Feedback equivalence of nonlinear control systems: a survey on formal approach

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Abstract

This paper is a survey devoted to the formal approach to the feedback equivalence problem for nonlinear control systems. We show how classical Poincaré’s approach, developed for dynamical systems, generalizes to control systems for continuous and discrete-time. We present normal forms and canonical forms for nonlinear control systems (single-input and multi-input, control-affine and general). We use the formal approach to study symmetries of nonlinear control systems as well as discuss special forms: linear and feedforward. We illustrate presented forms by various examples in dimensions 3 and 4.
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1 Introduction

In this survey we will be dealing with nonlinear control systems of the form

$$\Pi : \dot{x} = F(x, u),$$

where $x \in X$, an open subset of $\mathbb{R}^n$ and $u \in U \subset \mathbb{R}^m$, and $F(x, u)$ is a family of vector fields, $C^\infty$-smooth with respect to $(x, u)$. The variables $x = (x_1, \ldots, x_n)^T$ represent the state of the system and the variables $u = (u_1, \ldots, u_m)^T$ represent the control, that is, an external influence on the system. $\Pi$ can be understood as an underdetermined system of ordinary differential equations: $n$ equations for $n + m$ variables.

We will be interested in equivalence problems for the system $\Pi$. Consider another system of the same form

$$\tilde{\Pi} : \dot{\tilde{x}} = \tilde{F}(\tilde{x}, \tilde{u}),$$

where $\tilde{x} \in \tilde{X}$, an open subset of $\mathbb{R}^n$ and $\tilde{u} \in \tilde{U} \subset \mathbb{R}^m$. A natural equivalence can be defined as follows. Assume that $U = \tilde{U}$. We say that $\Pi$ and $\tilde{\Pi}$ are state-space equivalent, shortly $S$-equivalent, if there exist a diffeomorphism

$$\tilde{x} = \phi(x)$$
$$\tilde{u} = u$$

transforming solutions into solutions. More precisely, if $(x(t), u(t))$ is a solution of $\Pi$, then $(\phi(x(t)), u(t))$ is a solution of $\tilde{\Pi}$, which is equivalent to

$$D\phi(x) \cdot F(x, u) = \tilde{F}(\phi(x), u),$$

for any $u \in U$, where $D\phi(x)$ denotes the derivative of $\phi$ at $x$. It means that the state-space equivalence establishes a diffeomorphic correspondence of the right hand sides of differential equations corresponding to the constant controls, which can be expressed as

$$(\phi_*F)(\tilde{x}, u) = \tilde{F}(\tilde{x}, u), \quad u \in U,$$

where for any vector field $f$ and any diffeomorphism $\tilde{x} = \phi(x)$, we denote $(\phi_*f)(\tilde{x}) = D\phi(\phi^{-1}(\tilde{x})) \cdot f(\phi^{-1}(\tilde{x}))$.

$S$-equivalence is well understood. It establishes a one-to-one smooth correspondence between the trajectories of equivalent systems (corresponding to the same measurable, not necessarily constant, controls). For accessible systems (see, e.g., [36] and [64] for definition) the set of complete invariants for the local $S$-equivalence is formed by all iterative Lie-brackets evaluated at a nominal point (in the analytic case) or in its neighborhood (in the smooth case), see, e.g., [38].

Since the system $\Pi$ has state and control variables, another natural transformation is to apply to $\Pi$ a diffeomorphism $Y = (\phi, \psi)^T$ of $X \times U$ onto $\tilde{X} \times \tilde{U}$ that changes both $x$ and $u$, that is,

