1 The $\mathcal{L}_2$ gain of a stable linear system

Consider a linear system described by equations of the form

\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$.

Let $\gamma > 0$ be a fixed number. Suppose there exists a positive definite symmetric $n \times n$ matrix $P$ such that the quadratic function $V(x) = x^TPx$ satisfies, some $\epsilon > 0$, the inequality

$$
\frac{\partial V}{\partial x}(Ax + Bu) \leq -\epsilon \|x\|^2 + \gamma^2 \|u\|^2 - \|Cx + Du\|^2
$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$. Note, to begin with, that this property can only hold if the system is asymptotically stable. For $u = 0$ this reduces in fact to

$$
\frac{\partial V}{\partial x}Ax \leq -\epsilon \|x\|^2
$$

which implies asymptotic stability.

Suppose the input $u(\cdot)$ of (1) is a function in $L_2[0, \infty)$. Integration of the inequality (2) on the interval $[0, T]$ yields, for any initial state $x(0)$,

$$
V(x(T)) \leq V(x(0)) + \gamma^2 \int_0^T \|u(t)\|^2dt - \int_0^T \|y(t)\|^2dt
$$

from which it is deduced that the response $x(t)$ of the system is defined for all $t \in [0, \infty)$ and bounded. Now, suppose $x(0) = 0$ and observe that the previous inequality yields

$$
V(x(T)) \leq \gamma^2 \int_0^T \|u(t)\|^2dt - \int_0^T \|y(t)\|^2dt
$$

A inequality of this kind is often called a dissipation inequality.

The space $L_2[0, \infty)$ is the space of all piecewise-continuous inputs defined on $[0, \infty)$ satisfying

$$
\int_0^\infty \|u(t)\|^2dt < \infty
$$

The nonnegative number

$$
\|u(\cdot)\|_2 := \left( \int_0^\infty \|u(t)\|^2dt \right)^{\frac{1}{2}}
$$

is the $\mathcal{L}_2$ norm of $u(\cdot)$.
for any $T > 0$. Since $V(x(T)) \geq 0$, we deduce that
\[ \int_0^T \|y(t)\|^2 dt \leq \gamma^2 \int_0^T \|u(t)\|^2 dt \leq \gamma^2 \left[ \|u(\cdot)\|_2 \right]^2 \]
for any $T > 0$ and therefore
\[ \left[ \|y(\cdot)\|_2 \right]^2 \leq \gamma^2 \left[ \|u(\cdot)\|_2 \right]^2 , \]
i.e.
\[ \|y(\cdot)\|_2 \leq \gamma \|u(\cdot)\|_2 . \]

In other words, for any $u(\cdot) \in L_2[0, \infty)$, the response of the system from the initial state $x(0) = 0$ is defined for all $t \geq 0$, produces an output $y(\cdot)$ which is a function in $L_2[0, \infty)$, and the ratio between the $L_2$ norm of the output and the $L_2$ norm of the input is bounded by $\gamma$. For this reason, the system is said to have a finite $L_2$ gain, bounded above by the number $\gamma$.

Note that functions in $L_2[0, \infty)$ represent signals having finite energy over the infinite time interval $[0, \infty)$, and therefore the number $\gamma$ in the inequality (2) can be given the interpretation of (an upper bound of the) ratio between the energies of output and input.

Another interpretation, which does not necessarily require the consideration of inputs having finite energy, is possible. Suppose the input is a periodic function of time, with period $T$, i.e.
\[ u(t + kT) = u^\circ(t), \quad \text{for all } t \in [0, T), \quad k \geq 0 \]
for some piecewise continuous function $u^\circ(t)$, defined on $[0, T)$. Also, suppose that, for some suitable initial state $x(0) = x^\circ$, the state response $x^\circ(t)$ of the system is defined for all $t \in [0, T]$ and satisfies
\[ x^\circ(T) = x^\circ. \]
Then, it is obvious that $x^\circ(t)$ exists for all $t \geq 0$, and is a periodic function, having the same period $T$ of the input, namely
\[ x^\circ(t + kT) = x^\circ(t), \quad \text{for all } t \in [0, T), \quad k \geq 0 \]
and so is the corresponding output response $y(t)$.

For the triplet $\{u(t), x^\circ(t), y(t)\}$ thus defined, integration of the inequality (2) over an interval $[t_0, t_0 + T]$, with arbitrary $t_0 \geq 0$, yields
\[ V(x^\circ(t_0 + T)) - V(x^\circ(t_0)) \leq \gamma^2 \int_{t_0}^{t_0+T} \|u(s)\|^2 ds - \int_{t_0}^{t_0+T} \|y(s)\|^2 ds , \]
i.e., since $V(x^\circ(t_0 + T)) = V(x^\circ(t_0))$,
\[ \int_{t_0}^{t_0+T} \|y(s)\|^2 ds \leq \gamma^2 \int_{t_0}^{t_0+T} \|u(s)\|^2 ds . \] (3)

Observe that the integrals on both sides of this inequality are independent of $t_0$, because the integrands are periodic functions having period $T$, and recall that the root mean square
value of any (possibly vector-valued) periodic function $f(t)$ (which is usually abbreviated as r.m.s. and characterizes the average power of the signal represented by $f(t)$) is defined as

$$\|f(\cdot)\|_{r.m.s.} = \left(\frac{1}{T} \int_{t_0}^{t_0+T} \|f(s)\|^2 ds\right)^{\frac{1}{2}}.$$ 

With this in mind, (3) yields

$$\|y(\cdot)\|_{r.m.s.} \leq \gamma \|u(\cdot)\|_{r.m.s.}. \quad (4)$$

In other words, in a system having finite $L_2$ gain system, the number $\gamma$ (which appears in the inequality (2)) happens to be also an upper bound for the ratio between the r.m.s. value of the output and the r.m.s value of the input, whenever a periodic input is producing (from an appropriate initial state) a periodic (state and output) response.

We can therefore conclude that, in a system which satisfies an inequality of the form (2) the number $\gamma$ can be given these two interpretations. If the input represents a signal whose energy over the infinite interval $[0, \infty)$ is finite, then the corresponding output from the initial state $x(0) = 0$ is a function having finite energy over the interval $[0, \infty)$ and the ratio between the energies of output and input is bounded from above by the number $\gamma$. On the other hand, if the input is a periodic function which produces, from some appropriate initial state $x(0) = x^0$, a periodic state and output response, the number $\gamma$ provides a bound for the ratio between the average powers of the output and input.

We will see in the next section that the fulfillment of an inequality of the form (2) is equivalent to the fulfillment of a linear matrix inequality involving the system data $A, B, C, D$, the number $\gamma$, and the matrix $P$.

## 2 An LMI characterization of the $L_2$ gain

In this section, we derive alternative characterizations of the inequality (2).

**Lemma 1** Let $V(x) = x^T P x$, with $P$ a positive definite symmetric matrix and let $\gamma > 0$ be a fixed number. There exists a number $\tilde{\gamma} < \gamma$ and a number $\epsilon > 0$ such that

$$\frac{\partial V}{\partial x}(Ax + Bu) \leq -\epsilon \|x\|^2 + \tilde{\gamma}^2 \|u\|^2 - \|Cx + Du\|^2 \quad (5)$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$ if and only if

$$D^T D - \gamma^2 I < 0 \quad (6)$$

$$PA + A^T P + C^T C + [PB + C^T D][\gamma^2 I - D^T D]^{-1}[PB + C^T D]^T < 0 \quad (7)$$

**Proof.** Necessity. Recall that

$$\frac{\partial V}{\partial x} = 2x^T P$$

and observe that the inequality (5) becomes

$$2x^T P[Ax + Bu] + \epsilon x^T x - \tilde{\gamma}^2 u^T u + x^T C^T C x + 2x^T C D u + u^T D^T D u \leq 0 \quad (8)$$
for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$. For $x = 0$ this implies, in particular,

$$-\tilde{\gamma}^2 I + D^T D \leq 0.$$ 

Since $\gamma > \tilde{\gamma}$ we see that (5) implies

$$D^T D < \gamma^2 I,$$

which is condition (6) and

$$2x^T P[ Ax + Bu ] + \epsilon x^T x - \gamma^2 u^T u + x^T C^T C x + 2x^T C^T D u + u^T D^T D u \leq 0 \quad (9)$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$.

Now, observe that, for each fixed $x$, the right-hand side of (9) is a quadratic form in $u$, expressible as

$$M(x) + N(x) u - u^T W u.$$ 

in which

$$W = \gamma^2 I - D^T D$$

and

$$M(x) = x^T (PA + A^T P + C^T C + \epsilon I) x , \quad N(x) = 2x^T (PB + C^T D).$$

Since $W$ is positive definite, this form has a unique maximum point, at

$$u^*(x) = \frac{1}{2} W^{-1} N(x)^T.$$ 

Hence, (9) holds if and only if the value of the form (10) at $u = u^*(x)$ is non positive, which yields

$$M(x) + \frac{1}{4} N(x) W^{-1} N(x)^T \leq 0.$$ 

Using the expressions of $M(x), N(x), W$, this reads as

$$x^T (PA + A^T P + C^T C + \epsilon I + [PB + C^T D] [\gamma^2 I - D^T D]^{-1} [PB + C^T D]^T) x$$

and this, since $\epsilon > 0$, implies condition (7).

