High Order Sliding Mode, Relative Degree, Finite Time Convergence and Disturbance Rejection
THE CHATTERING PROBLEM

Unmodeled dynamics

Continuous control: \( v = v(x) \)

Sliding mode control: \( v = M \text{ sign } s \)
Second Order Sliding Mode, Relative Degree, Finite Time Convergence and Disturbance Reject

\[ \dot{x} = f(x, t) + b(x, t)u, \quad x, f, b \in \mathbb{R}^n \]

The conventional control

\[ \dot{s} = [\text{grad}(s)]^T f + [\text{grad}(s)]^T bu \]

\[ u(x) = \begin{cases} 
  u^+(x) & \text{if } s(x) > 0 \\
  u^-(x) & \text{if } s(x) < 0 
\end{cases} \]

\( (\text{grad}(s))^T b \neq 0 \), then sliding mode can be enforced on surface \( s(x) = 0 \)

and relative degree is equal to 1.
The Problem:

Can the similar effect be obtained with control as a Continuous State Function?

Then the range of applications of sliding mode control will be increased since not all actuators can operate with discontinuous inputs. For example high frequency switching in an output may result in damage of valves in hydraulic actuators.

Yes, if control is a non-Lipschitzian function
For the system with  \((\text{grad}(s))^T b = 1\) and continuous control

\[ u = -\sqrt{s} \text{sign}(s) - (\text{grad}(s))^T f \]

\[ \dot{s} = -\sqrt{|s|} \text{sign}(s) \]

the time derivative of Lyapunov function \( V = \frac{1}{2} s^2 > 0 \)

\[ \dot{V} = -\sqrt{|s|} < 0 \quad \text{and} \quad \dot{V} = -(2V)^{3/4} \]

with the solution

\[ V(t) = (V_0^{1/4} - 2^{-5/4} t)^4 \text{ equal to zero for } t \geq 2^{5/4} V_0^{1/4} \]

It means that state trajectories belong to the surface \( s(x) = 0 \) after a finite time interval.

The motion is similar to that of the systems with discontinuous control: it is reasonable to call this motion \textbf{SLIDING MODE}.

Unfortunately this approach does not work in systems with disturbances, since the control tends to zero with state tending to zero and is unable to reject them.
SECOND ORDER SLIDING MODE
AND RELATIVE DEGREE

design methods were developed to enforce sliding mode in the subspace of dimension $n-2$
directly such that the control is continuous function. The authors referred to their methods as the high order sliding mode

$v \rightarrow \int \rightarrow u$

Control is a continuous function as an output of integrator with a discontinuous state function as an input

Then sliding mode can be enforced with $V$ as a discontinuous function of and $s$ and $\dot{s}$
For example if sliding mode exists on line $S = cs + \dot{s} = 0$
then $s$ tends to zero asymptotically and sliding mode exists in the origin of two dimensional subspace $s$ and $\dot{s}$

It is hardly reasonable to call this conventional sliding mode as the second order sliding mode. For slightly modified switching line $S = c\sqrt{s} + \dot{s} = 0$, $s>0$ the state reaches the origin after a finite time interval. The finiteness of reaching time served for several authors as the argument to label this motion in the point $s = \dot{s} = 0$

"second order sliding mode"
Short Discussion

1-2 reaching phase
2-3 sliding mode of the 1st order
Point 3 sliding mode of the 2nd order

Finite times of 1-2 and 2-3

System of the 3rd order

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u \\
u &= -M \text{sign}(s), \\
S &= \dot{s} + \sqrt{|s|} \text{sign}(s), \\
s &= x_2 + \sqrt{|x_1|} \text{sign}(x_1)
\end{align*}
\]

1st phase - reaching surface \( S=0 \)
2nd phase - reaching curve \( s=0 \) in \( S=0 \)
sliding mode of the 1st order
3rd phase – reaching the origin
sliding mode of the 2nd order
4th phase – sliding mode of the 3rd order in the origin

