The Steady-State Behavior of a Nonlinear System

A. Isidori and C.I. Byrnes

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1 Limit Sets

In linear system theory, if the variables which characterize the behavior of a system are either constant or periodic functions of time, the system is said to be in *steady state*. In a stable linear system, the steady state can be seen as a *limit* behavior, approached either as the *actual* time t tends to $+\infty$ or, alternatively, as the *initial* time t_0 tends to $-\infty$ (the two viewpoints being in fact equivalent). For a general nonlinear dynamical system, concepts yielding to a notion of steady state repose on certain fundamental ideas dating back to the works of H. Poincaré and G.D. Birkhoff. [1] In particular, a fundamental role is played by the concept of ω-limit set of a given point, which is defined as follows. Consider an *autonomous* ordinary differential equation

$$\dot{x} = f(x) \tag{1}$$

in which $x \in \mathbb{R}^n$. Suppose that $f(x)$ is locally Lipschitz. Then, it is well-known that, for any $x_0 \in \mathbb{R}^n$, the solution of (1) with initial condition $x(0) = x_0$, denoted in what follows by $x(t, x_0)$, exists on some open interval of the point $t = 0$ and is unique.

Suppose that, for some x_0, the solution $x(t, x_0)$ is defined for all $t \geq 0$, that is for all forward times. A point x is said to be an ω-limit *point* of the trajectory $x(t, x_0)$ if there exists a sequence of times $\{t_k\}$, with $\lim_{k \to \infty} t_k = \infty$, such that

$$\lim_{k \to \infty} x(t_k, x_0) = x \,.$$

The ω-limit *set* of a point x_0, denoted $\omega(x_0)$, is *the union* of all ω-limit points of the trajectory $x(t, x_0)$.

It is obvious from this definition that an ω-limit point *is not* necessarily a limit of $x(t, x_0)$ as $t \to \infty$, since the solution in question may not admit any limit as $t \to \infty$. However, it

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[1] Relevant, in this regard, are the concepts introduced by G.D. Birkhoff in his classical 1927 essay, where he asserts that "with an arbitrary dynamical system ... there is associated always a closed set of 'central motions' which do possess this property of regional recurrence, towards which all other motions of the system in general tend asymptotically" [1, page 190].

happens that if the motion $x(t, x_0)$ is *bounded*, then $x(t, x_0)$ asymptotically approaches *the set $\omega(x_0)$*. In fact, the following property holds [1, page 198].

Lemma 1 *Suppose $x(t, x_0)$ is bounded in forward time, i.e. there is a $M > 0$ such that $\|x(t, x_0)\| \leq M$ for all $t \geq 0$. Then, $\omega(x_0)$ is a nonempty compact connected set, invariant under the dynamics (1).*[2] *Moreover, the distance of $x(t, x_0)$ from $\omega(x_0)$ tends to 0 as $t \to \infty$.*[3]

One of the features of the set $\omega(x_0)$, highlighted in this Lemma, is that this set is *invariant* for (1). Hence, the set $\omega(x_0)$ is filled by trajectories of (1) which are defined for all backward and forward times, and also bounded, because so is the set $\omega(x_0)$. The other relevant feature is that $x(t, x_0)$ asymptotically *approaches* the set $\omega(x_0)$ as $t \to \infty$, in the sense that the distance of the point $x(t, x_0)$ from the set $\omega(x_0)$ tends to 0 as $t \to \infty$.

Suppose now that the initial conditions of (1) range over some set B and that $x(t, x_0)$ is bounded, in forward time, for any $x_0 \in B$. Since any trajectory which is bounded in forward time asymptotically approaches its own ω-limit set $\omega(x_0)$, it concluded that any trajectory obtained by picking x_0 in B asymptotically approaches the set

$$\psi(B) = \bigcup_{x_0 \in B} \omega(x_0) \,.$$

The set in question is filled by trajectories of (1) which are bounded in forward an backward time. Since the set $\psi(B)$ is approached asymptotically by any trajectory with initial condition in B, it seems plausible to regard to the set of all trajectories evolving in $\psi(B)$ as the set of steady state "behaviors" of (1). There is, however, a problem in taking this as definition of steady state behavior of a nonlinear system: the convergence of $x(t, x_0)$ to $\psi(B)$ is not guaranteed to be *uniform* in x_0, even if the set B is compact (see, e.g. [2]).