**Sufficiency.** Observe that the left-hand sides of (6) and (7), which are negative definite, by continuity remain negative definite if $\gamma$ is replaced by a number $\tilde{\gamma} < \gamma$ such that the difference $\gamma - \tilde{\gamma}$ is small enough. Thus, for some $\epsilon > 0$, the matrix $P$ satisfies

$$PA + A^T P + C^T C - \epsilon I + [PB + C^T D] [\gamma^2 I - D^T D]^{-1} [PB + C^T D] < 0.$$ 

As a consequence, the form $M(x) + N(x) u - u^T W u$, in which $W = \tilde{\gamma}^2 I - D^T D$, is non positive and (5) holds. \(\triangleright\)

**Lemma 2** Let $\gamma$ be a fixed number. There exists a positive definite symmetric matrix $P$ satisfying (6) and (7) if and only if there exists a positive definite symmetric matrix $X$ satisfying

$$\begin{pmatrix}
    A^T X + X A & X B & C^T \\
    B^T X & -\gamma I & D^T \\
    C & D & -\gamma I
\end{pmatrix} < 0. \quad (11)$$
Proof. Consider the matrix inequality
\[
\begin{pmatrix}
A^T X + X A + \frac{1}{\gamma} C^T C & X B + \frac{1}{\gamma} C^T D \\
X^T P + \frac{1}{\gamma} D^T C & -\gamma I + \frac{1}{\gamma} D^T D
\end{pmatrix} < 0.
\] (12)

This inequality holds if and only if the lower right block
\[-\gamma I + \frac{1}{\gamma} D^T D\]
is negative definite, which is equivalent to condition (6), and so is its the Schur’s complement
\[A^T X + X A + \frac{1}{\gamma} C^T C - [X B + \frac{1}{\gamma} C^T D][-\gamma I + \frac{1}{\gamma} D^T D]^{-1}[X B + \frac{1}{\gamma} C^T D].\]

This, having replaced \(X\) by \(\frac{1}{\gamma} P\), is identical to condition (7).

Rewrite now (12) as
\[\begin{pmatrix} A^T X + X A & X B \\ D^T X & -\gamma I \end{pmatrix} + \begin{pmatrix} C^T \\ \frac{1}{\gamma} \end{pmatrix} \begin{pmatrix} C & D \end{pmatrix} < 0\]
and use again (backward) Schur’s complement to arrive at (11). \(\Box\)

3 The \(H_\infty\) norm of a transfer function

We have seen in Section 1 that, in a system which satisfies an inequality of the form (2), if the input is a periodic function which produces, from some appropriate initial state \(x(0) = x^0\), a periodic state and output response, the number \(\gamma\) provides a bound for the ratio between the average powers of the output and input. This observation can be further pursued in the following terms. Let the input \(u(\cdot)\) be a sinusoidal function of time, of period \(T = 2\pi/\omega_0\), e.g.,
\[\tilde{u}(t) = U u_0 \cos(\omega_0 t)\] (15)
where \(U > 0\) and \(u_0 \in \mathbb{R}^m\) has unitary norm. It is well-know that system (1), which is asymptotically stable, exhibits a well-defined steady-state response, which is itself a sinusoidal

\[3\text{Schur’s complement Lemma. The symmetric matrix}\]
\[\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}\]
is negative definite if and only if
\[R < 0 \quad \text{and} \quad Q - SR^{-1}S^T < 0.\] (14)

In fact, observe that a necessary condition for (13) to be negative definite is \(R < 0\). Hence \(R\) is nonsingular and (13) can be transformed, by congruence, as
\[\begin{pmatrix} I \\ -R^{-1}S^T \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} I \\ -R^{-1}S^T \end{pmatrix} = \begin{pmatrix} Q - SR^{-1}S^T & 0 \\ 0 & R \end{pmatrix},\]
from which the conditions (14) follow.
function of period $t$. The response in question can be easily characterized by means of a simple geometric construction. Observe that the input thus defined can be viewed as generated by an autonomous system of the form $w \in \mathbb{R}^2$,

$$\begin{align*}
\dot{w} &= \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} w := Sw \\
u &= u_0 \begin{pmatrix} 1 & 0 \end{pmatrix} w := Qw
\end{align*}$$

(16)

with $w \in \mathbb{R}^2$, from in the initial state

$$w(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} U .$$

Since $A$ has all eigenvalues with negative real part and $S$ has eigenvalues on the imaginary axis, the Sylvester equation

$$\Pi S = A \Pi + BQ$$

(17)

has a unique solution $\Pi$. The composite system

$$\begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} S & 0 \\ BQ & A \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix}$$

possesses two complementary invariant subspaces: a stable subspace and a center subspace, respectively spanned by the columns of the matrices

$$V^s = \begin{pmatrix} 0 \\ I \end{pmatrix} , \\
V^c = \begin{pmatrix} I \\ \Pi \end{pmatrix} .$$

The latter, in particular, shows that the center subspace is the set of all pairs $(w, x)$ such that

$$x = \Pi w .$$

Consider now the change of variables

$$\tilde{x} = x - \Pi w$$

which, after a simple calculation using (17), yields

$$\begin{align*}
\dot{w} &= Sw \\
\dot{x} &= A\tilde{x} .
\end{align*}$$

We see from this that, for any initial condition,

$$\lim_{t \to \infty} \tilde{x}(t) = 0 ,$$

which is to say that the (unique) projection of the trajectory along the stable subspace asymptotically tends to zero. In the original coordinates, this reads as

$$\lim_{t \to \infty} [x(t) - \Pi w(t)] = 0 ,$$
from which we see that the steady-state response of system (1) to any input generated by (16) can be expressed as
\[ x_{ss} = \Pi w(t). \]  
(18)

The calculation of the solution \( \Pi \) of the Sylvester equation (17) is straightforward. Set
\[ \Pi = (\Pi_1 \quad \Pi_2) \]
and observe that the equation in question reduces to
\[ \Pi \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} = A\Pi + Bu_0 \begin{pmatrix} 1 & 0 \end{pmatrix}. \]
An elementary calculation (multiply first both sides on the right by the vector \( \begin{pmatrix} 1 & j \end{pmatrix}^T \)) yields
\[ \Pi_1 + j\Pi_2 = (j\omega_0 I - A)^{-1}Bu_0, \]
i.e.
\[ \Pi = (\text{Re}((j\omega_0 I - A)^{-1}B)u_0 \quad \text{Im}((j\omega_0 I - A)^{-1}B)u_0). \]
As shown above, the steady state response has the form (18). Hence, in particular, the periodic input
\[ u(t) = u_0 \cos(\omega_0 t) \]
produces the periodic state response
\[ x_{ss}(t) = \Pi w(t) = \Pi_1 \cos(\omega_0 t) - \Pi_2 \sin(\omega_0 t), \]  
(19)
and the periodic output response
\[ y_{ss}(t) = Cx_{ss}(t) + Du_0 \cos(\omega_0 t) \]
\[ = \text{Re}[T(j\omega_0)]u_0 \cos(\omega_0 t) - \text{Im}[T(j\omega_0)]u_0 \sin(\omega_0 t), \]  
(20)
in which
\[ T(j\omega) = C(j\omega I - A)^{-1}B + D. \]

Observe now that
\[ \int_0^{2\pi} \|\tilde{u}(s)\|^2 ds = \frac{\pi}{\omega_0} \|u_0\|^2 \]
and that, in view of the specific form of \( \Pi \),
\[ \int_0^{2\pi} \|\tilde{y}_{ss}(s)\|^2 ds = \frac{\pi}{\omega_0} \|T(j\omega_0)u_0\|^2. \]
In other words
\[ \|\tilde{u}(\cdot)\|_{\text{r.m.s.}}^2 = \frac{1}{2} \|u_0\|^2 \]
\[ \|\tilde{y}_{ss}(\cdot)\|_{\text{r.m.s.}}^2 = \frac{1}{2} \|T(j\omega_0)u_0\|^2. \]
Thus, from the interpretation illustrated above one can conclude that, if the system satisfies (2), then
\[ \|T(j\omega_0)u_0\|^2 = 2\|\tilde{y}_{ss}(\cdot)\|_{\text{r.m.s.}}^2 \leq \gamma^2 2\|\tilde{u}(\cdot)\|_{\text{r.m.s.}}^2 = \gamma^2 \|u_0\|^2 \]
\[ \|T(j\omega_0)u_0\| \leq \gamma \|u_0\| . \]

Observing that both \( u_0 \) and \( \omega_0 \) are arbitrary, it is seen from this that

\[ \sup_{\omega \in \mathbb{R}} \max_{\|u\|=1} \|T(j\omega)u\| \leq \gamma . \]

The quantity on the left-hand side is, by definition, the so-called \( H_\infty \) norm of the matrix \( T(j\omega) \). Therefore, one can conclude that a linear system that satisfies (2) is asymptotically stable and the \( H_\infty \) norm of its frequency response matrix is bounded from above by the number \( \gamma \).

