Introduction to Nonlinear Control
Lecture # 3

Time-Varying
and
Perturbed Systems
Time-varying Systems

\[ \dot{x} = f(t, x) \]

\( f(t, x) \) is piecewise continuous in \( t \) and locally Lipschitz in \( x \) for all \( t \geq 0 \) and all \( x \in D \). The origin is an equilibrium point at \( t = 0 \) if

\[ f(t, 0) = 0, \quad \forall t \geq 0 \]

While the solution of the autonomous system

\[ \dot{x} = f(x), \quad x(t_0) = x_0 \]

depends only on \( (t - t_0) \), the solution of

\[ \dot{x} = f(t, x), \quad x(t_0) = x_0 \]

may depend on both \( t \) and \( t_0 \)
Comparison Functions

A scalar continuous function $\alpha(r)$, defined for $r \in [0, a)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if it defined for all $r \geq 0$ and $\alpha(r) \to \infty$ as $r \to \infty$.

A scalar continuous function $\beta(r, s)$, defined for $r \in [0, a)$ and $s \in [0, \infty)$ is said to belong to class $\mathcal{K}\mathcal{L}$ if, for each fixed $s$, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to $r$ and, for each fixed $r$, the mapping $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \to 0$ as $s \to \infty$. 
Example

- $\alpha(r) = \tan^{-1}(r)$ is strictly increasing since $\alpha'(r) = 1/(1 + r^2) > 0$. It belongs to class $\mathcal{K}$, but not to class $\mathcal{K}_\infty$ since $\lim_{r \to \infty} \alpha(r) = \pi/2 < \infty$

- $\alpha(r) = r^c$, for any positive real number $c$, is strictly increasing since $\alpha'(r) = cr^{c-1} > 0$. Moreover, $\lim_{r \to \infty} \alpha(r) = \infty$; thus, it belongs to class $\mathcal{K}_\infty$

- $\alpha(r) = \min\{r, r^2\}$ is continuous, strictly increasing, and $\lim_{r \to \infty} \alpha(r) = \infty$. Hence, it belongs to class $\mathcal{K}_\infty$
\[ \beta(r, s) = \frac{r}{(ksr + 1)}, \text{ for any positive real number } k, \]

is strictly increasing in \( r \) since

\[
\frac{\partial \beta}{\partial r} = \frac{1}{(ksr + 1)^2} > 0
\]

and strictly decreasing in \( s \) since

\[
\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ksr + 1)^2} < 0
\]

Moreover, \( \beta(r, s) \to 0 \) as \( s \to \infty \). Therefore, it belongs to class \( \mathcal{K} \mathcal{L} \)

\[ \beta(r, s) = r^c e^{-s}, \text{ for any positive real number } c, \text{ belongs to class } \mathcal{K} \mathcal{L} \]
Definition: The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is

- uniformly stable if there exist a class $\mathcal{K}$ function $\alpha$ and a positive constant $c$, independent of $t_0$, such that
  $$
  \|x(t)\| \leq \alpha(\|x(t_0)\|), \ \forall \ t \geq t_0 \geq 0, \ \forall \ \|x(t_0)\| < c
  $$

- uniformly asymptotically stable if there exist a class $\mathcal{KL}$ function $\beta$ and a positive constant $c$, independent of $t_0$, such that
  $$
  \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \ \forall \ t \geq t_0 \geq 0, \ \forall \ \|x(t_0)\| < c
  $$

- globally uniformly asymptotically stable if the foregoing inequality is satisfied for any initial state $x(t_0)$
exponentially stable if there exist positive constants $c$, $k$, and $\lambda$ such that

$$\|x(t)\| \leq k\|x(t_0)\| e^{-\lambda(t-t_0)}, \quad \forall \|x(t_0)\| < c$$

globally exponentially stable if the foregoing inequality is satisfied for any initial state $x(t_0)$
Theorem: Let the origin $x = 0$ be an equilibrium point for $\dot{x} = f(t, x)$ and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Suppose $f(t, x)$ is piecewise continuous in $t$ and locally Lipschitz in $x$ for all $t \geq 0$ and $x \in D$. Let $V(t, x)$ be a continuously differentiable function such that

\begin{align*}
(1) \quad W_1(x) &\leq V(t, x) \leq W_2(x) \\
(2) \quad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) &\leq 0
\end{align*}