Finite times of the first 3 phases
More interesting versions of the second order sliding mode control known as **twisting** and **super twisting algorithms** were proposed under the assumption that the point $s = \dot{s} = 0$ is reached after a finite time interval as well, but sliding mode does not occur at this first stage of motion.

$$\dot{s} = (\text{grad}(s))^T (f + Bu), \quad [\text{grad}(s)]^T b = 1$$

**TWISTING ALGORITHM**

Again control is a continuous function as an output of integrator

$$u = v, \quad v = -M_0 \text{sign}(s) - M_1 \text{sign}(\dot{s}), \quad \ddot{s} = [\text{grad}(s)]^T f + [\text{grad}(s)]^T bu$$

$$M_0 > M_1 + F_0, \quad M_1 > F_0. \quad |F(x,t,u)| \leq F_0,$$

$$\dot{s} = F(x,t,u) - M_0 \text{sign}(s) - M_1 \text{sign}(\dot{s}),$$

Of course relative degree between discontinuous input $v$ and output $s$ is still equal to 1 and the conventional sliding mode can be enforced, since $ds/dt$ is used.
**SUPER TWISTING ALGORITHM**

\[ \dot{s} = (\nabla s)^T (f + Bu), \]

\[ u = -a \sqrt{s} \text{sign}(s) + v, \quad \dot{v} = -M \text{sign}(s), \quad a > 0, M > 0, \quad a - \text{const}. \]

\[ \ddot{s} = -a \frac{\dot{s}}{2\sqrt{|s|}} - M \text{sign}(s) + f(t) \]

Control \( u \) is continuous, no \( \dot{s} \), relative degree of the open loop system from \( v \) to \( s \) is equal to 2!

**FINITE TIME CONVERGENCE AND BOUNDED DISTURBANCE CAN BE REJECTED**

\[ |f| \leq f_0 - \text{const}. \]

**HOWEVER IT WORKS FOR THE SYSTEMS FOR SPECIAL CONTINUOUS PART WITH NON-LIPSCHIZIAN FUNCTION.**
**ASYMPTOTIC STABILITY**

The stability follows from the evident fact: equilibrium point of the second order systems

\[ \ddot{s} + q_1(s, \dot{s}, t) + q_0(s) = 0 \quad \text{with} \quad \dot{s}q_1(s, \dot{s}, t) > 0, \quad sq_0(s) > 0 \]

is asymptotically stable, as a model of mass-spring-damper system with total - kinetic and potential - energy as a Lyapunov function

\[ V = 0.5\dot{s}^2 + \int_0^s q_0(\alpha)d\alpha > 0, \quad \dot{V} = -\dot{s}q_1(s, \dot{s}, t) < 0. \]

The both systems satisfy these conditions with arbitrary

\[ M_1(t) > 0, \quad M_0(s) > 0 \quad \text{or} \quad M(s) > 0 \]

AND ZERO DISTURBANCES

\[ f \text{ and } F. \]
**FINITE TIME CONVERGENCE**

**Homogeneity property**

\[ \dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n \]

is asymptotically stable and

\[ f(Cx) = \beta Cf(x), \quad C = \text{diag}(c_i), \quad \beta \] is a scalar parameter.

**Theorem:** the state converges to the equilibrium point \( x = 0 \) in a finite time interval, if

\[ 0 < c_i < 1, \quad \beta > 1. \]

There exists \( T \) such that

\[ \|x(T)\| < c_{\min} \|x(0)\|, \quad c_{\min} = \min_i c_i. \] (2)

If \( x = Cy_1 \), then

\[ \frac{dy_1}{d(\beta t)} = f(y_1), \quad \|y_1(T)\| \leq \frac{\|x(T)\|}{c_{\min}} \leq \|x(0)\| \] and according (2) \[ \|y_1(T + \frac{T}{\beta})\| \leq c_{\min} \|x(0)\|. \] The condition

\[ x = Cy_1 \] means that \[ \|y_1\| \geq \frac{\|x\|}{c_{\max}}, \quad c_{\max} = \max_i c_i \] and \[ \|x(T + \frac{T}{\beta})\| \leq c_{\min} c_{\max} \|x(0)\|. \]
Similarly for \( y_1 = C y_2 \),