\section*{4 The Bounded Real Lemma}

We have shown above that, for a system of the form (1), if (and only if) there exists a number \( \gamma > 0 \) and a symmetric positive definite matrix \( X \) satisfying (11), then there exists a number \( \tilde{\gamma} \) and a positive definite matrix \( P \) satisfying (2) for some \( \epsilon \). This, in view of the interpretation provided above, proves that the fulfillment of (11) for some \( \gamma \) (with \( X \) positive definite) implies that the system is asymptotically stable and that:

(i) its \( L_2 \) gain is strictly less than \( \gamma \),
(ii) the \( H_\infty \) norm of its transfer function is strictly less than \( \gamma \).

However, put in these terms, we have only said that (2) implies both (i) and (ii) and we have not investigated yet whether the reverse implications might hold. In this section, we complete the analysis, by showing that the two properties (i) and (ii) are, in fact, two different characterizations of the same property and both imply (11).

This will be done by means of a circular proof involving another equivalent version of the property that \( \gamma \) is an upper bound for the \( H_\infty \) norm of the transfer function matrix of the system, which is very useful for practical purposes, since it can be rather easily verified. More precisely, the fact that \( \gamma \) is an upper bound for the \( H_\infty \) norm of the transfer function matrix of the system can be checked by looking at the spectrum of a matrix of the form

\[ H = \begin{pmatrix} A_0 & R_0 \\ -Q_0 & -A_0^T \end{pmatrix} , \]

in which \( R_0 \) and \( Q_0 \) are symmetric matrices which, together with \( A_0 \), depend on the matrices \( A, B, C, D \) which characterize the system and on the number \( \gamma \).

\section*{Theorem 1}

Consider the linear system (1) and let \( \gamma > 0 \) be a fixed number. The following are equivalent:

(i) there exists \( \tilde{\gamma} < \gamma, \epsilon > 0 \) and a symmetric positive definite matrix \( P \) such that (5) holds for \( V(x) = x^TPx \),

\footnote{A matrix with this structure is called an Hamiltonian matrix and has the property that its spectrum is symmetric with respect to the imaginary axis (see Section 7). Thus, if \( \lambda \) is an eigenvalue of \( H \), so is also \(-\lambda\). Since the entries of \( H \) are real numbers, this shows that if \( a + jb \) is an eigenvalue of \( H \), so is also \(-a + jb\).}
(ii) all the eigenvalues of \( A \) have negative real part and the frequency response matrix of the system \( T(j\omega) = C(j\omega I - A)^{-1}B + D \) satisfies
\[
\sup_{\omega \in \mathbb{R}} \max_{\|u\|=1} \|T(j\omega)u\| < \gamma, \tag{23}
\]

(iii) all the eigenvalues of \( A \) have negative real part, the matrix \( W = \gamma^2 I - D^T D \) is positive definite, and the Hamiltonian matrix
\[
H = \begin{pmatrix}
A + BW^{-1}D^T C & BW^{-1}B^T \\
-C^T C + C^T D W^{-1} D^T C & -A^T - C^T D W^{-1} B^T
\end{pmatrix} \tag{24}
\]
has no eigenvalues on the imaginary axis,

(iv) there exists a positive definite symmetric matrix \( X \) satisfying
\[
\begin{pmatrix}
A^T X + XA & XB & C^T \\
B^T X & -\gamma I & D^T \\
C & D & -\gamma I
\end{pmatrix} < 0. \tag{25}
\]

Proof. We have already shown, in the previous sections, that (i) implies that (1) is an asymptotically stable system, with a frequency response matrix satisfying
\[
\sup_{\omega \in \mathbb{R}} \max_{\|u\|=1} \|T(j\omega)u\| \leq \tilde{\gamma}. \]
Thus, (i) \( \Rightarrow \) (ii).

To show that (ii) \( \Rightarrow \) (iii), first of all that note, since \[
\lim_{\omega \to \infty} T(j\omega) = D
\]
it necessarily follows that \( \|Du\| < \gamma \) for all \( u \) with \( \|u\| = 1 \) and this implies \( \gamma^2 I > D^T D \), i.e. the matrix \( W \) is positive definite.

Now observe that the Hamiltonian matrix (24) can be expressed in the form
\[
H = L + MN
\]
for
\[
L = \begin{pmatrix}
A & 0 \\
-C^T C & -A^T
\end{pmatrix}, \quad M = \begin{pmatrix}
B \\
-C^T D
\end{pmatrix}, \quad N = \begin{pmatrix}
W^{-1} D^T C & W^{-1} B^T
\end{pmatrix}.
\]

Suppose, by contradiction, that the matrix \( H \) has eigenvalues on the imaginary axis. By definition, there exist a \( 2n \)-dimensional vector \( x_0 \) and a number \( \omega_0 \in \mathbb{R} \) such that
\[
(j\omega_0 I - L)x_0 = MNx_0.
\]

Observe now that the matrix \( L \) has no eigenvalues on the imaginary axis, because its eigenvalues coincide with those of \( A \) and \(-A^T\), and \( A \) is by hypothesis stable. Thus \( (j\omega_0 I - L) \) is nonsingular. Observe also that the vector \( u_0 = N x_0 \) is nonzero because otherwise \( x_0 \) would
be an eigenvector of $L$ associated with an eigenvalue at $j\omega_0$, which is a contradiction. A simple manipulation yields

$$u_0 = N(j\omega_0 I - L)^{-1} Mu_0. \tag{26}$$

It is easy to check that

$$N(j\omega_0 I - L)^{-1} M = W^{-1}[T^T(-j\omega_0)T(j\omega_0) - D^T D] \tag{27}$$

where $T(s) = C(sI - A)^{-1}B + D$. In fact, it suffices to compute the transfer function of

$$\dot{x} = Lx + Mu$$
$$y = Nx$$

and observe that $N(sI - L)^{-1} M = W^{-1}[T^T(-s)T(s) - D^T D]$.

Multiply (27) on the left by $u_0^T W$ and on the right by $u_0$, and use (26), to obtain

$$u_0^T W u_0 = u_0^T [T^T(-j\omega_0)T(j\omega_0) - D^T D] u_0,$$

which in turn, in view of the definition of $W$, yields

$$\gamma^2 \|u_0\|^2 = \|T(j\omega_0)u_0\|^2$$

which contradicts (ii), thus completing the proof.

To show that (iii) $\Rightarrow$ (iv), set

$$F = (A + BW^{-1}D^T C)^T$$
$$Q = -BW^{-1}B^T$$
$$GG^T = C^T(I + DW^{-1}D^T)C$$

(the latter is indeed possible because $I + DW^{-1}D^T$ is a positive definite matrix: in fact, it is a sum of the positive definite matrix $I$ and of the positive semidefinite matrix $DW^{-1}D^T$; hence there exists a nonsingular matrix $M$ such that $I + DW^{-1}D^T = M^T M$; in view of this, the previous expression holds with $G^T = MC$).

The pair $(F, GG^T)$ thus defined is stabilizable. In fact, suppose that this is not the case. Then, there is a vector $x \neq 0$ such that

$$x^T (F - \lambda I GG^T) = 0$$

for some $\lambda$ with non-negative real part. Then,

$$0 = \left( A + BW^{-1}D^T C - \lambda I \
C^T M^T MC \right) x .$$

This implies in particular $0 = x^T C^T M^T M C x = \|MCx\|^2$ and hence $Cx = 0$, because $M$ is nonsingular. This in turn implies $Ax = \lambda x$, and this is a contradiction because all the eigenvalues of $A$ have negative real part.

Moreover, it is easy to check that

$$H^T = \begin{pmatrix} F & -GG^T \\ -Q & -F^T \end{pmatrix}$$
and this matrix by hypothesis has no eigenvalues on the imaginary axis. Therefore, there is a unique solution $Y^-$ of
\[ Y^F + F^TY^- - Y^-GG^TY^- + Q = 0, \quad \sigma(F - GG^TY^-) \subset \mathbb{C}^- \] (28)

Moreover, the set of solutions $Y$ of the inequality
\[ YF + F^TY - YGG^TY + Q > 0 \] (29)
is nonempty and any $Y$ in this set is such that $Y < Y^-$. Observe now that
\[
0 = Y^F + F^TY^- - Y^-GG^TY^- + Q \\
= Y^-(A^T + C^TDW^{-1}B^T) + (A + BW^{-1}D^TC)Y^- \\
- Y^C^T(I + DW^{-1}D^T)CY^- - BW^{-1}B^T \\
\]
which yields
\[ Y^-A^T + AY^- \geq 0. \]
Setting $V(x) = x^TY^x$, this inequality shows that, along the trajectories of
\[ \dot{x} = A^Tx, \] (30)
the function $V(x(t))$ is non-decreasing, i.e., $V(x(t)) \geq V(x(0))$ for any $x(0)$ and any $t \geq 0$. On the other hand, system (30) is by hypothesis asymptotically stable, i.e., $\lim_{t \to \infty} x(t) = 0$. Therefore, necessarily, $V(x(0)) \leq 0$, i.e., the matrix $Y^-$ is negative semi-definite. From this, it is concluded that any solution $Y$ of (29), that is of the inequality
\[ YA^T + AY - [YC^TD - B]W^{-1}[D^TCY - B^T] - YC^TCY > 0, \] (31)
which necessarily satisfies $Y < Y^- \leq 0$, is a negative definite matrix. Take any of the solutions $Y$ of (31) and consider $P = -Y^{-1}$. By construction, this matrix is a positive definite solution of the inequality in (7). Thus, by Lemma 2, (iv) holds.