$$\tilde{x} = \phi(x, u)$$
$$\tilde{u} = \psi(x, u)$$

and transforms the solutions of $\Pi$ into those of $\tilde{\Pi}$. Taking a $C^1$-solution $(x(t), u(t))$ of $\Pi$ and using the fact that its image $(\phi(x(t), u(t)), \psi(x(t), u(t)))$ is assumed to be a solution of $\tilde{\Pi}$, we
conclude that $\frac{\partial \phi}{\partial x} F(x, u) + \frac{\partial \phi}{\partial u} \dot{u} = \tilde{F}(\phi(x, u), \psi(x, u))$. Now it is easy to see that, since $F$ and $\tilde{F}$ do not depend, respectively, on $\dot{u}$ and $\dot{\tilde{u}}$, the map $\phi$ cannot depend on $u$. This implies that any $\Upsilon$ preserving the system solutions must actually be a triangular diffeomorphism

$$\Upsilon: \begin{array}{l}
\dot{x} = \phi(x) \\
\dot{u} = \psi(x, u),
\end{array}$$

satisfying

$$D\phi(x) \cdot F(x, u) = \tilde{F}(\phi(x), \psi(x, u)),$$

which is called a feedback transformation. Systems $\Pi$ and $\tilde{\Pi}$, equivalent via $\Upsilon$, are called feedback equivalent, shortly $F$-equivalent. The states $x$ and $\tilde{x}$ of two feedback equivalent systems are thus related by a diffeomorphism $\phi$ between the corresponding state spaces $X$ and $\tilde{X}$ while the controls $u$ and $\tilde{u}$ are related by a diffeomorphism $\psi$ between $U$ and $\tilde{U}$ which depends on the state $x$. We will call $\Pi$ and $\tilde{\Pi}$ locally feedback equivalent at $(x_0, u_0)$ and $(\tilde{x}_0, \tilde{u}_0)$, respectively, if $(\phi, \psi)$ is a local diffeomorphism satisfying $(\phi, \psi)(x_0, u_0) = (\tilde{x}_0, \tilde{u}_0)$.) It is the feedback equivalence and its local counterpart, which are the main topic of our survey.

Notice that the diffeomorphism $\phi$ establishes a one-to-one correspondence of $x$-trajectories of two feedback equivalent systems although equivalent trajectories are differently parameterized by controls. Indeed, a trajectory $x(t)$ of the first system corresponding to a control $u(t)$ is mapped into the curve $\phi(x(t))$, which is the trajectory of $\tilde{\Pi}$ corresponding to $\tilde{u}(t) = \psi(x(t), u(t))$. Based on this observation, one can define a weaker notion of equivalence of $\Pi$ and $\tilde{\Pi}$ asking that there exists a one-to-one correspondence between trajectories (corresponding to, say, $C^\infty$-controls) and omitting the assumption that the correspondence is given by a diffeomorphism. This leads to the important notion of dynamic feedback equivalence (see, e.g., [17], [18], [39], [66]), which we, however, will not study in this survey.

The main subject of this paper is feedback equivalence, which has been extensively studied during the last twenty years. Although being natural, this problem is very involved (mainly because of functional parameters appearing in the classification that we will describe briefly in a moment). Many existing results are devoted to systems that are affine with respect to controls, that is, are of the form

$$\Sigma: \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + g(x)u,$$

where $x \in X$, $f$ and $g_i$ are $C^\infty$-smooth control vector fields on $X$, $u = (u_1, \ldots, u_m)^T \in U = \mathbb{R}^m$ and $g = (g_1, \ldots, g_m)$. When studying feedback equivalence of control-affine systems, we will apply feedback transformations that are affine with respect to controls:

$$\Gamma: \begin{array}{l}
\dot{x} = \phi(x) \\
u = \alpha(x) + \beta(x)\tilde{u},
\end{array}$$

where $u = \psi^{-1}(x, \tilde{u}) = \alpha(x) + \beta(x)\tilde{u}$, with $\alpha$ and $\beta$ being $C^\infty$-smooth functions with values in $\mathbb{R}^m$ and $GL(m, \mathbb{R})$, respectively. Consider another control-affine system

$$\tilde{\Sigma}: \dot{x} = \tilde{f}(\tilde{x}) + \sum_{i=1}^{m} \tilde{g}_i(x)\tilde{u}_i = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\tilde{u},$$

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where \( \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_m)^T \in \tilde{U} = \mathbb{R}^m \) and \( \tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_m) \).