Uniform convergence to the steady state, which is automatically guaranteed in the case of linear systems, is an important feature to be kept in extending the notion of steady state from linear to nonlinear systems. In fact, the notion of steady state would lose much of its practical relevance if the convergence were not uniform, i.e. if the time needed to get within an ε-distance from the steady state could grow unbounded when the initial state is varied (even when the latter is picked within a fixed bounded set). Thus, the set $\psi(B)$ is not a good candidate for a definition of steady state in a nonlinear system. There is a larger set, though, which does have this property of uniform convergence. This set, known as the ω limit set *of a set*, is defined as follows. Suppose that, for all $x_0 \in B$, the trajectory $x(t, x_0)$ is defined for all $t \geq 0$. The ω-limit set of B, denoted $\omega(B)$, is the set of all points x for which there exists a sequence of pairs $\{x_k, t_k\}$, with $x_k \in B$ and $\lim_{k \to \infty} t_k = \infty$ such that

$$\lim_{k \to \infty} x(t_k, x_k) = x \,.$$

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[2] A set S is invariant for (1) if, for any $x_0 \in S$, the solution $x(t, x_0)$ exists for all $t \in \mathbb{R}$ and $x(t, x_0) \in S$ for all $t \in \mathbb{R}$.

[3] The distance of a point $x \in \mathbb{R}^n$ from a set $S \subset \mathbb{R}^n$, written as $\mathrm{dist}(x, S)$, is defined as $\inf_{y \in S} \|y - x\|$.

It is clear from this definition that if B consists of only one single point x_0, all x_k's in the definition above are necessarily equal to x_0 and the definition in question reduces to the definition of ω-limit set of a point. It is also clear form this definition that, if for some $x_0 \in B$ the set $\omega(x_0)$ is nonempty, all points of $\omega(x_0)$ are points of $\omega(B)$. In fact, all such points have the property indicated in the definition, if all the x_k's are taken equal to x_0. Thus, in particular, if all motions with $x_0 \in B$ are bounded in forward time, $\psi(B) \subset \omega(B)$. However, the converse inclusion is not true in general. The simplest example in which this fact can be checked is the the case of a stable Van der Pol Oscillator

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - \epsilon(1 - x^2)y \qquad \epsilon > 0 \,. \end{aligned}$$

If B is a disc of sufficiently large radius, centered at $(x, y) = (0, 0)$, the set $\psi(B)$ consists of the union of the (unstable) equilibrium at $(x, y) = (0, 0)$ and of the (stable) limit cycle. On the contrary, the set $\omega(B)$ consists of all points on the limit cycle and of all points inside this limit cycle.

The counterpart of Lemma 1 is the following result, which characterizes the relevant features of the concept of the ω-limit set of a set [3, page 8].

Lemma 2 *Let B be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number M such that $\|x(t, x_0)\| \leq M$ for all $t \geq 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under (1). Moreover, the distance of $x(t, x_0)$ from $\omega(B)$ tends to 0 as $t \to \infty$, uniformly in $x_0 \in B$. If B is connected, so is $\omega(B)$.*

As in the case of the set $\psi(B)$, it is seen that the set $\omega(B)$ is filled with trajectories which are defined for all backward and forward times, and bounded. But, above all, it is seen that the set in question is *uniformly* approached by trajectories with initial state $x_0 \in B$, a property that the set $\psi(B)$ does not have. The set $\omega(B)$ asymptotically attracts, as $t \to \infty$, all motions that start in B. Since the convergence to $\omega(B)$ is uniform in x_0, it is also true that, whenever $\omega(B)$ is contained in the interior of B, the set $\omega(B)$ is *asymptotically stable*, in the sense of Lyapunov, i.e. for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that, if the distance of x_0 from $\omega(B)$ is less than δ, then the distance of $x(t, x_0)$ from $\omega(B)$ is less than ε for all $t \geq 0$. This property is very important in nonlinear feedback design, because it is a key property in establishing the existence of Lyapunov functions, as often required in most results concerning the design of stabilizing feedback laws.

2 The Steady State Behavior of a System

Consider an autonomous finite-dimensional system

$$\dot{x} = f(x) \,. \tag{2}$$

with initial conditions in a closed subset $X \subset \mathbb{R}^n$, and suppose the set X is *forward invariant*, that is for any initial condition $x_0 \in X$, the solution $x(t, x_0)$ exists for all $t \geq 0$ and $x(t, x_0) \in X$ for all $t \geq 0$.