for all $t \geq 0$ and $x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on $D$. Then, the origin is uniformly stable.
Theorem: Suppose the assumptions of the previous theorem are satisfied with

\[ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x) \]

for all \( t \geq 0 \) and \( x \in D \), where \( W_3(x) \) is a continuous positive definite function on \( D \). Then, the origin is uniformly asymptotically stable. Moreover, if \( r \) and \( c \) are chosen such that \( B_r = \{ \|x\| \leq r \} \subset D \) and \( c < \min_{\|x\|=r} W_1(x) \), then every trajectory starting in \( \{ x \in B_r \mid W_2(x) \leq c \} \) satisfies

\[ \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall \ t \geq t_0 \geq 0 \]

for some class \( \mathcal{KL} \) function \( \beta \). Finally, if \( D = \mathbb{R}^n \) and \( W_1(x) \) is radially unbounded, then the origin is globally uniformly asymptotically stable.
Theorem: Suppose the assumptions of the previous theorem are satisfied with

\[ k_1 \|x\|^a \leq V(t, x) \leq k_2 \|x\|^a \]

\[ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -k_3 \|x\|^a \]

for all \( t \geq 0 \) and \( x \in D \), where \( k_1, k_2, k_3 \), and \( a \) are positive constants. Then, the origin is exponentially stable. If the assumptions hold globally, the origin will be globally exponentially stable.
Example:

\[
\dot{x} = -[1 + g(t)]x^3, \quad g(t) \geq 0, \quad \forall \ t \geq 0
\]

\[
V(x) = \frac{1}{2}x^2
\]

\[
\dot{V}(t, x) = -[1 + g(t)]x^4 \leq -x^4, \quad \forall \ x \in \mathbb{R}, \ \forall \ t \geq 0
\]

The origin is globally uniformly asymptotically stable

Example:

\[
\begin{align*}
\dot{x}_1 & = -x_1 - g(t)x_2 \\
\dot{x}_2 & = x_1 - x_2
\end{align*}
\]

\[
0 \leq g(t) \leq k \quad \text{and} \quad \dot{g}(t) \leq g(t), \quad \forall \ t \geq 0
\]
\[ V(t, x) = x_1^2 + [1 + g(t)]x_2^2 \]

\[ x_1^2 + x_2^2 \leq V(t, x) \leq x_1^2 + (1 + k)x_2^2, \quad \forall x \in \mathbb{R}^2 \]

\[ \dot{V}(t, x) = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2 \]

\[ 2 + 2g(t) - \dot{g}(t) \geq 2 + 2g(t) - g(t) \geq 2 \]

\[ \dot{V}(t, x) \leq -2x_1^2 + 2x_1x_2 - 2x_2^2 = -x^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x \]

The origin is globally exponentially stable
Perturbed Systems

Nominal System: \( \dot{x} = f(x), \quad f(0) = 0 \)

Perturbed System: \( \dot{x} = f(x) + g(t, x), \quad g(t, 0) = 0 \)

Case 1: The origin of the nominal system is exponentially stable

\[
c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2, \quad \| \frac{\partial V}{\partial x} \| \leq c_4 \|x\| \]

\[
\frac{\partial V}{\partial x} f(x) \leq -c_3 \|x\|^2
\]
Use $V(x)$ as a Lyapunov function candidate for the perturbed system

$$\dot{V}(t, x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t, x)$$

Assume that

$$\|g(t, x)\| \leq \gamma \|x\|, \quad \gamma \geq 0$$

$$\dot{V}(t, x) \leq -c_3 \|x\|^2 + \left\| \frac{\partial V}{\partial x} \right\| \|g(t, x)\|$$

$$\leq -c_3 \|x\|^2 + c_4 \gamma \|x\|^2$$
\[ \gamma < \frac{c_3}{c_4} \]

\[ \dot{V}(t, x) \leq -(c_3 - \gamma c_4) \|x\|^2 \]

The origin is an exponentially stable equilibrium point of the perturbed system.
Example

\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -4x_1 - 2x_2 + \beta x_2^3, \quad \beta \geq 0 \\
\dot{x} &= Ax + g(x)
\end{align*}

\[A = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ \beta x_2^3 \end{bmatrix}\]