The general definition implies that the control-affine systems \( \Sigma \) and \( \tilde{\Sigma} \) are feedback equivalent if and only if

\[
\phi_\ast (f + g\alpha) = \tilde{f} \quad \text{and} \quad \phi_\ast (g\beta) = \tilde{g},
\]

which we will write shortly

\[
\Gamma_\ast (\Sigma) = \tilde{\Sigma}.
\]

We will say that the control-affine systems \( \Sigma \) and \( \tilde{\Sigma} \) are locally feedback equivalent at \( x_0 \) and \( \tilde{x}_0 \), respectively, if \( \phi \) is a local diffeomorphism satisfying \( \phi(x_0) = \tilde{x}_0 \) and \( \alpha \) and \( \beta \) are defined locally around \( x_0 \). Notice that local feedback equivalence is local in the state-space \( X \) but global in the control space \( U = \mathbb{R}^m \).

Given two control-affine systems \( \Sigma \) and \( \tilde{\Sigma} \), the problem of their (local) feedback equivalence amounts thus to solving the system of 1st-order partial differential equations

\[
(CDE) \quad \frac{\partial \phi}{\partial x}(x)(f(x) + g(x)\alpha(x)) = \tilde{f}(\phi(x)) \quad \frac{\partial \phi}{\partial x}(x)(g(x)\beta(x)) = \tilde{g}(\phi(x)).
\]

It turns out that feedback equivalence of general systems \( \Pi \) under \( \Upsilon \) and of control-affine systems \( \Sigma \) under \( \Gamma \) are very closely related. Consider a general nonlinear control system

\[
\Pi: \quad \dot{x} = F(x, u),
\]

where \( x \in X \), an open subset of \( \mathbb{R}^n \), \( u \in U \), an open subset of \( \mathbb{R}^m \). Together with \( \Pi \), we consider its extension (preintegration)

\[
\Sigma^e: \quad \dot{x}^e = f^e(x^e) + g^e(x^e)u^e,
\]

where \( x^e = (x, u) \in X^e = X \times U \), \( u^e \in U^e = \mathbb{R}^m \), and the dynamics are given by

\[
\dot{x} = F(x, u) \quad \dot{u} = u^e,
\]

that is \( f^e(x^e) = (F(x, u), 0)^T \) and \( g^e(x^e) = (0, Id)^T \). Notice that \( \Sigma^e \) is a control-affine system controlled by the derivatives \( \dot{u}_i = u_i^e \) of the original controls \( u_i \), for \( 1 \leq i \leq m \).

**Proposition 1.1** Two control systems \( \Pi \) and \( \tilde{\Pi} \) are equivalent (resp. locally equivalent at \( (x_0, u_0) \) and \( (\tilde{x}_0, \tilde{u}_0) \)) under a general feedback transformation \( \Upsilon \) if and only if their respective extensions \( \Sigma^e \) and \( \tilde{\Sigma}^e \) are equivalent (resp. locally equivalent at \( x_0^e = (x_0, u_0) \) and \( \tilde{x}_0^e = (\tilde{x}_0, \tilde{u}_0) \)) under an affine feedback \( \Gamma \).

As a consequence, many problems concerning feedback equivalence are studied and solved for control-affine systems and their extension to the general case can be done by an appropriate application of Proposition 1.1.