The motions of this system are said to be *ultimately bounded* if there is a bounded subset B with the property that, for every compact subset $\bar{X}$ of X, there is a time $\bar{T} > 0$ such that $x(t, x_0) \in B$ for all $t \geq \bar{T}$ and all $x_0 \in \bar{X}$. In other words, the motions are ultimately bounded if any admissible trajectory in finite time (this finite time possibly being dependent on the chosen initial condition) enters a bounded set B, and remains in this set for all future times.

If the motions of a system are ultimately bounded, in particular all trajectories with initial conditions in B are bounded. As a consequence, the limit set $\omega(B)$ is non empty. Since all trajectories with initial conditions in X eventually enter the set B, the set $\omega(B)$ – which by Lemma 2 attracts all trajectories with initial condition in B – also attracts all trajectories with initial conditions in X. It is therefore natural to regard $\omega(B)$ a set to which all admissible trajectories of (2) converge. While the set B, in the definition of ultimate boundedness, is not uniquely characterized, it is easy to see that the set $\omega(B)$ is, on the contrary, a uniquely defined object. In fact, it is possible to prove that if the motions of (2) are ultimately bounded and if $B' \neq B$ is any other bounded subset with the property that, for every compact subset $\bar{X}$ of X, there is a time $\bar{T} > 0$ such that $x(t, x_0) \in B'$ for all $t \geq \bar{T}$ and all $x_0 \in \bar{X}$, then $\omega(B') = \omega(B)$ (see e.g. [2]).

In view of these properties, it is concluded that if the motions of a system are ultimately bounded, any trajectory asymptotically approaches a uniquely defined compact invariant set, the set $\omega(B)$. The latter is nonempty, compact and invariant. In other words, any trajectory of the system approaches a nonempty set, which is in turn filled by other trajectories, which are defined and bounded in forward and backward time. Thus, is natural to look at any of such trajectories as a steady state trajectory and to regard the set $\omega(B)$ as the set in which the steady state behavior of the system takes place. This leads to the following definition (see [2]).

Definition. Suppose the motions of system (2), with initial conditions in a closed and forward invariant set X, are ultimately bounded. A *steady state* motion is any motion with initial condition $x(0) \in \omega(B)$. The set $\omega(B)$ is the *steady state locus* of (2) and the *restriction* of (2) to $\omega(B)$ is the *steady state behavior* of (2). ◁

3 The Steady State Response

The definition given in the previous Section recaptures the classical notion of steady state for linear systems and provides a powerful tool to deal with similar issues in the case of nonlinear systems. We discuss in what follows a number of relevant cases.

Consider a n-dimensional, single-input, *asymptotically stable* linear system

$$\dot{z} = Az + Bu, \tag{3}$$

forced by the harmonic input $u(t) = u_0 \sin(\Omega t + \phi_0)$. It is well known that, regardless of what the initial condition is, the response $z(t)$ converges to a unique, well-defined, steady state

response, a periodic function of period $2\pi/\Omega$. A simple (geometric) method to determine such response consists in viewing the forcing input $u(t)$ as provided by an autonomous "signal generator" of the form

$$\dot{w} = Sw \qquad u = Qw$$

in which

$$S = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

and in analyzing the state state behavior of the associated "augmented" system

$$\begin{aligned} \dot{w} &= Sw \\ \dot{z} &= BQw + Az \,. \end{aligned} \tag{4}$$

Since the matrices S and A do not have common eigenvalues, the Sylvester equation $\Pi S = A\Pi + BQ$ has a unique solutions. The augmented system possesses two complementary invariant subspaces: a *stable invariant subspace* and a *center invariant subspace*. The former is the set of all pairs (w, z) in which $w = 0$, while the latter is the graph of a linear map

$$\begin{aligned} \pi \ : \ \mathbb{R}^2 &\to \ \mathbb{R}^n \\ w &\mapsto \ \Pi w \,. \end{aligned}$$

All trajectories of (4) approach the center invariant subspace as $t \to \infty$, and hence the limit behavior of (4) is determined by the restriction of its motions to this invariant subspace. As a consequence, the steady state response of (3) to the periodic input $u(t) = u_0 \sin(\Omega t + \phi_0)$ is given by

$$x(t) = \Pi w(t) = \Pi \begin{pmatrix} u_0 \sin(\Omega t + \phi_0) \\ u_0 \cos(\Omega t + \phi_0) \end{pmatrix} \,.$$

Revisiting this analysis from the viewpoint of the more general notion of steady state introduce above, let $W \subset \mathbb{R}^2$ be a set of the form

$$W = \{w \in \mathbb{R}^2 : \|w\| \le c\} \tag{5}$$

in which c is a fixed number, and suppose the set of initial conditions for (4) is $W \times \mathbb{R}^n$. This is the case when the problem of evaluating the periodic response of (3) to harmonic inputs whose amplitude does not exceed a fixed number c is addressed. Note that the set W is compact and invariant for the upper subsystem of (4).