The proof that (iv) $\Rightarrow$ (i) is provided by Lemma 2 and 1. $\triangle$

5 Small gain theorem and robust stability

We discuss in this section the stability properties of feedback interconnected systems. More precisely, consider two systems $\Sigma_1$ and $\Sigma_2$, described by equations of the form
\[
\begin{align*}
\dot{x}_i &= A_ix_i + B_iu_i \\
y_i &= C_ix_i + D_iu_i
\end{align*}
\] (32)

5See Appendix C, Proposition 1
6See Appendix C, Proposition 2
with $i = 1, 2$, in which we assume that
\[
\dim(u_2) = \dim(y_1) \\
\dim(u_1) = \dim(y_2).
\]

Suppose that the matrices $D_1$ and $D_2$ are such that the constraint
\[
\begin{align*}
  u_2 &= y_1 \\
  u_1 &= y_2
\end{align*}
\]
makes sense. In other words, suppose that for each $x_1, x_2$ there is a unique pair $u_1, u_2$ satisfying
\[
\begin{align*}
  u_2 &= C_1 x_1 + D_1 u_1 \\
  u_1 &= C_2 x_2 + D_2 u_2.
\end{align*}
\]
This is the case if the system of equations
\[
\begin{align*}
  u_2 &= C_1 x_1 + D_1 u_1 \\
  u_1 &= C_2 x_2 + D_2 u_2
\end{align*}
\]
as a unique solution $u_1, u_2$, and this occurs if and only if the matrix
\[
\begin{pmatrix}
  -D_1 & I \\
  I & -D_2
\end{pmatrix}
\]
is invertible, i.e. the matrix $I - D_2 D_1$ is nonsingular.

The system defined by (33) is the pure feedback interconnection of $\Sigma_1$ and $\Sigma_2$.

Suppose now there exist two positive definite matrices $P_1, P_2$ and a two real numbers $\gamma_1, \gamma_2$ such that $\Sigma_1$ and $\Sigma_2$ satisfy inequalities of the form (2), namely
\[
\begin{align*}
  \frac{\partial V_i}{\partial x_i}(A_i x_i + B_i u_i) &\leq -\epsilon_i \|x_i\|^2 + \gamma_i^2 \|u_i\|^2 - \|C_i x_i + D_i u_i\|^2 
\end{align*}
\]
in which $V_i(x_i) = x_i^T P_i x_i$ and $\epsilon_i > 0$.

Consider, for the pure feedback interconnection of $\Sigma_1$ and $\Sigma_2$ the candidate Lyapunov function
\[
W(x_1, x_2) = V(x_1) + a V(x_2)
\]
in which $a > 0$ is a number to be determined. A simple calculation shows that
\[
\begin{align*}
\frac{\partial W}{\partial x_1} \dot{x}_1 + \frac{\partial W}{\partial x_2} \dot{x}_2 &\leq -\epsilon_1 \|x_1\|^2 - \epsilon_2 \|x_1\|^2 + \gamma_1^2 \|u_1\|^2 - \|y_1\|^2 + a \gamma_2^2 \|u_2\|^2 - a \|y_2\|^2 \\
&\leq -\epsilon_1 \|x_1\|^2 - \epsilon_2 \|x_1\|^2 + \gamma_1^2 \|y_2\|^2 - \|y_1\|^2 + a \gamma_2^2 \|y_1\|^2 - a \|y_2\|^2 \\
&= -\epsilon_1 \|x_1\|^2 - \epsilon_2 \|x_1\|^2 + (y_1^T y_2^T) \begin{pmatrix}
-1 + a \gamma_2^2 I \\
0
\end{pmatrix} \begin{pmatrix}
0 \\
(\gamma_1^2 - a) I
\end{pmatrix} \begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}.
\end{align*}
\]
If
\[
\begin{pmatrix}
-1 + a \gamma_2^2 I \\
0
\end{pmatrix} \begin{pmatrix}
0 \\
(\gamma_1^2 - a) I
\end{pmatrix} \leq 0,
\]
we have
\[ \frac{\partial W}{\partial x_1} \dot{x}_1 + \frac{\partial W}{\partial x_2} \dot{x}_2 \leq -\alpha_1(\|x_1\|) - a\alpha_2(\|x_2\|) \]
and hence the interconnected system is asymptotically stable. Condition (35), on the other hand, is can be fulfilled for some \( a > 0 \) if (and only if)
\[ -1 + a\gamma_2^2 \leq 0, \quad \gamma_1^2 - a \leq 0 \]
i.e. if
\[ \gamma_1^2 \leq a \leq \frac{1}{\gamma_2^2}. \]

All of the above can be summarized in the following important result.

**Theorem 2** Consider a pair of systems (32) and suppose that the matrix \( I - D_2D_1 \) is non-singular. Suppose (32) satisfy inequalities of the form (34), with \( P_1, P_2 \) positive definite and \( \gamma_1 \gamma_2 \leq 1 \).

Then, the pure feedback interconnection of \( \Sigma_1 \) and \( \Sigma_2 \) is asymptotically stable.

Note also that, in view of the Bounded Real Lemma, the Theorem above can be rephrased in terms of \( H_\infty \) norms of the transfer functions of the two component subsystems, as follows.

**Theorem 3** Consider a pair of systems (32) and suppose that the matrix \( I - D_2D_1 \) is non-singular. Suppose both systems are asymptotically stable. Let
\[ T_i(s) = C_i(sI - A_i)^{-1}B_i + D_i \]
denote the respective transfer functions and suppose
\[ \|T_1\|_\infty < \gamma_1, \quad \|T_2\|_\infty < \gamma_2. \]

If
\[ \gamma_1 \gamma_2 \leq 1 \]
the pure feedback interconnection of \( \Sigma_1 \) and \( \Sigma_2 \) is asymptotically stable.

This theorem is the point of departure for the study of robust stability via \( H_\infty \) methods.

A problem of robust stabilization can be cast, in rather general terms, as follows. A plant with control input \( u \) and measurement output \( y \) whose model is uncertain can be, from a rather general viewpoint, thought of as the interconnection of a nominal system modeled by equations of the form
\[ \begin{align*}
\dot{x} &= Ax + B_1v + B_2u \\
z &= C_1x + D_{11}v + D_{12}u \\
y &= C_2x + D_{21}v
\end{align*} \tag{36} \]
in which the “additional” input \( v \) and the “additional” output \( z \) are seen seen as output and, respectively, input of a system
\[ \begin{align*}
\dot{x}_p &= A_p x_p + B_p v \\
v &= C_p x_p + D_p z
\end{align*} \tag{37} \]
whose parameters are uncertain.\footnote{Note that, for the interconnection (36)–(37) to be well-defined, the matrix $I - D_p D_{11}$ is required to be invertible.}

This setup includes the special case of a plant of fixed dimension whose parameters are uncertain, that is a plant modeled as

$$
\begin{align*}
\dot{x} &= (A_0 + \delta A)x + (B_0 + \delta B)u \\
y &= (C_0 + \delta C)
\end{align*}
$$

in which $A_0, B_0, C_0$ represent nominal values and $\delta A, \delta B, \delta C$ uncertain perturbations. In fact, the latter can be seen as interconnection of a system of the form (36) in which

$$
A = A_0, \quad B_1 = (I \ 0), \quad B_2 = B_0
$$

$$
C_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad D_{11} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix}
$$

$$
C_2 = C_0, \quad D_{21} = (0 \ I)
$$

with a system of the form

$$
v = \begin{pmatrix} \delta A \\ \delta B \\ \delta C \\ 0 \end{pmatrix} z
$$

which is indeed a special case of a system of the form (37). The interconnection (36)–(37) of a nominal model and a of dynamic perturbation is more general, though, because it accommodates for perturbations which possibly include unmodelled dynamics. In this setup, all modeling uncertainties are confined in the model (37), including the dimension itself of $x_p$.

Suppose now that (37) $A_p$ is a Hurwitz matrix and that the transfer function

$$
T_p(s) = C_p(s I - A_p)^{-1}B_p + D_p
$$

has an $H_\infty$ norm which is bounded by a known number $\gamma_p$. That is, assume that, no matter what the perturbations are, the perturbing system (37) is a stable system satisfying

$$
\|T_p\|_\infty < \gamma_p.
$$

for some $\gamma_p$.