In order to geometrize the problem of feedback equivalence, we associate to the system \( \Pi \) its field of admissible velocities

\[
\mathcal{F}(x) = \{F(x, u) : u \in U\} \subset T_xX.
\]
The field of admissible velocities of the control-affine system $\Sigma$ is the following field of affine subspaces (equivalently, an affine distribution):

$$A(x) = \{ f(x) + \sum_{i=1}^{m} g_i(x) u_i : u_i \in \mathbb{R} \} = f(x) + \mathcal{G}(x) \subset T_x X,$$

where $\mathcal{G}$ denotes the distribution spanned by the vector fields $g_1, \ldots, g_m$. Now it is easy to see that if two control affine-systems $\Sigma$ and $\tilde{\Sigma}$ are feedback equivalent, then the corresponding affine distributions are equivalent, that is,

$$\phi_* A = \tilde{A}.$$

Moreover, the converse holds if the distributions $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are of constant ranks. Analogous implications (the converse under the constant rank assumption) are true for local feedback equivalence. Notice that attaching the field of admissible velocities to an affine system results in eliminating controls from the description: what remains is a geometric object, which is the affine distribution $A$ while the choice of controls (equivalently, the choice of sections of $A$) becomes irrelevant.

**Example 1.2** To illustrate the notion of feedback equivalence, let us recall the first (historically) studied feedback classification problem, which is that for linear control systems of the form

$$\Lambda : \dot{x} = Ax + Bu = Ax + \sum_{i=1}^{m} u_i b_i,$$

where $x \in \mathbb{R}^n$, $Ax$ and $b_1, \ldots, b_m$ are, respectively, linear and constant vector fields on $\mathbb{R}^n$, and $u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m$. In order to preserve the linear form of the system, we apply to it the linear feedback transformation.

$$\tilde{x} = Tx$$

$$u = Kx + Lu,$$

where $T$, $K$, and $L$ are matrices of appropriate sizes, $T$ and $L$ being invertible. The system $\Lambda$ is transformed into

$$\tilde{\Lambda} : \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u} = T(A + BK)T^{-1}\tilde{x} + TBLu.$$

It is a classical result of the linear control theory (see, e.g., [47]) that any linear controllable system is feedback equivalent to the following system (called Brunovsky canonical form):

$$\begin{align*}
\dot{\tilde{x}}_{i,j} &= \tilde{x}_{i,j+1} & 1 \leq j \leq \rho_i - 1 \\
\dot{\tilde{x}}_{i,\rho_i} &= u_i & 1 \leq i \leq m,
\end{align*}$$

where $\tilde{x} = (\tilde{x}_{1,1}, \tilde{x}_{1,2}, \ldots, \tilde{x}_{1,\rho_1}, \ldots, \tilde{x}_{m,\rho_m})^T$.

The integers $\rho_1 \geq \cdots \geq \rho_m$, $\sum_{i=1}^{m} \rho_i = n$ (called controllability, Brunovsky or Kronecker indices) form a set of complete feedback invariants of the linear feedback linear group action on controllable systems and are defined as follows

$$\rho_i = \max\{ r_j \mid r_j \geq i \}, \quad (1.1)$$

where $d_0 = 0$, $d_i = \text{rank}(B, \ldots, A^{i-1}B)$ and $r_i = d_i - d_{i-1}$ for $1 \leq i \leq n$.

Notice that for the linear control system $\Lambda$, the field of admissible velocities is given by the field of $m$-dimensional affine subspaces $\mathcal{A}(x) = Ax + B$ of $\mathbb{R}^n$, where $B$ is the image of $\mathbb{R}^m$ under the linear map $B : \mathbb{R}^m \to \mathbb{R}^n$. The Brunovsky canonical form thus gives a canonical form for the field $\mathcal{A}$ under linear invertible transformations $\tilde{x} = Tx$. □
Observe that the dimension of the space of linear systems (of pairs \((A, B)\)) is \(n^2 + nm\) and the dimension of the group of linear feedback (of the triples \((T, K, L)\)) is \(n^2 + nm + m^2\). We can thus expect open orbits to exist and, indeed, open orbits exist (those of systems with the maximal vector of \(d_i\)'s).