\[
\frac{dy_2}{d(\beta^2 t)} = f(y_2), \quad \left\| y_2 \left( T + \frac{T}{\beta} + \frac{T}{\beta^2} \right) \right\| \leq c_{\text{min}} \left\| y_2 \left( T + \frac{T}{\beta} \right) \right\| \leq c_{\text{min}} \left\| x(0) \right\|
\]

and

\[
\left\| y_2 \left( T + \frac{T}{\beta} + \frac{T}{\beta^2} \right) \right\| \geq \frac{c_{\text{max}}}{c_{\text{max}}} \left\| y_1 \left( T + \frac{T}{\beta} + \frac{T}{\beta^2} \right) \right\| \geq \frac{c_{\text{max}}}{c_{\text{max}}} \left\| x(T + \frac{T}{\beta} + \frac{T}{\beta^2}) \right\|
\]

Comparing (4) and (5) results in

\[
\left\| x(T + \frac{T}{\beta} + \frac{T}{\beta^2}) \right\| \leq c_{\text{min}} c_{\text{max}}^2 \left\| x(0) \right\|
\]

After \( k \) steps

\[
\left\| x(T + \frac{T}{\beta} + \ldots + \frac{T}{\beta^{k-1}}) \right\| \leq c_{\text{min}} c_{\text{max}}^{k-1} \left\| x(0) \right\|
\]

The result means convergence of the state to zero after finite time.

**Convergence time:**

\[
T + \frac{T}{\beta} + \ldots + \frac{T}{\beta^{k-1}} \quad \rightarrow \quad T \left/ \left( 1 - \frac{1}{\beta} \right) \right.
\]
EXAMPLES OF SYSTEMS WITH NO DISTURBANCES

**Example 1** (twisting)

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = -M_0 \text{sign}(x_1) - M_1 \text{sign}(x_2), \quad M_0 > M_1.
\]

The equations are the model of a mass-damper-nonlinear spring mechanical system with the only equilibrium point therefore it is asymptotically stable. The total (kinetic and potential) energy may serve as a Lyapunov function:

\[
V = M_1 |x_1| + 0.5x_2^2 > 0, \quad \dot{V} = -M_2 |x_2| \leq 0, \text{ asymptotically stable (by LaSalle)}.
\]

\[f(Cx) = \beta Cf(x), \quad \beta = 2, \quad c_1 = 0.25, \quad c_2 = 0.5 \quad (c_2 = \beta c_1, 1 = \beta c_2), \text{ which means finite time convergence.}\]

**Example 2** (super twisting)

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = -\frac{x_2}{\sqrt{|x_1|}} - M \text{sign}(x_1) \quad (\text{from} \quad \dot{x}_1 = -2\sqrt{|x_1|} \text{sign}(x_1) + \nu). \\
\dot{\nu} = -M \text{sign}(x_1)
\]

Similarly the equilibrium point is asymptotically stable. It can be confirmed by the same Lyapunov function with \(\dot{V} = -M \frac{x_2^2}{\sqrt{|x_1|}} \leq 0, \text{ (again by LaSalle)}.\)

\[f(Cx) = \beta Cf(x), \quad \beta = 2, \quad c_1 = 0.25, \quad c_2 = 0.5 \quad (c_2 = \beta c_1, \frac{c_2}{\sqrt{c_1}} = \beta c_2 = 1), \text{ which also means finite time convergence.}\]
HOMOGENEITY PROPERTY

for the systems with zero disturbances and constant $M_i$. **Motion Equations:**

\[ \dot{S} = f(S), \quad S^T = (s, \dot{s}), \quad f^T = (\dot{s}, -M_0 \text{sign}(s) - M_1 \text{sign}(\dot{s})) \]

for twisting and

\[ f^T = (\dot{s}, -a \frac{\dot{s}}{2\sqrt{|s|}} - Ms\text{sign}(s)) \]

for super twisting.