The eigenvalues of $A$ are $-1 \pm j\sqrt{3}$

\[PA + A^TP = -I \quad \Rightarrow \quad P = \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix}\]
\[ V(x) = x^T P x, \quad \frac{\partial V}{\partial x} A x = -x^T x \]

\[ c_3 = 1, \quad c_4 = 2 \| P \| = 2\lambda_{\text{max}}(P) = 2 \times 1.513 = 3.026 \]

\[ \| g(x) \| = \beta |x_2|^3 \]

\( g(x) \) satisfies the bound \( \| g(x) \| \leq \gamma \| x \| \) over compact sets of \( x \). Consider the compact set

\[ \Omega_c = \{ V(x) \leq c \} = \{ x^T P x \leq c \}, \quad c > 0 \]

\[ k_2 = \max_{x^T P x \leq c} |x_2| \Rightarrow |x_2^3| \leq k_2^2 |x_2| \]
\[ k_2 = \max_{x^T P x \leq c} |[0 1] x| = \max_{y^T y \leq c} \sqrt{c} |[0 1] P^{-1/2} y| \]
\[ = \sqrt{c} ||[0 1] P^{-1/2}|| = 1.8194\sqrt{c} \]
\[ \|g(x)\| \leq \beta \ c \ (1.8194)^2 \|x\|, \ \forall \ x \in \Omega_c \]
\[ \|g(x)\| \leq \gamma \|x\|, \ \forall \ x \in \Omega_c, \ \gamma = \beta \ c \ (1.8194)^2 \]
\[ \gamma < \frac{c_3}{c_4} \ \Leftrightarrow \ \beta < \frac{1}{3.026 \times (1.8194)^2 c} \approx \frac{0.1}{c} \]
\[ \beta < 0.1/c \ \Rightarrow \ \dot{V}(x) \leq -(1 - 10\beta c)\|x\|^2 \]

Hence, the origin is exponentially stable and \( \Omega_c \) is an estimate of the region of attraction
Alternative Bound on $\beta$

\[
\dot{V}(x) = -\|x\|^2 + 2x^T P g(x) \\
\leq -\|x\|^2 + \frac{1}{8}\beta x_2^3 ([2 \ 5] x) \\
\leq -\|x\|^2 + \frac{\sqrt{29}}{8}\beta x_2^2 \|x\|^2
\]

Over $\Omega_c$, $x_2^2 \leq (1.8194)^2 c$

\[
\dot{V}(x) \leq -\left(1 - \frac{\sqrt{29}}{8}\beta (1.8194)^2 c\right) \|x\|^2 = \left(1 - \frac{\beta c}{0.448}\right) \|x\|^2
\]

If $\beta < 0.448/c$, the origin will be exponentially stable and $\Omega_c$ will be an estimate of the region of attraction
Remark: The inequality $\beta < \frac{0.448}{c}$ shows a tradeoff between the estimate of the region of attraction and the estimate of the upper bound on $\beta$
Application to Linearization

\[ \dot{x} = f(x) = [A + G(x)]x \]

\[ A = \frac{\partial f}{\partial x}(x) \bigg|_{x=0}, \quad G(x) \to 0 \text{ as } x \to 0 \]

Theorem: The origin of \( \dot{x} = f(x) \) is exponentially stable if and only if \( A \) is Hurwitz
Proof Of sufficiency: Suppose $A$ is Hurwitz. Choose $Q = Q^T > 0$ and solve the Lyapunov equation $PA + A^TP = -Q$ for $P$

Use $V(x) = x^TPx$ as a Lyapunov function candidate for $\dot{x} = f(x)$

\[
\dot{V}(x) = x^TPf(x) + f^T(x)Px \\
= x^TP[A + G(x)]x + x^T[A^T + G^T(x)]Px \\
= x^T(PA + A^TP)x + 2x^TPG(x)x \\
= -x^TQx + 2x^TPG(x)x
\]
\[ \dot{V}(x) \leq -x^T Q x + 2\|P\| \|G(x)\| \|x\|^2 \]

For any \( \gamma > 0 \), there exists \( r > 0 \) such that

\[ \|G(x)\| < \gamma, \quad \forall \|x\| < r \]

\[ x^T Q x \geq \lambda_{\text{min}}(Q)\|x\|^2 \iff -x^T Q x \leq -\lambda_{\text{min}}(Q)\|x\|^2 \]

\[ \dot{V}(x) < -[\lambda_{\text{min}}(Q) - 2\gamma\|P\|]\|x\|^2, \quad \forall \|x\| < r \]