The picture gets completely different for nonlinear control systems under the action of nonlinear feedback. Although both are infinite dimensional, the group of (local) feedback transformations is much "smaller" than the space of all (local) control systems and, as a consequence, functional parameters must necessarily appear in the feedback classification. To see this, notice that the space of general systems \(\Pi\) is parameterized by \(n\) components of the map \(\psi(x, u)\), each being a function of \(m\) variables and \(n\) components of \(\phi(x)\), each being a function of \(n\) variables. Notice that \((k + 1)\)-jet of a diffeomorphism act on the \(k\)-jet of the system. Thus

\[
d_U(k) = n \frac{(k + n + m)!}{k!(n + m)!}, \quad d_\Gamma(k) = n \frac{(k + 1 + n)!}{(k + 1)!n!} + m \frac{(k + n + m)!}{k!(n + m)!}.
\]

The codimension of any orbit, of the feedback group action on the space of systems, is bounded from below by the difference \(d_U(k) - d_\Gamma(k)\), which is a polynomial of \(k\) of degree \(n + m\), whose coefficient multiplying \(k^{n+m}\) is

\[
\frac{n - m}{(n + m)!}.
\]

This coefficient is positive when \(m < n\) and thus the polynomial and the codimension of any orbit tend to infinity when \(k\) tends to infinity. As a consequence, functional moduli must appear in the feedback classification if \(m < n\) (which exhausts all interesting cases).

Observe that a control-affine system is defined by \(m + 1\) vector fields \(f, g_1, \ldots, g_m\) and the feedback group by the diffeomorphism \(\phi\) and \(m + m^2\) components of the pair \((\alpha, \beta)\). Therefore in the case of control-affine systems the corresponding dimensions are:

\[
d_\Sigma = n(m + 1) \frac{(k + n)!}{k!(n)!}, \quad d_\Gamma = n \frac{(k + 1 + n)!}{(k + 1)!(n)!} + m(m + 1) \frac{(k + n)!}{k!(n)!}.
\]

The codimension of any orbit in the \(k\)-jets space is bounded below by the difference \(d_\Sigma - d_\Gamma\), which is a polynomial of \(k\) of degree \(n\), whose coefficient multiplying \(k^n\) is

\[
\frac{m(n - m - 1)}{n}.
\]
If \( m < n - 1 \), then this coefficient is positive and thus the polynomial and the codimension of any orbit under the feedback group action tend to infinity as \( k \) tends to infinity. As a consequence, functional moduli must appear in feedback classification of control-affine systems if \( m < n - 1 \). In the case \( m = n - 1 \) we can hope, however, for normal forms without functional parameters and, indeed, such normal forms have been obtained by Respondek and Zhitomirskii for \( m = 2, n = 3 \) in [73] and in the general case in [94].

It is the existence of functional moduli, which causes one of the main difficulties of the feedback equivalence problem. Four basic methods have been proposed to study various aspects of feedback equivalence. The first method, used for control-affine systems, is based on studying invariant properties of two geometric objects attached to the system: the distribution \( \mathcal{G} \) and the affine distribution \( \mathcal{A} \). Notice that feedback equivalence of control-linear systems (that is control-affine system \( \Sigma \) with \( f \equiv 0 \)) coincides with equivalence, under a diffeomorphism, of the corresponding distributions \( \mathcal{G} \) and \( \mathcal{G} \). So this approach is linked, in a natural way, with the classification and with singularities of vector fields and distributions, and their invariants. Using that method a large variety of feedback classification problems have been solved, see e.g. [7], [12], [35], [38], [43], [44], [54], [67], [73], [94].

The second approach, proposed by Gardner [19], uses Cartan’s method of equivalence [11]. To the control system \( \Pi \), we can associate the Pfaffian system given by the differential forms \( dx_i - F_i(x, u) dt \), for \( 1 \leq i \leq n \), on \( X \times U \times \mathbb{R} \) and the feedback equivalence of \( \Pi \) and \( \tilde{\Pi} \) is analyzed by studying the equivalence of the corresponding Pfaffian systems and their geometry, see [23], [21], [22], [60].