Then finite time convergence follows from

\[ f(Cx) = \beta Cf(x), \quad \beta = 2, \quad c_1 = 0.25, \quad c_2 = 0.5 \quad (c_2 = \beta c_1, 1 = \beta c_2) \]

for twisting and from

\[ f(Cx) = \beta Cf(x), \quad \beta = 2, \quad c_1 = 0.25, \quad c_2 = 0.5 \quad (c_2 = \beta c_1, \frac{c_2}{\sqrt{c_1}} = \beta c_2 = 1) \]

for super twisting.

A. Levant, A. Polyakov and A. Poznyak, Yu. Orlov - twisting algorithms with time varying disturbances

In what follows

finite time convergence will be demonstrated at the presence of disturbances.
Finite Time Convergence of Super-Twisting Algorithm.
(the domain of initial conditions is bounded)

\[ \dot{x} = -a \sqrt{|x|} \text{sign}(x) + y \]  \hfill (1)

\[ \dot{y} = -M_0 \text{sign}(x) + f(t), \]  \hfill (2)

| \[ f(t) \] | < f_0, M_0 > f_0, m = M_0 - f_0, M = M_0 - f_0.

For initial conditions

\[ y(0) = 0, x(0) = -x_0, x_0 > 0 : \quad \dot{x}(0) > 0 \quad \text{and} \quad x(t) \text{ is increasing.} \]

Let \( t_1^* = x(t_1^*) = 0, x(t) < 0 \) for \( t \in (0, t_1^*) \).

Compare two systems

\[ \dot{x} = -\frac{a}{\sqrt{|x|}}x + y \]  \hfill (3)

\[ \dot{z} = -\frac{a}{\sqrt{|x_0|}}z + mt, \quad z(0) = -x_0 \]  \hfill (4)

\[ y(t) = \int_0^t [M_0 + f(\gamma)]d\gamma > mt. \]

Equation (3) is equivalent to (1).

Since \( |x| \) is decreasing, \[ \frac{1}{\sqrt{|x|}} > \frac{1}{\sqrt{|x_0|}} = k_0, \]

if \( x = z < 0, \) then \( \dot{x} > \dot{z}, \) hence \( x(t) > z(t) \) and

if \( z(t') = 0, \) then \( t_1^* < t'. \)
Solution to (4):

\[ z(t) = \frac{m}{ak_0} (t - \frac{1}{ak_0}) + (-x_0 + \frac{m}{(ak_0)^2})e^{-ak_0t} \]

\[ z(t) = 0 : \]

\[ \frac{m}{a} x_0 \left( \frac{t}{\sqrt{x_0}} - \frac{1}{a} \right) + (-x_0 + x_0 \frac{m}{a^2})e^{-a\frac{t}{\sqrt{x_0}}} = 0, \quad \tau = \frac{t}{\sqrt{x_0}} \]

\[ m(\tau - \frac{1}{a}) + (-a + \frac{m}{a})e^{-a\tau} = 0 \] (5)

Solution to (5) \( \tau' \):

\[ \lim_{a \to \infty} \tau' = 0, \quad \tau' = \left[ 0 \left( \frac{1}{a} \right) \right] \text{ DOSSES NOT DEPEND ON } x_0 !!! \]

\[ t' = \sqrt{x_0} \left[ 0 \left( \frac{1}{a} \right) \right], \quad t_1^* < t', \text{ hence } y(t_1^*) < M \sqrt{x_0} \left[ 0 \left( \frac{1}{a} \right) \right] \]

and \( x(t) > 0 \) for \( t > t_1^* \).

\[ y(t) < M \sqrt{x_0} \left[ 0 \left( \frac{1}{a} \right) \right] - mt \text{ for } t > t_1^* \] (6)

Now \( y(t) \) is a decaying function and \( y(t_1^{**}) = 0 \) for

\[ t_1^{**} < t_1^* + \frac{M}{m} \sqrt{x_0} \left[ 0 \left( \frac{1}{a} \right) \right]. \]