Choose

\[ \gamma < \frac{\lambda_{\text{min}}(Q)}{2\|P\|} \]

The origin of \( \dot{x} = f(x) \) is exponentially stable
Proof of Necessity: Suppose the origin of $\dot{x} = f(x)$ is exponentially stable. View the system

$$\dot{x} = Ax = f(x) - G(x)x$$

as a perturbation of $\dot{x} = f(x)$. Recall that

$$\|G(x)\| < \gamma, \ \forall \|x\| < r$$

Because the origin of $\dot{x} = f(x)$ is exponentially stable, let $V(x)$ be the function provided by the converse Lyapunov theorem over a domain $\{\|x\| < r_0\}$. Use $V(x)$ as a Lyapunov function candidate for $\dot{x} = Ax$. 
In the domain \( \{ \| x \| < \min\{r_0, r\}\} \), we have

\[
\frac{\partial V}{\partial x} Ax = \frac{\partial V}{\partial x} f(x) - \frac{\partial V}{\partial x} G(x)x
\leq -c_3\|x\|^2 + c_4 \gamma \|x\|^2
= -(c_3 - c_4 \gamma)\|x\|^2
\]

Take \( \gamma < c_3/c_4 \), and set \( \lambda = (c_3 - c_4 L) > 0 \) \( \Rightarrow \)

\[
\frac{\partial V}{\partial x} Ax \leq -\lambda\|x\|^2, \quad \forall \|x\| < \min\{r_0, r\}
\]

The origin of \( \dot{x} = Ax \) is exponentially stable
Case 2: The origin of the nominal system is asymptotically stable

\[ \dot{V}(t, x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t, x) \leq -W_3(x) + \left\| \frac{\partial V}{\partial x} g(t, x) \right\| \]

Under what condition will the following inequality hold?

\[ \left\| \frac{\partial V}{\partial x} g(t, x) \right\| < W_3(x) \]

Special Case: Quadratic-Type Lyapunov function

\[ \frac{\partial V}{\partial x} f(x) \leq -c_3 \phi^2(x), \quad \left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \phi(x) \]
\[ \dot{V}(t, x) \leq -c_3 \phi^2(x) + c_4 \phi(x) \| g(t, x) \| \]

If \( \| g(t, x) \| \leq \gamma \phi(x) \), with \( \gamma < \frac{c_3}{c_4} \)

\[ \dot{V}(t, x) \leq -(c_3 - c_4 \gamma) \phi^2(x) \]
Example

\[ \dot{x} = -x^3 + g(t, x) \]

\( V(x) = x^4 \) is a quadratic-type Lyapunov function for the nominal system \( \dot{x} = -x^3 \)

\[ \frac{\partial V}{\partial x}(-x^3) = -4x^6, \quad \left| \frac{\partial V}{\partial x} \right| = 4|x|^3 \]

\( \phi(x) = |x|^3, \quad c_3 = 4, \quad c_4 = 4 \)

Suppose \( |g(t, x)| \leq \gamma |x|^3, \quad \forall \ x, \quad \text{with} \ \gamma < 1 \)

\[ \dot{V}(t, x) \leq -4(1 - \gamma)\phi^2(x) \]

Hence, the origin is a globally uniformly asymptotically stable
Remark: A nominal system with asymptotically, but not exponentially, stable origin is not robust to smooth perturbations with arbitrarily small linear growth bounds.

Example

\[ \dot{x} = -x^3 + \gamma x \]

The origin is unstable for any \( \gamma > 0 \)
**Ultimate Boundedness**

**Definition:** The solutions of $\dot{x} = f(t, x)$ are

- uniformly bounded if $\exists c > 0$ and for every $0 < a < c$, $\exists \beta = \beta(a) > 0$ such that
  $$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta, \quad \forall t \geq t_0 \geq 0$$

- uniformly ultimately bounded with ultimate bound $b$ if $\exists b$ and $c$ and for every $0 < a < c$, $\exists T = T(a, b) \geq 0$ such that
  $$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b, \quad \forall t \geq t_0 + T$$

  “Globally” if $a$ can be arbitrarily large

  Drop “uniformly” if $\dot{x} = f(x)$
**Lyapunov Analysis:** Let $V(x)$ be a cont. diff. positive definite function and suppose that the sets

$$
\Omega_c = \{ V(x) \leq c \}, \quad \Omega_\varepsilon = \{ V(x) \leq \varepsilon \}, \quad \Lambda = \{ \varepsilon \leq V(x) \leq c \}
$$

are compact for some $c > \varepsilon > 0$
Suppose

\[ \dot{V}(t, x) = \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \quad \forall x \in \Lambda, \forall t \geq 0 \]