The third method, inspired by the hamiltonian formalism for optimal control problems, has been developed by Bonnard and Jakubczyk [6], [7], [42], [40] and has led to a very nice description of feedback invariants in terms of singular extremals. Another approach based also on the hamiltonian formalism for optimal control has been proposed by Agrachev [?] and has led to a construction of a fundamental geometric invariant of feedback equivalence: the curvature of control systems.

Finally, a very fruitful approach was proposed by Kang and Krener [52] and then followed by Kang [48], [49]. Their idea, which is closely related with classical Poincaré’s technique for linearization of dynamical systems (see e.g. [1]), is to analyze the system \( \Pi \) and the feedback transformation \( \Upsilon \) (the system \( \Sigma \) and the transformation \( \Gamma \), respectively, in the control-affine case) step by step and, as a consequence, to produce a simpler equivalent system \( \tilde{\Pi} \) also step by step. It is this approach, and various classification results obtained using it, which form the subject of this survey.

This survey is organized as follows. We will present in Section 2 classical Poincaré’s approach to the problem of formal equivalence of dynamical systems. In Section 3 we will generalize, following Kang and Krener, that formal approach to nonlinear control systems. We will present a normal form for homogeneous systems, their invariants, explicit normalizing transformations and, finally, a normal form under a formal feedback. We will also extend the normal form to general (non-affine systems). Then, in Section 4, we will propose a canonical form for nonlinear control systems. In the two following Sections 5 and 6 we will dualize results of preceding sections and give a dual normal form (together with dual invariants and explicit normalizing transformations) and a dual canonical form. Then in Section 7 we will pass to systems with uncontrollable linearization, introduce weighted homogeneity and we will give: a normal form, invariants, explicit normalizing transformations, and a formal normal form. This section gen-
eralizes, from the one hand side, results for systems with controllable linearization (presented in earlier sections) and, on the other hand side, results on dynamical systems from Section 2. Section 8 will be devoted to multi-input normal forms (for space related reasons we just treat the controllable case): it generalizes results on normal forms obtained in Section 3. A discrete-time version of Section 3 will be given in Section 10. In Section 9 we compare well known results devoted to feedback linearization with their counterpart obtained via the formal feedback. We will also discuss systems that are feedback equivalent to linear uncontrollable systems. Then the two following Sections present applications of the formal approach to the classification of control systems. We discuss symmetries of control systems in Section 11 and show an enormous difference between the group symmetries of feedback linearizable and nonlinearizable systems. In Section 12 we characterize, using the formal approach, systems that are feedback equivalent to feedforward and strict feedforward forms. Finally, in Section 13 we give a class of analytic strict feedforward forms that can be transformed to a normal form via constructive analytic transformations.

Because of space limits, this survey does not touch many important results. To mention just a few: we do not discuss analysis of bifurcations based on formal approach (e.g., [50], [51], [57], [58]), bifurcations of discrete-time systems, normal forms for observed dynamics. Each of those subjects requires its own survey proving efficiency of the formal approach.

2 Equivalence of dynamical systems: Poincaré theorem

In this section we will summarize very briefly Poincaré’s approach to the problem of (formal) equivalence of dynamical systems. The goal of this section is three-folds. Firstly, to make our survey complete and self-contained. Secondly, to show how the formal approach to the equivalence of dynamical systems generalizes to the formal approach to feedback equivalence of control systems. Thirdly, some of results on formal normal forms for dynamical systems and of formal linearization (Theorems 2.3 and 2.4 stated at the end of this section) will be used in Sections 7 and 9 of the survey.