Interval between two instants, when \( y(t) \) is equal to zero

\[ t_1^{***} < \sqrt{x_0} \left( M + \frac{M}{m} \right) \left[ 0 \left( \frac{1}{a} \right) \right] \] (7)
As it follows from (1) and (6), \( x(t) > 0 \) for \( t_1^* < t < t_1^{**} \) and
\[
a \sqrt{|x|} \text{ can not exceed } M \sqrt{x_0} \left[ 0 \left( \frac{1}{a} \right) \right],
\]
\[
x(t_1^{**}) < x_0 \left( \frac{M}{a} \right)^2 \left[ 0 \left( \frac{1}{a} \right) \right]^2,
\]
select \( a \) such that
\[
\left( \frac{M}{a} \right)^2 \left[ 0 \left( \frac{1}{a} \right) \right]^2 < \gamma^2, \quad 0 < \gamma < 1
\]
then
\[
x(t_1^{**}) < \gamma^2 x_0
\]  
(8)
\( x(t_1^{**}) \) is an initial value for the second interval \((t_1^{**}, t_2^{**})\), \( y(t_2^{**}) = 0 \).

Similarly to (7), (8)
\[
\Delta t_2 < \gamma \sqrt{x_0} \left( M + \frac{M}{m} \right) \left[ 0 \left( \frac{1}{a} \right) \right], \quad \Delta t_2 = t_2^{**} - t_1^{**}, \quad x(t_1^{**}) < \gamma^2 x(t_2^{**})
\]
\[
| x(t_2^{**}) | < \gamma^4 x_0
\]
and
\[
\Delta t_i < \gamma^{i-1} \sqrt{x_0} \left( M + \frac{M}{m} \right) \left[ 0 \left( \frac{1}{a} \right) \right]
\]
\[
| x(t_i^{**}) | < \gamma^{2i} x_0
\]
The last two equations mean
\[
\lim_{t \to \infty} x(t) = 0,
\]
\[
t_1^{**} + \sum_{i=2}^{\infty} \Delta t_i \text{ is bounded, or finite time convergence.}
\]
The above property can be provided for any disturbance with upper estimate \( f_0 < M_0 \).
It is evident that convergence does not take place for \( f_0 \geq M_0 \).

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*) Note, that \( a \to \infty \text{ with } m \to 0 \). Indeed, if \( m \to 0 \) with \( a = \text{const} \), then \( t' \to \infty \).
Twisting Algorithm

\[ \dot{s} = F(x,t,u) - M_0 \text{sign}(s) - M_1 \text{sign}(\dot{s}) , \]

\[ \text{Lyapunov function} \quad V = 2\sqrt{\frac{1}{2} \dot{s}^2 + M_0 |s|} \quad \text{(square root of energy (V>0))} \]

\[ \dot{V} = -\frac{M'}{\sqrt{\frac{1}{2} + M_0 \frac{|s|}{\dot{s}^2}}} < 0 \quad \text{with} \quad M' = M_1 - F_0 > 0 \]

Beyond domain D with

\[ \frac{|s|}{\dot{s}^2} \leq k, \quad k = \text{const} \]

\[ S_I = \dot{s} - \varepsilon \sqrt{s} = 0 \]

\[ S_{II} = \dot{s} + \varepsilon \sqrt{s} = 0, \quad \varepsilon = 1/\sqrt{k} \]

\[ \dot{V} \leq -M' \frac{1}{\sqrt{\frac{1}{2} + M_0 k}} = -\alpha < 0 , \]

Lyapunov function decays at finite rate

Trajectories can penetrate into D through \( S_I = 0 \) and leave it through \( S_{II} = 0 \) only.
Based on analysis of solutions to
\[ \ddot{s} = -m \]
with upper and low estimates of \( m \) it can be shown that for initial conditions \( s_0 \) and \( S_I(0) = 0 \) (\( \dot{s}_0 = -\varepsilon \sqrt{s_0} \)):

\[ t_1 + t_2 \leq \lambda_{1,2} \sqrt{s_0}, \quad t_3 \geq \lambda_3 \sqrt{s_0}, \]
\[ \dot{s}(t_1) = 0, \quad S_{II}(t_1 + t_2) = 0, \quad s(t_1 + t_2 + t_3) = 0. \]

Positive coefficients \( \lambda_{1,2} \) and \( \lambda_3 \) do not depend on the system state.