\( W_3(x) \) is continuous and positive definite

\( \Omega_c \) and \( \Omega_\varepsilon \) are positively invariant

\[ k = \min_{x \in \Lambda} W_3(x) > 0 \]

\[ \dot{V}(t, x) \leq -k, \quad \forall x \in \Lambda, \forall t \geq t_0 \geq 0 \]

\[ V(x(t)) \leq V(x(t_0)) - k(t - t_0) \leq c - k(t - t_0) \]

\( x(t) \) enters the set \( \Omega_\varepsilon \) within the interval \([t_0, t_0 + (c - \varepsilon)/k]\)
Suppose

\[ \dot{V}(t, x) \leq -W_3(x), \quad \forall \mu \leq \|x\| \leq r, \quad \forall t \geq 0 \]

Choose \( c \) and \( \varepsilon \) such that \( \Lambda \subset \{ \mu \leq \|x\| \leq r \} \)
Let $\alpha_1$ and $\alpha_2$ be class $\mathcal{K}$ functions such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$V(x) \leq c \Rightarrow \alpha_1(\|x\|) \leq c \iff \|x\| \leq \alpha_1^{-1}(c)$$

$$c = \alpha_1(r) \Rightarrow \Omega_c \subset B_r$$

$$\|x\| \leq \mu \Rightarrow V(x) \leq \alpha_2(\mu)$$

$$\varepsilon = \alpha_2(\mu) \Rightarrow B_{\mu} \subset \Omega_\varepsilon$$

What is the ultimate bound?

$$V(x) \leq \varepsilon \Rightarrow \alpha_1(\|x\|) \leq \varepsilon \iff \|x\| \leq \alpha_1^{-1}(\varepsilon) = \alpha_1^{-1}(\alpha_2(\mu))$$
Theorem (special case of Thm 4.18): Suppose

\[ \alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \]

\[ \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \ \forall \|x\| \geq \mu > 0 \]

\[ \forall t \geq 0 \text{ and } \|x\| \leq r, \text{ where } \alpha_1, \alpha_2 \in \mathcal{K}, W_3(x) \text{ is continuous \& positive definite, and } \mu < \alpha_2^{-1}(\alpha_1(r)). \]

Then, for every initial state \( x(t_0) \in \{\|x\| \leq \alpha_2^{-1}(\alpha_1(r))\} \), there is \( T \geq 0 \) (dependent on \( x(t_0) \) and \( \mu \)) such that

\[ \|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\mu)), \ \forall t \geq t_0 + T \]

If the assumptions hold globally and \( \alpha_1 \in \mathcal{K}_\infty \), then the conclusion holds for any initial state \( x(t_0) \)
Remarks:

- The ultimate bound is independent of the initial state.
- The ultimate bound is a class $\mathcal{K}$ function of $\mu$; hence, the smaller the value of $\mu$, the smaller the ultimate bound. As $\mu \to 0$, the ultimate bound approaches zero.
Example

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -(1 + x_1^2)x_1 - x_2 + M \cos \omega t, \quad M \geq 0 \]

With \( M = 0 \), \( \dot{x}_2 = -(1 + x_1^2)x_1 - x_2 = -h(x_1) - x_2 \)

\[ V(x) = x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + 2 \int_0^{x_1} (y + y^3) \, dy \quad \text{(Example 4.5)} \]

\[ V(x) = x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \frac{1}{2} x_1^4 \overset{\text{def}}{=} x^T \mathbf{P} x + \frac{1}{2} x_1^4 \]
\[
\lambda_{\text{min}}(P)\|x\|^2 \leq V(x) \leq \lambda_{\text{max}}(P)\|x\|^2 + \frac{1}{2}\|x\|^4
\]

\[
\alpha_1(r) = \lambda_{\text{min}}(P)r^2, \quad \alpha_2(r) = \lambda_{\text{max}}(P)r^2 + \frac{1}{2}r^4
\]

\[
\dot{V} = -x_1^2 - x_4^4 - x_2^2 + (x_1 + 2x_2)M\cos\omega t
\]
\[
\leq -\|x\|^2 - x_1^4 + M\sqrt{5}\|x\|
\]
\[
= -(1 - \theta)\|x\|^2 - x_1^4 - \theta\|x\|^2 + M\sqrt{5}\|x\|
\]
\[
(0 < \theta < 1)
\]
\[
\leq -(1 - \theta)\|x\|^2 - x_1^4, \quad \forall \|x\| \geq M\sqrt{5}/\theta \equiv \mu
\]