Consider the uncontrolled dynamical system

\[ \dot{x} = f(x), \]

where \( x \in X \), an open subset of \( \mathbb{R}^n \) and \( f \) is a \( C^\infty \)-smooth vector field on \( X \). A \( C^\infty \)-smooth diffeomorphism

\[ \tilde{x} = \phi(x) \]

brings the considered dynamical system into

\[ \dot{\tilde{x}} = \tilde{f}(\tilde{x}) = (\phi_* f)(\tilde{x}), \]

where

\[ (\phi_* f)(\tilde{x}) = \frac{\partial \phi}{\partial x}(\phi^{-1}(\tilde{x})) \cdot f(\phi^{-1}(\tilde{x})). \]

Now given two dynamical systems \( \dot{x} = f(x) \) and \( \dot{\tilde{x}} = \tilde{f}(\tilde{x}) \), the problem of establishing their equivalence is to find a diffeomorphism \( \tilde{x} = \phi(x) \) satisfying

\[ (DE) \quad \frac{\partial \phi}{\partial x}(x) f(x) = \tilde{f}(\phi(x)), \]
which is a system of $n$ first order partial differential equations for the components of $\phi(x)$. Notice that in the most interesting case of $f(x_0) = \tilde{f}(\tilde{x}_0) = 0$, this is a system of singular partial differential equations.

Consider the infinite Taylor series expansion of our dynamical system

$$\dot{x} = f(x) = Jx + \sum_{m=2}^{\infty} f^{[m]}(x)$$

around an equilibrium, which is assumed to be $x_0 = 0 \in \mathbb{R}^n$, where $f^{[m]}$ denotes a polynomial vector field, whose all components are homogenous polynomials of degree $m$. Apply to it a formal change of coordinates given by an invertible formal transformation of the form

$$\tilde{x} = \phi(x) = x + \sum_{m=2}^{\infty} \phi^{[m]}(x),$$

which preserves $0 \in \mathbb{R}^n$ and starts with the identity, where all components of $\phi^{[m]}$ are homogenous polynomials of degree $m$. In order to study the action of $\phi(x)$ on $f(x)$, we will see how its homogenous part of degree $m$ acts on terms of degree $m$ of $f$. To this end, apply to

$$\dot{x} = Jx + f^{[m]}(x)$$

the transformation

$$\tilde{x} = x + \phi^{[m]}(x),$$

where $m \geq 2$. We have, modulo terms of higher degree,

$$\dot{\tilde{x}} = J\tilde{x} - J\phi^{[m]}(x) + f^{[m]}(x) + \frac{\partial \phi^{[m]}}{\partial x}(x)(Jx + f^{[m]}(x))$$

$$= J\tilde{x} - J\phi^{[m]}(x) + f^{[m]}(x) + \frac{\partial \phi^{[m]}}{\partial x}(x)Jx$$

$$= J\tilde{x} + f^{[m]}(x) + [Jx, \phi^{[m]}(x)]$$

$$= J\tilde{x} + \tilde{f}^{[m]}(\tilde{x}),$$

where $[v, w](x) = \frac{\partial w}{\partial x}(x)v(x) - \frac{\partial v}{\partial x}w(x)$ is the Lie bracket of two vector fields $v$ and $w$. Using the notation $ad_vw = [v, w]$, we obtain

$$(HE) \quad ad_{Jx}\phi^{[m]}(x) = \tilde{f}^{[m]}(x) - f^{[m]}(x),$$

which we will call a homological equation.

Consider the action of $ad_{Jx}$ on the space $P^{[m]}$ of polynomial vector fields whose all components are homogeneous polynomials of degree $m$. For a multi-index $k = (k_1, \ldots, k_n)$, denote $x^k = x_1^{k_1} \cdots x_n^{k_n}$.

**Lemma 2.1** Assume that $J$ is diagonal, say $J = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Then $ad_{Jx}$ is a diagonal operator on the space $P^{[m]}$ in the eigenbasis formed by the eigenvectors $x^k \frac{\partial}{\partial x_i}$, for all multi-indices $k$ such that $k_1 + \cdots + k_n = m$ and $1 \leq i \leq n$. The eigenvalues of $ad_{Jx}$ depend linearly on the eigenvalues of $J$, more precisely, we have

$$ad_{Jx}(x^k \frac{\partial}{\partial x_i}) = (k, \lambda)(x^k \frac{\partial}{\partial x_i}),$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$, and $(k, \lambda) = k_1\lambda_1 + \cdots + k_n\lambda_n$.  