Let

$$
\begin{align*}
\dot{x}_c &= A_c x_c + B_c y \\
u &= C_c x_c + D_c y
\end{align*}
$$

be a controller for the nominal plant (36) yielding a closed loop system

$$
\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} A + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + \begin{pmatrix} B_1 + B_2 D_c D_{21} \\ B_c D_{21} \end{pmatrix} w
$$

$$
z = \begin{pmatrix} C_1 + D_c C_2 & D_{12} C_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + (D_{11} + D_c D_{21}) w
$$

(41)
which is asymptotically stable and whose transfer function, between the input \( w \) and output \( z \) has an \( H_\infty \) norm bounded by a number \( \gamma \) satisfying

\[
\gamma \gamma_p < 1.
\] (42)

If this is the case, thanks to the small gain theorem, it can be concluded that the controller (44) stabilizes any of the perturbed plants (36) – (37), so long as the perturbation is such that (37) is asymptotically stable and the bound (39) holds.

In this way, the problem of robust stabilization is reduced to a problem of stabilizing a nominal plant and to simultaneously enforce an \( H_\infty \) bound.

6 The coupled LMIs approach to the problem of \( \gamma \)-suboptimal \( H_\infty \) feedback design

We summarize in this Section some relevant results about the so-called problem of \( \gamma \)-suboptimal \( H_\infty \) feedback design. Consider a linear system described by equations of the form

\[
\begin{align*}
\dot{x} &= Ax + B_1 v + B_2 u \\
z &= C_1 x + D_{11} v + D_{12} u \\
y &= C_2 x + D_{12} v .
\end{align*}
\] (43)

Let \( \gamma > 0 \) be a fixed number. The problem of \( \gamma \)-suboptimal \( H_\infty \) feedback design consists in finding a controller

\[
\begin{align*}
\dot{\xi} &= A_c \xi + B_c y \\
u &= C_c \xi + D_c y
\end{align*}
\] (44)

yielding a closed loop system

\[
\begin{align*}
\begin{pmatrix}
\dot{x} \\
\dot{\xi}
\end{pmatrix} &= \begin{pmatrix}
A + B_2 D_c C_2 & B_2 C_c \\
B_c C_2 & A_c
\end{pmatrix}
\begin{pmatrix}
x \\
\xi
\end{pmatrix} + \begin{pmatrix}
B_1 + B_2 D_c D_{21} \\
B_c D_{21}
\end{pmatrix} v \\
z &= (C_1 + D_{12} D_c C_2) (D_{12} C_c) \begin{pmatrix}
x \\
\xi
\end{pmatrix} + (D_{11} + D_{12} D_c D_{21}) v
\end{align*}
\] (45)

which is asymptotically stable and whose transfer function, between the input \( v \) and output \( z \) has an \( H_\infty \) norm strictly less than \( \gamma \).

Rewrite system (45) as

\[
\begin{align*}
\dot{x} &= A x + B v \\
z &= C x + D v
\end{align*}
\]

where

\[
A = \begin{pmatrix}
A + B_2 D_c C_2 & B_2 C_c \\
B_c C_2 & A_c
\end{pmatrix}, \quad B = \begin{pmatrix}
B_1 + B_2 D_c D_{21} \\
B_c D_{21}
\end{pmatrix} \\
C = (C_1 + D_{12} D_c C_2) (D_{12} C_c), \quad D = D_{11} + D_{12} D_c D_{21}.
\]

In view of the Bounded Real Lemma, the properties in question are achieved if and only if there exists a symmetric matrix \( \mathcal{X} \) satisfying

\[
\mathcal{X} > 0
\] (46)
Thus, the problem is to try to find a quadruplet \( \{A_c, B_c, C_c, D_c\} \) such that (46) and (47) hold for some symmetric \( X \).

The basic inequality (47) will be now transformed as follows. Let 
\[
x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^k, \quad v \in \mathbb{R}^{m_1}, \quad u \in \mathbb{R}^{m_2}, \quad z \in \mathbb{R}^{p_1}, \quad y \in \mathbb{R}^{p_2}.
\]

Set
\[
A_0 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} C_1 & 0 \end{pmatrix},
\]
\[
\Psi_X = \begin{pmatrix} A^T X + X A & X B & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix},
\]
\[
\mathcal{P} = \begin{pmatrix} 0 & I_k & 0 & 0 \\ B_2^T & 0 & 0_{m_2 \times m_1} & D_{12}^T \end{pmatrix},
\]
\[
\mathcal{Q} = \begin{pmatrix} 0 & I_k & 0 \\ C_2 & 0 & D_{21} & 0_{p_2 \times p_1} \end{pmatrix},
\]
\[
\Xi_X = \begin{pmatrix} X & 0 & 0 \\ 0 & I_{m_1} & 0 \\ 0 & 0 & I_{p_1} \end{pmatrix}.
\]

Collecting the parameters of the controller (44) in the \((n + m_2) \times (n + p_2)\) matrix
\[
K = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix}
\]
the inequality (47) can be rewritten as
\[
\Psi_X + \mathcal{Q}^T K^T (\mathcal{P} \Xi_X) + (\mathcal{P} \Xi_X)^T K \mathcal{Q} < 0.
\]

Thus, the problem of \(\gamma\)-suboptimal \(H_\infty\) feedback design can be cast as the problem of finding a symmetric \(X > 0\) and a matrix \(K\) such that (53) holds. In this context, the following result is very useful.\(^8\)

**Lemma 3** Given a symmetric \(m \times m\) matrix \(\Psi\) and two matrices \(P, Q\) having \(m\) columns, consider the problem of finding some matrix \(K\) of compatible dimensions such that
\[
\Psi + Q^T K P + P^T K^T Q < 0.
\]
Let \(W_P\) be a matrix such that \(\text{Im}(W_P) = \text{Ker}(P)\) and let \(W_Q\) be a matrix such that \(\text{Im}(W_Q) = \text{Ker}(Q)\). Then (54) is solvable in \(K\) if and only if
\[
W_Q^T \Psi W_Q < 0, \quad W_P^T \Psi W_P < 0.
\]

---

\(^8\)For a proof, see: P. Gahinet and P. Apkarian, *A Linear Matrix Inequality Approach to \(H_\infty\) Control*, in *Int. J. Robust and Nonlinear Control*, vol. 4, pp 421-448 (1994).
This Lemma can be used to eliminate \( K \) from (53) and obtain existence conditions depending only on \( X \) and on the parameters of the plant (43). Let \( W_P \) be a matrix whose columns span \( \text{Ker}(P \Xi) \) and let \( W_Q \) be a matrix whose columns span \( \text{Ker}(Q) \). According to Lemma 3, there exists \( K \) for which (53) holds if and only if

\[
W_Q^T \Psi_X W_Q < 0 \\
W_P^T \Psi_X W_P < 0 .
\]  

(56)

The second of these two inequalities can be further manipulated observing that if \( W_P \) is a matrix whose columns span \( \text{Ker}(P) \), the columns of the matrix \( \Xi^{-1} W_P \) span \( \text{Ker}(P \Xi) \). Thus, having set

\[
\Phi_X = \begin{pmatrix}
A_0 \chi^{-1} + \chi^{-1} A_0^T B_0 & \chi^{-1} C_0^T \\
B_0^T & -\gamma I & D_{11} \\
C_0 \chi^{-1} & D_{11} & -\gamma I
\end{pmatrix} ,
\]

(57)

the second of (56) can be rewritten as

\[
W_P^T \Xi^{-1} \Psi_X \Xi^{-1} W_P = W_P^T \Phi_X W_P < 0 .
\]

(58)

We summarize this fact as follows.

**Proposition 1** There exists a \( k \)-dimensional controller that solves the problem of \( \gamma \)-suboptimal \( H_\infty \) feedback design if and only if there exists a \((n + k) \times (n + k)\) symmetric matrix \( X \) such that

\[
W_Q^T \Psi_X W_Q < 0 \\
W_P^T \Phi_X W_P < 0,
\]

in which \( \Psi_X \) and \( \Phi_X \) are the matrices defined in (49) and (57). The two inequalities (58) thus found can be further simplified. To this end, it is convenient to partition \( X \) and \( \chi^{-1} \) as

\[
X = \begin{pmatrix}
S & N \\
N^T & *
\end{pmatrix} , \quad \chi^{-1} = \begin{pmatrix}
R & M \\
M^T & *
\end{pmatrix}
\]

(59)

in which \( R \) and \( S \) are \( n \times n \) and \( N \) and \( M \) are \( n \times k \). With the partition (59), the matrix (49) becomes

\[
\Psi_X = \begin{pmatrix}
SA + A^T S & A^T N & SB_1 & C_1^T \\
N^T A & 0 & N^T B_1 & 0 \\
B_1^T S & B_1^T N & -\gamma I & D_{11} \\
C_1 & 0 & D_{11} & -\gamma I
\end{pmatrix}
\]

(60)

and the matrix (57) becomes

\[
\Phi_X = \begin{pmatrix}
AR + R A^T & AM & B_1 & RC_1^T \\
M^T A^T & 0 & 0 & M^T C_1^T \\
B_1^T & 0 & -\gamma I & D_{11} \\
C_1 R & C_1 M & D_{11} & -\gamma I
\end{pmatrix}
\]

(61)
From the definition of $Q$, it is readily seen that a matrix $W_Q$ whose columns span $\text{Ker}(Q)$ can be expressed as 

$$
W_Q = \begin{pmatrix}
Z_1 & 0 \\
0 & 0 \\
Z_2 & 0 \\
0 & I_{p_1}
\end{pmatrix}
$$

in which 

$$
\begin{pmatrix}
Z_1 \\
Z_2
\end{pmatrix}
$$

is a matrix whose columns span $\text{Ker} (C_2 \quad D_{21})$.