The set $W \times \mathbb{R}^n$ is closed and forward invariant for the full system (4) and, moreover, since the lower subsystem of (4) is a linear asymptotically stable system driven by a bounded input, it is immediate to check that the motions of system (4), with initial conditions taken in $W \times \mathbb{R}^n$, are ultimately bounded. In particular,

$$\omega(B) = \{(w, z) \in \mathbb{R}^2 \times \mathbb{R}^n : w \in W, z = \Pi w\} \,,$$

i.e. $\omega(B)$ is the graph of the restriction of the map π to the set W. The restriction of (4) to the invariant set $\omega(B)$ characterizes the steady state behavior of (3) under the family of all harmonic inputs of fixed angular frequency ω, and amplitude not exceeding c.

A similar result holds if $u(t)$ is provided by a nonlinear "signal generator" of the form

$$\dot{w} = s(w), \qquad u = q(w). \tag{6}$$

If fact, consider an augmented system of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= Bq(w) + Az, \end{aligned} \tag{7}$$

in which $w \in W \subset \mathbb{R}^s$, with W a compact set, invariant for the the upper subsystem of (7). Suppose, as before, that the matrix A has eigenvalues with negative real part.

As in the previous example, since the lower subsystem of (7) is a linear asymptotically stable system driven by the bounded input $u(t) = q(w(t, w_0))$, the motions of system (7), with initial conditions taken in $W \times \mathbb{R}^n$, are ultimately bounded. A simple calculation (see [2]) shows that the steady state locus of (7) is the graph of the map

$$\begin{aligned} \pi \;:\; W &\to \quad \mathbb{R}^n \\ w &\mapsto \quad \pi(w), \end{aligned}$$

defined by

$$\pi(w) = \lim_{T \to \infty} \int_{-T}^{0} e^{-A\tau} Bq(w(\tau, w)) d\tau. \tag{8}$$

As a consequence, it is concluded that the steady state response of (3) to an input $u(t)$ produced by a signal generator of the form (6) can be expressed as $z(t) = \pi(w(t))$.

There are various ways in which the result discussed in the previous example can be generalized. For instance, it can be extended to describe the steady state response of a nonlinear system

$$\dot{z} = f(z, u) \tag{9}$$

to an input provided by a nonlinear signal generator of the form (6), if system (9) is input-to-state stable.[4] In this case, in fact, the composition of (9) and (6)

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(z, q(w)), \end{aligned} \tag{10}$$

with initial conditions $(w_0, z_0) \in W \times \mathbb{R}^n$ is a system whose motions are ultimately bounded and hence a well-defined steady state locus exists.

A common feature of the two examples (4) and (7) is the fact that the steady state locus of the system can be expressed as the *graph of a map*, defined on the set W. This means that, so long as this is the case, the system has a *unique* well defined *steady state response* to the input $u(t) = q(w(t))$, expressible as $z(t) = \pi(w(t))$. Of course, in general, this may not be the case and the global structure of the steady state locus could be very complicated. In particular, the set $\omega(B)$ may fail to be the graph of a map defined on W and *multiple*

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[4]For a definition of input-to-state stability, see e.g. XXX

steady state responses to a given input may occur. This is the counterpart – in the context of forced motions – of the fact that, in general, a nonlinear system may possess multiple equilibria. In these cases, the steady state response is determined not only by the forcing input, but also by the initial state of the system to which the input is applied.

Even though, in general, uniqueness of the steady-state response of a system (9) to inputs generated by a system of the form (6) cannot be guaranteed, if the set W is compact and invariant (as assumed above), for each $w \in W$ there is always at least one initial condition z of (9) such that the pair (w, z) produces a steady state response.

Lemma 3 *Let W be a compact set, invariant under the flow of (6). Let Z be a closed set and suppose that the motions of (10) with initial conditions in $W \times Z$ are ultimately bounded. Then, the steady state locus of (10) is the graph of a set-valued map defined on the whole of W.*

This result is particularly useful in establishing certain necessary conditions in the analysis of the problem of nonlinear output regulation.

References

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