The solutions are GUUB by

\[
b = \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\frac{\lambda_{\text{max}}(P)\mu^2 + \mu^4/2}{\lambda_{\text{min}}(P)}}
\]
Perturbed Systems: Nonvanishing Perturbation

Nominal System:
\[ \dot{x} = f(x), \quad f(0) = 0 \]

Perturbed System:
\[ \dot{x} = f(x) + g(t, x), \quad g(t, 0) \neq 0 \]

Case 1: The origin of \( \dot{x} = f(x) \) is exponentially stable

\[ c_1 \| x \|^2 \leq V(x) \leq c_2 \| x \|^2 \]

\[ \frac{\partial V}{\partial x} f(x) \leq -c_3 \| x \|^2, \quad \left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \| x \| \]

\[ \forall x \in B_r = \{ \| x \| \leq r \} \]
Use $V(x)$ to investigate ultimate boundedness of the perturbed system

$$\dot{V}(t, x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t, x)$$

Assume

$$\|g(t, x)\| \leq \delta, \quad \forall t \geq 0, \ x \in B_r$$

$$\dot{V}(t, x) \leq -c_3 \|x\|^2 + \left\|\frac{\partial V}{\partial x}\right\| \|g(t, x)\|$$

$$\leq -c_3 \|x\|^2 + c_4 \delta \|x\|$$

$$= -(1 - \theta)c_3 \|x\|^2 - \theta c_3 \|x\|^2 + c_4 \delta \|x\|$$

$$0 < \theta < 1$$

$$\leq -(1 - \theta)c_3 \|x\|^2, \quad \forall \|x\| \geq \delta c_4 / (\theta c_3) \overset{\text{def}}{=} \mu$$
Apply Theorem 4.18

\[ \|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r)) \iff \|x(t_0)\| \leq r \sqrt{\frac{c_1}{c_2}} \]

\[ \mu < \alpha_2^{-1}(\alpha_1(r)) \iff \frac{\delta c_4}{\theta c_3} < r \sqrt{\frac{c_1}{c_2}} \iff \delta < \frac{c_3}{c_4} \sqrt{\frac{c_1}{c_2}} \theta r \]

\[ b = \alpha_1^{-1}(\alpha_2(\mu)) \iff b = \mu \sqrt{\frac{c_2}{c_1}} \iff b = \frac{\delta c_4}{\theta c_3} \sqrt{\frac{c_2}{c_1}} \]

For all \( \|x(t_0)\| \leq r \sqrt{c_1/c_2} \), the solutions of the perturbed system are ultimately bounded by \( b \)
Example

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -4x_1 - 2x_2 + \beta x_2^3 + d(t)
\end{align*}
\]

\[\beta \geq 0, \quad |d(t)| \leq \delta, \forall t \geq 0\]

\[
V(x) = x^T P x = x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix} x
\]

\[
\dot{V}(t, x) = -\|x\|^2 + 2\beta x_2^2 \left( \frac{1}{8} x_1 x_2 + \frac{5}{16} x_2^2 \right) \\
+ 2d(t) \left( \frac{1}{8} x_1 + \frac{5}{16} x_2 \right)
\leq -\|x\|^2 + \sqrt{29} \frac{\beta k_2^2}{8} \|x\|^2 + \sqrt{29} \delta \|x\|
\]
\[ k_2 = \max_{x^T P x \leq c} |x_2| = 1.8194\sqrt{c} \]

Suppose \( \beta \leq 8(1 - \zeta)/(\sqrt{29} k_2^2) \) \( (0 < \zeta < 1) \)

\[ \dot{V}(t, x) \leq -\zeta \|x\|^2 + \frac{\sqrt{29}\delta}{8} \|x\| \]
\[ \leq -(1 - \theta)\zeta \|x\|^2, \quad \forall \|x\| \geq \frac{\sqrt{29}\delta}{8 \zeta \theta} \overset{\text{def}}{=} \mu \]
\[ (0 < \theta < 1) \]