Then, an easy calculation shows that the first inequality in (58) can be rewritten as 

$$
\begin{pmatrix}
Z_1^T \\
Z_2^T \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
A^T S + S A^T & S B_1 & C_1^T  \\
B_1^T S & -\gamma I & D_{11}^T  \\
C_1 & D_{11} & -\gamma I
\end{pmatrix}
\begin{pmatrix}
Z_1 \\
Z_2 \\
0 \\
0
\end{pmatrix} < 0.
$$

Likewise, from the definition of $P$, it is readily seen that a matrix $W_P$ whose columns span $\text{Ker}(P)$ can be expressed as 

$$
W_P = \begin{pmatrix}
V_1 & 0 \\
0 & 0 \\
0 & I_{m_1} \\
V_2 & 0
\end{pmatrix}
$$

in which 

$$
\begin{pmatrix}
V_1 \\
V_2
\end{pmatrix}
$$

is a matrix whose columns span $\text{Ker} (B_2^T \quad D_{12}^T)$.

Then, an easy calculation shows that the second inequality in (58) can be rewritten as 

$$
\begin{pmatrix}
V_1^T \\
V_2^T \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
A R + R A^T & R C_1^T & B_1  \\
C_1 R & -\gamma I & D_{11}  \\
B_1^T & D_{11} & -\gamma I
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2 \\
0 \\
0
\end{pmatrix} < 0.
$$

In this way, the two inequalities (58) have been transformed into two inequalities involving submatrix $S$ of $X$ and the submatrix $R$ of $X^{-1}$. To complete the analysis, it remains to connect these matrices to each other. This is achieved via the following Lemma.

**Lemma 4** Let $S$ and $R$ be symmetric $n \times n$ matrices. There exist a $n \times k$ matrix $N$ and $k \times k$ symmetric matrix $Z$ a such that 

$$
\begin{pmatrix}
S & N \\
N^T & Z
\end{pmatrix} > 0
$$

(62)
and
\[
\begin{pmatrix}
S & N \\
N^T & Z
\end{pmatrix}^{-1} = \begin{pmatrix} R & * \\
* & *
\end{pmatrix}
\]
(63)

if and only if
\[
\text{rank}(I - SR) \leq k
\]
(64)

and
\[
\begin{pmatrix}
S & I \\
I & R
\end{pmatrix} \geq 0.
\]
(65)

Proof. To prove necessity, write
\[
\begin{pmatrix}
S & N \\
N^T & Z
\end{pmatrix} = \begin{pmatrix} R & * \\
* & *
\end{pmatrix} = (R^* M^*)^T
\]
(63)

Then, by definition we have
\[
SR + NM^T = I \\
N^T R + Z M^T = 0.
\]
(66)

Thus \( I - SR = NM^T \) and this implies (64), because \( N \) has \( k \) columns. Now, set
\[
\mathcal{T} = \begin{pmatrix} I & R \\
0 & M^T
\end{pmatrix}
\]
Using (66), the first of which implies \( MN^T = I - RS \) because \( S \) and \( R \) are symmetric, observe that
\[
\mathcal{T}^T \mathcal{X} = \begin{pmatrix} S & I \\
I & R
\end{pmatrix}
\]
(66)

Pick \( y \in \mathbb{R}^{2n} \) and define \( x \in \mathbb{R}^{n+k} \) as \( x = \mathcal{T} y \). Then, it is seen that
\[
y^T \begin{pmatrix} S & I \\
I & R
\end{pmatrix} y = x^T \mathcal{X} x \geq 0
\]
because the matrix \( \mathcal{X} \) is positive definite by assumption. This concludes the proof of the necessity.

To prove sufficiency, let \( \hat{k} \leq k \) be the rank of \( I - SR \) and let \( \hat{N}, \hat{M} \) two \( n \times \hat{k} \) matrices of rank \( k \) such that
\[
I - SR = \hat{N} \hat{M}^T
\]
(67)

Using the property (67) it is possible to show that the equation
\[
\hat{N}^T R + \hat{Z} \hat{M}^T = 0
\]
(68)

has a solution \( Z \). In fact, observe that
\[
\hat{M} \hat{N}^T R - R \hat{N} \hat{M}^T = (I - RS) R - R (I - SR) = 0.
\]

Let \( L \) be any matrix such that \( \hat{M}^T L = -I_k \) (which exists because the \( k \) rows of \( \hat{M}^T \) are independent) and, from the identity above, obtain
\[
\hat{M} \hat{N}^T R L = -R \hat{N}.
\]
This shows that the matrix \( \hat{Z} = (\hat{N}^T R L)\) satisfies (68). The matrix \( \hat{Z} \) is symmetric. In fact, note that
\[
0 = \hat{M}(\hat{N}^T R + \hat{Z} \hat{M}^T) = (I - R S) R + \hat{M} \hat{Z} \hat{M}^T = R - R S R + \hat{M} \hat{Z} \hat{M}^T.
\]
and hence \( \hat{M} \hat{Z} \hat{M}^T \) is symmetric. This yields
\[
\hat{M} \hat{Z} \hat{M}^T - \hat{M} \hat{Z} \hat{M}^T \hat{M} = \hat{M}(\hat{Z} - \hat{Z}^T) \hat{M} = 0
\]
from which it is seen that \( \hat{Z} = \hat{Z}^T \) because \( \hat{M}^T \) has independent rows.

As a consequence of (67) and (68), the symmetric matrix
\[
\hat{X} = \left( \begin{array}{cc} S & \hat{N} \\ \hat{N}^T & \hat{Z} \end{array} \right)
\]
is a solution of
\[
\left( \begin{array}{cc} S & I \\ \hat{N}^T & 0 \end{array} \right) = \hat{X} \left( \begin{array}{cc} I & R \\ 0 & \hat{M}^T \end{array} \right).
\]
(70)

The symmetric matrix \( \hat{X} \) thus found is invertible because, otherwise, the independence of the rows of the matrix on the left-hand side of (70) would be contradicted. It is easily seen that
\[
\hat{X}^{-1} = \left( \begin{array}{cc} R & * \\ \hat{M}^T & * \end{array} \right).
\]
(71)

Moreover, it is also possible to prove that \( \hat{X} > 0 \). In fact, letting
\[
\hat{T} = \left( \begin{array}{cc} I & R \\ 0 & \hat{M}^T \end{array} \right)
\]
observe that
\[
\hat{T}^T \hat{X} \hat{T} = \left( \begin{array}{cc} S & I \\ I & R \end{array} \right).
\]
Assumption (65) implies that \( \hat{X} \geq 0 \). In fact, suppose \( x^T \hat{X} x < 0 \) for some \( x \). Using the fact that the rows of \( \hat{T} \) are independent, find \( y \) such that \( x = \hat{T} y \). This would make
\[
y^T \left( \begin{array}{cc} S & I \\ I & R \end{array} \right) y < 0,
\]
a contradiction. But we have shown before that \( \hat{X} \) is nonsingular and hence, knowing that this matrix is positive semidefinite, we conclude it is positive definite.

If \( k = k \) the sufficiency is proven. Otherwise, set \( \ell = k - \hat{k} \) and
\[
\hat{X} = \left( \begin{array}{cc} S & \hat{N} \\ \hat{N}^T & \hat{Z} \\ 0 & 0 \end{array} \right) I_{\ell \times \ell}
\]
and observe that \( \hat{X} \) is positive definite and
\[
\hat{X}^{-1} = \left( \begin{array}{cc} R & * & 0 \\ \hat{M}^T & * & 0 \\ 0 & 0 & I_{\ell \times \ell} \end{array} \right)
\]
Remark. Note that condition (65), alone, implies $S > 0$ and $R > 0$. In fact, in the proof of sufficiency, using this condition we have shown that (69) is positive definite. Hence $P > 0$. Its inverse has the structure indicated in (71) and hence also $R > 0$. Finally, note that the condition (64) is immaterial if $k \geq n$ because the matrix on the right is $n \times n$. <

This Lemma establishes a coupling condition between the two submatrices $S$ and $R$ identified in the previous analysis, that determines the positivity of the matrix $X$. Using this Lemma it is therefore possible to arrive at the following conclusion.

