (39) is strictly negative for all \( b(x, w, v) \) in the range \([b_{\text{min}}, b_{\text{max}}]\).

Recall now that the target was to achieve the inequality (37). With \( g \) chosen as indicated above (this, as we have seen, only depends on \( b_{\text{min}} \) and \( b_{\text{max}} \)) pick and fix \( \kappa \) small enough so that (37) holds, for the assigned \( \Omega \). At this point, all parameters in \( q_{11}, q_{12}, q_{22} \) have been fixed, except \( \epsilon \). Bearing in mind the fact that \( b(x, w, v) \) ranges on a compact set, it is readily seen that there are numbers \( q^* > 0 \) and \( a^* > 0 \), and a number \( \epsilon^* > 0 \) such that, if \( 0 < \epsilon \leq \epsilon^* \)
\[ q_{11} \leq -\alpha \epsilon^2 < 0, \]
\[ \text{det}(Q) \geq \epsilon a^* > 0 \]
for all \( (x, w, v) \) as requested.

We have shown, in this way, that, with the indicated choices of \( \kappa \) and \( \epsilon \), system (34) with output (36) is strictly dissipative with a quadratic storage function, with respect to the supply rate \( q(v, y) = g^2v^2 - y^2 \) and \( g \) satisfies (37). In view of the above, this means that the system obtained by interconnecting, as in (14), the auxiliary plant (30), the controller (32) and the filter (12), is strictly dissipative, with a quadratic storage function, with respect to the supply rate \( q(v, y) = g^2v^2 - y^2 \) and \( g < 1 \). This, in turn, implies that the system obtained by setting \( v = y_1 \) as in (15) is globally asymptotically stable, actually with a quadratic Lyapunov function.

At this point, we use a result proven, e.g. in [16, Proposition 12.6.3], which claims that if a globally asymptotically stable feedback loop is perturbed by the insertion of system having transfer function
\[ W(s) = \frac{1}{1 + Ts} \]
and \( T > 0 \) is small enough, the stability of the equilibrium is preserved, with a domain of attraction that can be rendered arbitrarily large by lowering \( T \). Appealing (twice) to this result we replace – as anticipated – the controller (32) by a minimal realization of the transfer function (31) and obtain a controller that fits into the design framework presented earlier in the paper and solves the problem of output regulation.

5. Performance limitations

As a second example, consider the special case of a linear system in which \( \text{dim}(z) = 1 \), modelled by equations of the form
\[ \dot{w} = Sw \]
\[ \dot{z} = az + be + Pw \]
\[ \dot{e} = cz + de + Qw + u \]
in which \( a > 0 \), and \( S, Q \) are as in the example discussed earlier. This system, again, is a system possessing an unstable zero dynamics. Trivial calculations show that
\[ \pi_0(w) = -(al - S)^{-1}Pw \]
and that the auxiliary system (11) is
\[ \dot{w} = Sw, \]
\[ \dot{z}_a = az_a + bz_a, \]
\[ y_a = cz_a + du_a. \]
The latter has a transfer function of the form
\[ P_a(s) = \frac{d(s + \bar{z})}{s - a}. \]
Suppose
\[ a \in [a_0, a_1], \quad 0 < a_0 < a_1 \]
\[ d \in [d_0, d_1], \quad 0 < d_0 < d_1 \]
\[ \bar{z} \in [\bar{z}_0, \bar{z}_1], \quad 0 < \bar{z}_0 < \bar{z}_1, \]
Choose for (20) a transfer function
\[ G(s) = \frac{\kappa}{s + p}. \]
Pick
\[ p > \bar{z}_1, \quad \kappa > \frac{a_3 p}{d_0 \bar{z}_0} \]
in which case
\[ T(s) = \frac{G(s)P_a(s)}{1 - G(s)P_a(s)} = \frac{-\kappa d (s + \bar{z})}{(s + p_1)(s + p_2)} \]
and \( p_1, p_2 \) are real numbers satisfying
\[ p_2 > \bar{z} > p_1 > 0. \]
Choose \( F_0, G_0, \Psi \) as in the example discussed earlier, yielding
\[ \Phi(s) = -\frac{2 \Omega s}{s + \Omega} \]
Then, it is easily checked that
\[ |\Phi(s)T(s)|_{\infty} \leq \frac{2}{\Omega} \kappa d. \]
The required condition for stability of (10) is that
\[ 2\kappa d < \Omega. \]
In view of the constraint on \( \kappa \), this can be obtained if
\[ \Omega > 2a_1 \frac{d_1}{d_0} \frac{\bar{z}_1}{\bar{z}_0}. \tag{41} \]

**Remark.** In the result indicated above, we succeeded to obtain the inequality
\[ |\Phi(s)T(s)|_{\infty} < 1. \]
It is worth stressing, in this respect, that it is only because we have looked at the product \( \Phi(s)T(s) \) that this was possible. The more conservative approach of seeking separately
\[ |T(s)|_{\infty} \leq g_T, \quad |\Phi(s)|_{\infty} \leq g_\Phi \]
for some \( g_T \leq 1 \) and \( g_\Phi \leq 1 \) would not work.
In fact, as observed before, \( g_\Phi \) is necessarily larger than or equal to 1 (and the choice of \( F_0, G_0, \Psi \) above is precisely the one that makes \( g_\Phi = 1 \)). On the contrary, if the auxiliary system (11) is unstable (as is it the case when the original plant is not minimum phase), \( g_T \) is necessarily larger than 1. Considering, for instance, the case in which the parameters of \( P_a(s) \) are fixed, we see that stability of \( T(s) \) is achieved if
\[ kd\bar{z} > ap \]
in which case
\[ |T(s)|_{\infty} = \frac{k d \bar{z}}{kd\bar{z} - ap} \]
is a number larger than 1. Fig. 2 illustrates the example with \( d = 1, \bar{z} = 0.5, a = 1, p = 0.75, k = 1 \) and \( \Omega = 5 \). In this case,
\[ |T(s)|_{\infty} = 4, \quad |\Phi(s)|_{\infty} = 1 \]
but \( |\Phi(s)T(s)|_{\infty} \simeq 0.8 \).

6. Conclusions

Most of the design methods, proposed in recent years, for the design of controllers to the purpose of solving problems of asymptotic tracking and disturbance rejection, only address systems in normal form with a (globally) stable zero dynamics. In this paper, we propose a design approach that – to some extent – is able to overcome this limitation. A modification of the design strategy suggested in [13] shows that the problem in question can be solved – even in the presence of an unstable zero dynamics – if an appropriate associated stabilization problem with a supplemental “gain constraint” is solved. The remarkable feature of the proposed approach is that the associated stabilization problem thus introduced is independent of the choice of the internal model. It rather only depends on a subsystem related to the zero dynamics of the controlled plant and on certain parameters of the exosystem. A pair of relevant examples are thoroughly discussed, from which it is seen that, for a non-minimum phase system, tracking performance is restricted. Specifically, in the case of sinusoidal exogenous input, these restrictions take the form of a lower bound of the frequency of the exogenous input, and certain parameters associated with the zero dynamics of the controlled plant.
References