If \( \mu^2 \lambda_{\text{max}}(P) < c \), then all solutions of the perturbed system, starting in \( \Omega_c \), are uniformly ultimately bounded by

\[ b = \frac{\sqrt{29}\delta}{8 \zeta \theta} \sqrt{\frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)}} \]
Case 2: The origin of $\dot{x} = f(x)$ is asymptotically stable

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x}f(x) \leq -\alpha_3(\|x\|), \quad \left\| \frac{\partial V}{\partial x} \right\| \leq k$$

$$\forall \ x \in B_r = \{\|x\| \leq r\}, \quad \alpha_i \in \mathcal{K}, \ i = 1, 2, 3$$

$$\dot{V}(t, x) \leq -\alpha_3(\|x\|) + \left\| \frac{\partial V}{\partial x} \right\| \|g(t, x)\|$$

$$\leq -\alpha_3(\|x\|) + \delta k$$

$$\leq -(1 - \theta)\alpha_3(\|x\|) - \theta \alpha_3(\|x\|) + \delta k$$

$$0 < \theta < 1$$

$$\leq -(1 - \theta)\alpha_3(\|x\|), \quad \forall \ \|x\| \geq \alpha_3^{-1} \left( \frac{\delta k}{\theta} \right) \overset{\text{def}}{=} \mu$$
Apply Theorem 4.18

\[ \mu < \alpha^{-1}_2(\alpha_1(r)) \iff \alpha^{-1}_3 \left( \frac{\delta k}{\theta} \right) < \alpha^{-1}_2(\alpha_1(r)) \]

\[ \iff \delta < \frac{\theta \alpha_3(\alpha^{-1}_2(\alpha_1(r))))}{k} \]

Compare with \( \delta < \frac{c_3}{c_4} \sqrt{\frac{c_1}{c_2}} \theta r \)

Example

\[ \dot{x} = -\frac{x}{1 + x^2} \]

\[ V(x) = x^4 \quad \Rightarrow \quad \frac{\partial V}{\partial x} \left[ -\frac{x}{1 + x^2} \right] = -\frac{4x^4}{1 + x^2} \]

\[ \alpha_1(|x|) = \alpha_2(|x|) = |x|^4; \quad \alpha_3(|x|) = \frac{4|x|^4}{1 + |x|^2}; \quad k = 4r^3 \]
The origin is globally asymptotically stable

\[
\frac{\theta \alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{k} = \frac{\theta \alpha_3(r)}{k} = \frac{r \theta}{1 + r^2}
\]

\[
\frac{r \theta}{1 + r^2} \to 0 \text{ as } r \to \infty
\]

\[
\dot{x} = -\frac{x}{1 + x^2} + \delta, \quad \delta > 0
\]

\[
\delta > \frac{1}{2} \Rightarrow \lim_{t \to \infty} x(t) = \infty
\]
**Input-to-State Stability (ISS)**

**Definition:** The system \( \dot{x} = f(x, u) \) is input-to-state stable if there exist \( \beta \in \mathcal{KL} \) and \( \gamma \in \mathcal{K} \) such that for any initial state \( x(t_0) \) and any bounded input \( u(t) \)

\[
\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma \left( \sup_{t_0 \leq \tau \leq t} \|u(\tau)\| \right)
\]

ISS of \( \dot{x} = f(x, u) \) implies

- BIBS stability
- \( x(t) \) is ultimately bounded by a class \( \mathcal{K} \) function of \( \sup_{t \geq t_0} \|u(t)\| \)
- \( \lim_{t \to \infty} u(t) = 0 \Rightarrow \lim_{t \to \infty} x(t) = 0 \)
- The origin of \( \dot{x} = f(x, 0) \) is GAS
Theorem (Special case of Thm 4.19): Let $V(x)$ be a continuously differentiable function such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x} f(x, u) \leq -W_3(x), \quad \forall \|x\| \geq \rho(\|u\|) > 0$$

$\forall x \in \mathbb{R}^n, u \in \mathbb{R}^m$, where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{K}$, and $W_3(x)$ is a continuous positive definite function. Then, the system $\dot{x} = f(x, u)$ is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Proof: Let $\mu = \rho(\sup_{\tau \geq t_0} \|u(\tau)\|)$; then

$$\frac{\partial V}{\partial x} f(x, u) \leq -W_3(x), \quad \forall \|x\| \geq \mu$$
Choose $\varepsilon$ and $c$ such that

$$\frac{\partial V}{\partial x} f(x, u) \leq -W_3(x), \quad \forall x \in \Lambda = \{\varepsilon \leq V(x) \leq c\}$$