**Theorem 4** Consider a plant of modelled by equations of the form (43). Let $V_1, V_2, Z_1, Z_2$ be matrices such that

\[
\begin{align*}
    \text{Im} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} &= \ker \begin{pmatrix} C_2 \\ D_{21} \end{pmatrix}, \\
    \text{Im} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} &= \ker \begin{pmatrix} B_2^T \\ D_{12}^T \end{pmatrix}.
\end{align*}
\]

The problem of $\gamma$-suboptimal $H_\infty$ feedback design has a solution if and only if there exist symmetric matrices $S$ and $R$ satisfying the following system of linear matrix inequalities

\[
\begin{align*}
    \begin{pmatrix} Z_1^T & Z_2^T \\ 0 & 0 & I \end{pmatrix} &\begin{pmatrix} A^T S + S A^T & SB_1 & C_1^T \\ B_1^T S & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{pmatrix} \begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \end{pmatrix} < 0 \quad (72) \\
    \begin{pmatrix} V_1^T & V_2^T \\ 0 & 0 & I \end{pmatrix} &\begin{pmatrix} A R + R A^T & R C_1^T & B_1 \\ C_1 R & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ V_2 & 0 \end{pmatrix} < 0 \quad (73) \\
    \begin{pmatrix} S & I \\ I & R \end{pmatrix} &\geq 0 \quad (74)
\end{align*}
\]

In particular, there exists a solution of dimension $k < n$ if and only if there exist $R$ and $S$ satisfying (72), (73), (74) and, in addition,

\[
\text{rank}(I - RS) \leq k. \quad (75)
\]

The result above describes necessary conditions for the existence of a controller that solves the problem of problem of $\gamma$-suboptimal $H_\infty$ feedback design. For the actual construction of a controller, one may proceed as follows. Assuming that $S$ and $R$ are positive definite symmetric matrices satisfying the system of linear matrix inequalities (72), (73), (74), construct a matrix $X$ as indicated in the proof of Lemma 4, that is, find two matrices $n \times k$ matrices $N$ and $M$ such that

\[
    I - SR = NM^T
\]

with $k = \text{rank}(I - SR)$, and solve for $X$ the linear equation

\[
\begin{pmatrix} S & I \\ N^T & 0 \end{pmatrix} = X \begin{pmatrix} I & R \\ 0 & M^T \end{pmatrix}.
\]

By construction, the matrix in question satisfies (58) and their equivalent versions (56). Thus, according to Lemma 3, there exists a matrix $K$ satisfying (53). This is a linear matrix
inequality in the only unknown $K$. The solution of such inequality provides the required controller as

$$K = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix}.$$
A Hamiltonian Matrices and Algebraic Riccati Equations

In this section, we will study Riccati equations of the form

\[
A^T X + XA + Q + RX = 0 \tag{76}
\]

with \( R \) and \( Q \) symmetric matrices, which can also be rewritten in the equivalent form

\[
(X - I) \begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} I \\ X \end{pmatrix} = 0
\]

From either one of these expressions, it is easy to deduce the following identity

\[
\begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} I \\ X \end{pmatrix} = \begin{pmatrix} I \\ X \end{pmatrix} (A + RX) \tag{77}
\]

and to conclude that \( X \) is a solution of the Riccati equation (76) if and only if the subspace

\[
V = \text{Im} \begin{pmatrix} I \\ X \end{pmatrix}
\]

is an \((n\text{-dimensional})\) invariant subspace of the matrix

\[
H = \begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix} \tag{79}
\]

In particular, (77) also shows that if \( X \) is a solution of (76), the matrix \( A + RX \) characterizes the restriction of \( H \) to its invariant subspace (78). A matrix of the form (79), with real entries and in which \( R \) and \( Q \) are symmetric matrices, is called an Hamiltonian matrix. Some relevant features of the Hamiltonian matrix (79) and their relationships with the Riccati equation (76) are described in the following Lemmas.

**Lemma 5** The spectrum of the Hamiltonian matrix (79) is symmetric with respect to the imaginary axis.

Proof. Set

\[
J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}
\]

and note that

\[
J^{-1} H J = \begin{pmatrix} -A^T & Q \\ -R & A \end{pmatrix} = -H^T.
\]

Hence \( H \) and \(-H^T\) are similar. As a consequence, if \( \lambda \) is an eigenvalue of \( H \) so is also \(-\lambda\). Since the entries of \( H \) are real numbers, and therefore the spectrum of this matrix is symmetric with respect to the real axis, the result follows. □

Suppose now that the matrix (79) has no eigenvalues on the imaginary axis. Then, the matrix in question has exactly \( n \) eigenvalues in \( \mathbb{C}^- \) and \( n \) eigenvalues in \( \mathbb{C}^+ \). As a consequence, there exist two complementary \( n\text{-dimensional} \) invariant subspaces of \( H \): a subspace \( V^- \) on which the restriction of \( H \) has all eigenvalues with strictly negative real part, which is
called the stable eigenspace and a subspace $V^+$ on which the restriction of $H$ has all eigenvalues with strictly positive real part, which is called the unstable eigenspace. A situation of special interest in the subsequent analysis is the one in which the stable eigenspace (respectively, the unstable eigenspace) of the matrix (79) can be given the form (78); in this case in fact, as observed before, it is possible to associate with this subspace a particular solution of the Riccati equation (76). Observing that an $n$-dimensional subspace $V$ of $\mathbb{R}^{2n}$ can be given the form (78) if and only if

$$V \cap \text{Im} \begin{pmatrix} 0 \\ I \end{pmatrix} = \{0\},$$

it is easy to arrive at the following conclusions.

**Lemma 6** The following are equivalent:

(i) the Hamiltonian matrix (79) has no eigenvalue on the imaginary axis and its stable eigenspace $V^-$ (respectively its unstable eigenspace $V^+$) satisfies the condition (80),

(ii) there exists a unique symmetric solution $X^-$ (respectively, $X^+$) of the Riccati equation (76) such that $A + RX^-$ has all eigenvalues in $\mathbb{C}^-$ (A + RX$^+$ has all eigenvalues in $\mathbb{C}^+$).

Proof. The identity (77) already shows the equivalence between (i) and the existence of a solution $X$ of (76) such that $A + RX$ has all eigenvalues either in $\mathbb{C}^-$ or in $\mathbb{C}^+$. The uniqueness of such a solution follows from the uniqueness of the expression of an $n$-dimensional invariant subspace of (79) in the special form (78). Finally, to prove that such a solution is symmetric, multiply (77) on the left by $(I \ X^T)J$ to obtain

$$\begin{pmatrix} I & X^T \end{pmatrix}JH \begin{pmatrix} I \\ X \end{pmatrix} = (X^T - X)(A + RX)$$

and, taking the transpose,

$$\begin{pmatrix} I & X^T \end{pmatrix}H^TJ^T \begin{pmatrix} I \\ X \end{pmatrix} = (A + RX)^T(X - X^T).$$

Since $JH = H^TJ^T$, these yield

$$(X^T - X)(A + RX) + (A + RX)^T(X^T - X) = 0.$$ 

This is a Sylvester equation in $(X^T - X)$, which has a unique solution because $(A + RX)$ and $-(A + RX)^T$ have no common eigenvalues. This solution is necessarily the trivial solution $(X^T - X) = 0$ and the result follows. $\triangle$

The solutions $X^-$ and $X^+$ of the Riccati equation (76) thus characterized, whenever they exist, are called the stabilizing solution and, respectively, the antistabilizing solution of the equation in question. We describe a special cases in which these particular solutions exist. To this end all we need, in view of the previous Lemma, is to establish conditions under which the Hamiltonian matrix (79) has no eigenvalues on the imaginary axis and its stable eigenspace $V^-$ (respectively its unstable eigenspace $V^+$) satisfies the condition (80).
Lemma 7 Suppose the Hamiltonian matrix (79) has no eigenvalues on the imaginary axis and \( R \) is a (either positive or negative) semidefinite matrix. If the pair \((A, R)\) is stabilizable, the stable eigenspace \( V^- \) of (79) satisfies the condition (80). If the pair \((A, R)\) is antistabilizable, the unstable eigenspace \( V^+ \) of (79) satisfies the condition (80).

Proof. Express \( V^- \) in the form
\[
V^- = \text{Im} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}
\]
and note that, by definition,
\[
H \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} S
\]
where \( S \) is a stable matrix. Multiplying this relation on the left by
\[
\begin{pmatrix} X_1^T \\ X_2^T \end{pmatrix} J = \begin{pmatrix} X_2^T \\ X_1^T \end{pmatrix}
\]
one obtains
\[
\begin{pmatrix} X_1^T \\ X_2^T \end{pmatrix} J H \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = (X_2^T X_1 - X_1^T X_2) S
\]
Since the matrix on the left-hand side is symmetric, so is also \((X_2^T X_1 - X_1^T X_2) S\) and the conclusion
\[
X_2^T X_1 = X_1^T X_2 \quad (81)
\]
follows by the same type of arguments used in the proof of the previous Lemma.