The average rate of decaying of Lyapunov function is finite and negative, which means \( \text{FINITE CONVERGENCE TIME.} \)
SUPER-TWISTING ALGORITHM

\[ s_0 \text{ and } S_I(0) = 0 \quad (s_0 = -\varepsilon \sqrt{s_0}) \]

\[ \dot{s} = -a \frac{s^2}{2|s|} + \frac{s}{\sqrt{|s|}} \dot{f} \]

\[ \dot{V} = \frac{-a \frac{s^2}{2|s|} + \frac{s}{\sqrt{|s|}} \dot{f}}{\sqrt{s^2/2|s|} + M} \]

may be positive (!!!)

may be positive (!!!)

\[ t_1 + t_2 \leq \lambda_{1,2} \sqrt{s_0}, \quad \lambda_{1,2} = \varepsilon \lambda_0, \quad \lim \lambda_0 > 0 \text{ with } \varepsilon \to 0 \]

\[ t_3 \geq \lambda_3 \sqrt{s_0}, \quad \lim \lambda_3 > 0 \text{ with } \varepsilon \to 0 \]

\[ s(t_1) = 0, \quad S_{I}(t_1 + t_2) = 0, \quad s(t_1 + t_2 + t_3) = 0. \]

Beyond domain \( D \)

\[ \dot{V} < -\varepsilon^2 V_{out}, \quad \lim V_{out} > 0 \text{ with } \varepsilon \to 0 \]

In domain \( D \)

\[ \dot{V} < \varepsilon^2 V_{in}, \quad \lim V_{in} > 0 \text{ with } \varepsilon \to 0 \]

\[ V_{av} = \frac{(t_1 + t_2)V_{in} - \varepsilon^2 V_{out}}{t_1 + t_2 + t_3} = \frac{\varepsilon^2 (\varepsilon \lambda_0 V_{in} - V_{out})}{\lambda_3 + \varepsilon}. \]

Again finite convergence time

**UPPER ESTIMATE OF THE DISTURBANCE**

\[ F < M/2 \]
DIFFERENTIATORS

The first-order system

\[
\begin{align*}
\dot{x} &= u, \\
\mu \dot{z} + z &= u,
\end{align*}
\]

\[
\lim_{\mu \to 0, \sigma \to 0} z = u_{eq} = k \dot{f}(t), \quad |\sigma| < \delta.
\]

u = M\text{sign}(\sigma), \quad \sigma = f(t) - x

\dot{\sigma} = \dot{f} - M\text{sign}(\sigma), \quad \dot{\sigma} = 0 \quad \rightarrow \quad u_{eq} = k \dot{f}(t)

The second-order system

\[
\begin{align*}
\dot{x} &= -a \sqrt{s} + v, \\
\dot{v} &= M\text{sign}(s), \quad s = f(t) - x
\end{align*}
\]

\[
s(t) \equiv 0, \quad x(t) \equiv f(t)
\]

\[
u(t) = v - a \sqrt{s} = \dot{f}(t)
\]

U is continuous, low-pass filter is not needed.
• OBJECTIVE: CHATTERING REDUCTION

• METHOD: REDUCING THE MAGNITUDE OF THE DISCONTINUOUS CONTROL TO THE MINIMAL VALUE PRESERVING SLIDING MODE UNDER UNCERTAINTY CONDITIONS.
Motivating example: a simple model

\[ \dot{x}(t) = a + u \]
\[ u = -k \text{sign}(x(t)), \quad k > 0, \quad 0 < |a| \leq a_+ \]

The sliding mode exists for all values of unknown parameter \( k > a_+ \). The objective of adaptation is decreasing \( k \) to the minimal value preserving sliding mode, if the parameter \( a \) is a constant but unknown.

If sliding mode with \( x(t) \equiv 0 \) occurs then

\[ \dot{x}(t) = 0 = a + u_{eq} \]
\[ k \left[ \text{sign}(x(t)) \right]_{eq} = a \]

where the function \( [\text{sign}(x(t))]_{eq} \) is an average value (a slow component of discontinuous function \( \text{sign} x(t) \)) which can be easily obtained by a low pass filter filtering out the high frequency component.