Suppose $x(t_0) \in \Lambda$ and $x(t)$ reaches $\Omega_\varepsilon$ at $t = t_0 + T$. For $t_0 \leq t \leq t_0 + T$, $V$ satisfies the conditions for the uniform asymptotic stability. Therefore, the trajectory behaves as if the origin was uniformly asymptotically stable and satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \text{for some } \beta \in \mathcal{KL}$$

For $t \geq t_0 + T$,

$$\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\mu))$$
\[ \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \alpha_1^{-1}(\alpha_2(\mu)), \quad \forall \, t \geq t_0 \]

\[ \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma \left( \sup_{\tau \geq t_0} \|u(\tau)\| \right), \quad \forall \, t \geq t_0 \]

Since \( x(t) \) depends only on \( u(\tau) \) for \( t_0 \leq \tau \leq t \), the supremum on the right-hand side can be taken over \([t_0, t]\).
Example

\[ \dot{x} = -x^3 + u \]

The origin of \( \dot{x} = -x^3 \) is globally asymptotically stable

\[ V = \frac{1}{2}x^2 \]

\[ \dot{V} = -x^4 + xu \]

\[ = -(1 - \theta)x^4 - \theta x^4 + xu \]

\[ \leq -(1 - \theta)x^4, \quad \forall |x| \geq \left( \frac{|u|}{\theta} \right)^{1/3} \]

\[ 0 < \theta < 1 \]

The system is ISS with

\[ \gamma(r) = (r/\theta)^{1/3} \]
Example

\[ \dot{x} = -x - 2x^3 + (1 + x^2)u^2 \]

The origin of \( \dot{x} = -x - 2x^3 \) is globally exponentially stable

\[ V = \frac{1}{2}x^2 \]

\[ \dot{V} = -x^2 - 2x^4 + x(1 + x^2)u^2 \]
\[ = x^4 - x^2(1 + x^2) + x(1 + x^2)u^2 \]
\[ \leq -x^4, \quad \forall |x| \geq u^2 \]

The system is ISS with \( \gamma(r) = r^2 \)
Example

\[ \dot{x}_1 = -x_1 + x_2^2, \quad \dot{x}_2 = -x_2 + u \]

Investigate GAS of \( \dot{x}_1 = -x_1 + x_2^2, \quad \dot{x}_2 = -x_2 \)

\[ V(x) = \frac{1}{2} x_1^2 + \frac{1}{4} x_2^4 \]

\[ \dot{V} = -x_1^2 + x_1 x_2^2 - x_2^4 = -(x_1 - \frac{1}{2} x_2^2)^2 - \left(1 - \frac{1}{4}\right) x_2^4 \]

Now \( u \neq 0 \), \( \dot{V} = -\frac{1}{2} (x_1 - x_2^2)^2 - \frac{1}{2} (x_1^2 + x_2^4) + x_2^3 u \)

\[ \leq -\frac{1}{2} (x_1^2 + x_2^4) + |x_2|^3 |u| \]

\[ \dot{V} \leq -\frac{1}{2} (1 - \theta) (x_1^2 + x_2^4) - \frac{1}{2} \theta (x_1^2 + x_2^4) + |x_2|^3 |u| \]

\[ (0 < \theta < 1) \]
\[-\frac{1}{2} \theta (x_1^2 + x_2^4) + |x_2|^3 |u| \leq 0\]

if \( |x_2| \geq \frac{2|u|}{\theta} \) or \( |x_2| \leq \frac{2|u|}{\theta} \) and \( |x_1| \geq \left( \frac{2|u|}{\theta} \right)^2 \)

if \( \|x\| \geq \frac{2|u|}{\theta} \sqrt{1 + \left( \frac{2|u|}{\theta} \right)^2} \)

\[
\rho(r) = \frac{2r}{\theta} \sqrt{1 + \left( \frac{2r}{\theta} \right)^2}
\]

\[
\dot{V} \leq -\frac{1}{2} (1 - \theta) (x_1^2 + x_2^4), \quad \forall \|x\| \geq \rho(\|u\|)
\]

The system is ISS