We prove now that \( \text{Ker}(X_1) \) is invariant under \( S \). To this end, note that
\[
\begin{pmatrix} I \\ 0 \end{pmatrix} H \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} S
\]
yields
\[
AX_1 + RX_2 = X_1 S \quad (82)
\]
Let \( x \in \text{Ker}(X_1) \). Then, using (81), we obtain
\[
(X_2 x)^T R (X_2 x) = x^T X_2^T (AX_1 + RX_2) x = x^T X_2^T X_1 S x = x^T X_1^T X_2 S x = 0.
\]
Since \( R \) is sign semidefinite by hypothesis, then \( RX_2 x = 0 \) and (82) yields \( X_1 S x = 0 \) i.e. \( Sx \in \text{Ker}(X_1) \).

Finally, we prove that \( \text{Ker}(X_1) = \{0\} \). By contradiction, suppose is not. Then there exists \( \lambda \in \mathbb{C}^- \) (because \( S \) is stable), and \( x \in \text{Ker}(X_1) \) such that \( Sx = \lambda x \). Consider now
\[
\begin{pmatrix} 0 \\ I \end{pmatrix} H \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} S
\]
which yields
\[
-Q X_1 - A^T X_2 = X_2 S .
\]
Multiplication on the right by \( x \) yields, in turn,
\[
-A^T X_2 x = X_2 \lambda x
\]
i.e. \((\lambda I + A^T)X_2x = 0\). Recalling that \(RX_2x = 0\), we obtain
\[
(X_2x)^T (A + \lambda I \quad R) = 0
\]
which, if \(X_2x \neq 0\), contradicts the hypothesis that \((A, R)\) is stabilizable, because \(-\lambda \in \mathbb{C}^+\). Thus necessarily \(X_2x = 0\) and \(x\) is such that
\[
0 = \begin{pmatrix} X_1x \\ X_2x \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} x,
\]
but this contradicts the fact that \(V^-\) has dimension \(n\). Thus \(x = 0\), \(\text{Ker}(X_1) = \{0\}\), \(X_1\) is invertible and condition (80) holds. \(<\)

We can merge the previous two Lemma as follows.

**Proposition 2** Suppose the Hamiltonian matrix (79) has no eigenvalues on the imaginary axis and \(R\) is a (either positive or negative) semidefinite matrix.

If the pair \((A, R)\) is stabilizable, the stable eigenspace \(V^-\) of (79) can be expressed in the form
\[
V^- = \text{Im} \left( \begin{pmatrix} I \\ X^- \end{pmatrix} \right)
\]
in which \(X^-\) is a symmetric matrix, the (unique) stabilizing solution of the Riccati equation (76).

If the pair \((A, R)\) is antistabilizable, the unstable eigenspace \(V^+\) of (79) can be expressed in the form
\[
V^+ = \text{Im} \left( \begin{pmatrix} I \\ X^+ \end{pmatrix} \right)
\]
in which \(X^-\) is a symmetric matrix, the (unique) antistabilizing solution of the Riccati equation (76).

We discuss now the relation between solutions of the algebraic Riccati equation (76) and of the algebraic Riccati inequality
\[
A^T X + XA + Q + RX > 0.
\]

**Lemma 8** Suppose \(R\) is negative semidefinite. Let \(X^-\) (respectively \(X^+\)) be a solution of the Riccati equation (76) having the property that \(\sigma(A + RX^-) \in \mathbb{C}^-\) (respectively, \(\sigma(A + RX^+) \in \mathbb{C}^+\)). Let \(X\) be any symmetric matrix satisfying (83). Then \(X < X^-\) (respectively, \(X > X^+\)).

Proof. Set \(R = -BB^T\) and observe that
\[
A^T (X - X^-) + (X - X^-)A - XBB^TX + X^-BB^TX^- > 0.
\]
Set \(P = X - X^-\) and observe that the rate of change of the quadratic function \(V(x) = x^TPx\) along the trajectories of
\[
\dot{x} = (A + RX^-)x = (A - BB^TX^-)x
\]
satisfies
\[ \dot{V}(x) = 2x^T(X - X^-)(A + RX^-)x = x^T[A^T(X - X^-) + (X - X^-)A - 2(X - X^-)BB^TX^-]x \]

Note now that
\[ -2(X - X^-)BB^TX^- = -2(X - X^-)BB^TX^- + XBB^TX - XBB^T X = -XBB^TX + XBB^T X^- + (X - X^-)BB^T(X - X^-) \]

Hence, letting \( \tilde{P} \) denote the left-hand side of (84), we obtain
\[ \dot{V}(x) = x^T[\tilde{P} + x^TPBB^TP]x. \]

Since \( \tilde{P} \) is positive definite by assumption, so is also \( \tilde{P} + PBB^TP \) and hence the function \( V(x) \) is increasing along any nontrivial trajectory of (85). In other words,
\[ V(x(t)) > V(x(0)) \quad (86) \]
along any nontrivial trajectory of (85). The latter, on the other hand, is an asymptotically stable system by assumption and \( \lim_{t \to \infty} x(t) = 0 \) for any \( x(0) \). Therefore, (86) implies that \( V(x) \) is negative for any \( x \neq 0 \). This shows that \( P \) is negative definite, i.e. \( X < X^- \).

Lemma 9

Let \( X^- \) (respectively, \( X^+ \)) be a solution of the Riccati equation (76) having the property that \( \sigma(A + RX^-) \in \mathbb{C}^- \) (respectively, \( \sigma(A + RX^+) \in \mathbb{C}^+ \)). Then, there exist a real number \( \varepsilon_0 > 0 \) and a family of symmetric matrices \( X_\varepsilon \), defined for \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \) and continuously depending on \( \varepsilon \), satisfying
\[ A^TX_\varepsilon + X_\varepsilon A + Q + X_\varepsilon RX_\varepsilon = \varepsilon I. \]
and such that \( \lim_{\varepsilon \to 0} X_\varepsilon = X^- \) (respectively, \( X^+ \)).

Proof. Consider the function \( f : \mathbb{R}^{n \times n} \times \mathbb{R} \to \mathbb{R}^{n \times n} \) defined by
\[ f(X, \varepsilon) = A^TX + XA + Q + XRX - \varepsilon I. \]

By definition, this function is such that
\[ f(X^-, 0) = 0. \]

Moreover, it is easy to see that
\[ f(X^- + \delta X, 0) = f(X^-, 0) + \delta X(A + RX^-) + (A + RX^-)^T \delta X + \delta X R \delta X. \]
The linear term in \( \delta X \) on the right-hand side of this relation can be viewed as the value at \( \delta X \) of the linear mapping \( \mathcal{L} : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n} \) defined by
\[ \mathcal{L}(\delta X) = \delta X(A + RX^-) + (A + RX^-)^T \delta X \]

In fact, suppose \( V(\bar{x}) = c > 0 \) for some \( \bar{x} \neq 0 \) and set \( x(0) = \bar{x} \). Since \( \lim_{t \to \infty} x(t) = 0 \), there exists \( T > 0 \) such that \( |V(x(T))| \leq c/2 \) and this, in view of (86), would yield \( c/2 \geq |V(x(T))| \geq V(x(T)) \geq V(x(0)) = c \), i.e. a contradiction.
and this mapping is indeed invertible since \((A + RX^{-})\) has by hypothesis all eigenvalues in \(\mathbb{C}^{-}\) and thus the Sylvester equation
\[
\delta X(A + RX^{-}) + (A + RX^{-})^T \delta X = Y
\]
has a unique solution for each \(Y \in \mathbb{R}^{n \times n}\). One can therefore conclude, by the implicit function theorem, that there exist a positive number \(\varepsilon_0\) and a continuous function \(X_\varepsilon\) of \(\varepsilon\), defined for all \(|\varepsilon| < \varepsilon_0\), such that \(f(X_\varepsilon, \varepsilon) = 0\), satisfying
\[
A^T X_\varepsilon + X_\varepsilon A + Q + X_\varepsilon RX_\varepsilon = \varepsilon I \quad \text{for all } |\varepsilon|.
\]
(87)

By construction
\[
\begin{pmatrix}
A & R \\
-Q + \varepsilon I & -A^T
\end{pmatrix}
\begin{pmatrix}
I \\
X_\varepsilon
\end{pmatrix}
=
\begin{pmatrix}
I \\
X_\varepsilon
\end{pmatrix}
(A + RX_\varepsilon).
\]
Since \((A + RX^{-})\) has all eigenvalues in \(\mathbb{C}^{-}\) and \(\lim_{\varepsilon \to 0} X_\varepsilon = X^{-}\), then also \((A + RX_{\varepsilon})\) has all eigenvalues in \(\mathbb{C}^{-}\) for small \(|\varepsilon|\). Hence \(X_\varepsilon\) is the unique stabilizing solution of (87) and is necessarily symmetric. \(\triangleleft\)

We can merge the previous two Lemma as follows.

**Proposition 3** Suppose \(R\) is negative semidefinite. Let \(X^{-}\) (respectively \(X^{+}\)) be a solution of the Riccati equation (76) having the property that \(\sigma(A + RX^{-}) \in \mathbb{C}^{-}\) (respectively, \(\sigma(A + RX^{+}) \in \mathbb{C}^{+}\)). Then, the set of solutions of
\[
A^T X + XA + Q + XRX > 0,
\]
is not empty and any \(X\) in this set satisfies \(X < X^{-}\) (respectively, \(X > X^{+}\)).

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