Of course, the average value of \( [\text{sign}(x(t))]_{eq} \) is in the range \((-1, 1)\).
The design idea of adaptation looks now transparent: after sliding mode occurs the control parameter $k$ should be decreased until becomes close to $a$. Further decreasing will lead to ceasing sliding mode. As a result, the minimal possible value of discontinuity magnitude $k$ is found for the current value of parameter $a$ to reduce the amplitude of chattering.

$$\dot{k}(t) = k(t)\text{sign}(\delta(t)) - M[k(t) - k^+]_+ + M[\mu - k(t)]_+$$

$$\delta(t) := \left[\text{sign}(x(t))\right]_{eq} - \alpha, \alpha \in (0, 1), M > k^+ > a^+$$

$$[z]_+ := \begin{cases} z & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases} \quad 0 < 1 - \alpha << 1$$

$$[\text{sign}(x)]_{eq} = \alpha = \frac{a}{k}$$

The gain $k$ can vary in the range $[\mu, k^+]$, $\mu$ is a preselected minimal value of $k$.

$$V(\delta) = \frac{\delta^2}{2} \quad \dot{V}(\delta(t)) = \delta(t)\dot{\delta}(t) \leq -\sqrt{2}\frac{|a|}{k^+} \sqrt{V(\delta(t))}$$

$$0 \leq \sqrt{V(\delta(t))} \leq \sqrt{V(\delta(0))} - \frac{|a|}{\sqrt{2}k^+} t \quad t_f = \frac{k^+}{|a|} \sqrt{2V(\delta(0))} = \frac{k^+}{|a|} |\delta(0)|$$
Adaptive super-twisting algorithm

\[
\begin{cases}
\dot{x}_1 = x_2 - \bar{a} \sqrt{|x_1|} \text{sign}(x_1) \\
\dot{x}_2 = \phi(t) + u \\
u := -k \text{sign}(x_1)
\end{cases}
\]
\[|\phi(t)| \leq \phi_0 < k\]

First, it was shown that

1. \(|\phi(t)| \leq \phi_0 < k\) is necessary condition for convergence

2. For any \(\phi_0 < k\) there exists \(\bar{\alpha}_0 > 0\) such that finite-time convergence takes place for \(\bar{\alpha} \geq \bar{\alpha}_0\)

Then

Similarly

\[\dot{k} = k(t) \text{sign}(\delta(t)),\]
\[\delta(t) = |\text{sign}(x(t))|_{eq} - \alpha, \quad 0 < 1 - \alpha << 1.\]

In sliding mode

\[|\text{sign}(x(t))|_{eq} = \frac{|\phi(t)|}{k}, \quad \alpha = \frac{\phi(t)}{k},\]
\[|\phi(t)| \text{ is close to } k(t).\]
Remark Instead of Conclusion

**CHALLENGE:** to generalize twisting algorithm to get the third order sliding mode adding two integrators with input similar to that for the 2\textsuperscript{nd} order:

\[
\ddot{s} = -M_0 \text{sign}(s) - M_1 \text{sign}(\dot{s}) - M_2 \text{sign}(\ddot{s}) + F(x, u, \dot{u}, t), \quad |F| \leq F_0 = \text{const},
\]

\[
M_0 > M_1 + M_2 + F_0.
\]

Unfortunately the 3\textsuperscript{rd} order sliding mode without sliding modes of lower order can not be implemented, indeed time derivative of sign-varying Lyapunov function \( V = s\ddot{s} - \frac{1}{2}(\dot{s})^2 \)

\[
\frac{dV}{dt} = -M_0 |s| - s[M_1 \text{sign}(\dot{s}) + M_2 \text{sign}(\ddot{s}) - F(x, u, \dot{u}, t)]
\]

is negative, hence the origin in the space \((s, \dot{s}, \ddot{s})\) is unstable (Lyapunov theorem on instability).