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Corollary 2.2 The operator $\text{ad}_{Jx}$ is invertible on the space $P^{[m]}$ if there does not hold any relation of the form

$$\sum_{s=1}^{n} k_{s} \lambda_{s} = \lambda_{j},$$

where $k_{s}$ are nonnegative integers, $|k| = k_{1} + \cdots + k_{n} \geq 2$ and $1 \leq j \leq n$.

For any relation $\lambda_{j} = \sum_{s=1}^{n} k_{s} \lambda_{s}$, called resonance, we define

$$R_{j} = \{ k = (k_{1}, \ldots, k_{n}) : \lambda_{j} = k_{1} \lambda_{1} + \cdots + k_{n} \lambda_{n}, \ k_{i} \in \mathbb{N} \cup \{0\}, \ |k| \geq 2 \},$$

which will be called the resonant set associated with $\lambda_{j}$.

Theorem 2.3 Consider the differential equation

$$\dot{x} = f(x) = Jx + \sum_{m=2}^{\infty} f^{[m]}(x),$$

and assume that all eigenvalues are real and distinct, and that the spectrum of $J$ is nonresonant.

(i) For each $m \geq 2$ and any homogenous vector fields $f^{[m]}$ and $\tilde{f}^{[m]}$ of degree $m$, the homological equation (HE) is solvable within the class of $\mathbb{R}^{n}$-valued polynomials $\phi^{[m]}$ of degree $m$.

(ii) The differential equations $\dot{x} = f(x) = Jx + \sum_{m=2}^{\infty} f^{[m]}(x)$ and $\dot{\tilde{x}} = \tilde{f}(\tilde{x}) = J\tilde{x} + \sum_{m=2}^{\infty} \tilde{f}^{[m]}(\tilde{x})$

are equivalent via an invertible formal transformation of the form $\tilde{x} = x + \sum_{m=2}^{\infty} \phi^{[m]}(x)$.

(iii) The differential equation $\dot{x} = f(x) = Jx + \sum_{m=2}^{\infty} f^{[m]}(x)$ can be formally linearized, that is, can be brought to the form $\dot{\tilde{x}} = J\tilde{x}$ via an invertible formal transformation of the form

$$\tilde{x} = x + \sum_{m=2}^{\infty} \phi^{[m]}(x).$$

Item (i) is a direct consequence of Corollary 2.2. Item (ii) follows by a successive application of (i) for $m = 2, 3$ and so on. Finally, (iii) is an immediate consequence of (ii), applied for $\tilde{f} = J\tilde{x}$.

If the spectrum of $J$ is resonant, then using the $\text{ad}_{Jx}$-operator we can get rid of all nonresonant terms, which leads to the following:

Theorem 2.4 Consider the differential equation

$$\dot{x} = f(x) = Jx + \sum_{m=2}^{\infty} f^{[m]}(x).$$

Assume that $J$ is diagonal, that is, $J = \text{diag}(\lambda_{1}, \ldots, \lambda_{n})$. There exits a formal invertible transformation of the form $\tilde{x} = x + \sum_{m=2}^{\infty} \phi^{[m]}(x)$ bringing $\dot{x} = f(x)$ into $\dot{\tilde{x}} = \tilde{f}(\tilde{x})$ of the form

$$\tilde{f}_{j}(\tilde{x}) = \lambda_{j} \tilde{x}_{j} + \sum_{k \in R_{j}} \gamma_{j}^{k} \tilde{x}_{1}^{k_{1}} \cdots \tilde{x}_{n}^{k_{n}},$$

where $\gamma_{j}^{k} \in \mathbb{R}$ and the summation is taken over all resonances $(k_{1}, \ldots, k_{n})$ forming the resonant set $R_{j}$ associated with $\lambda_{j}$. 

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If the eigenvalues of $J$ are distinct but not necessarily real, then an analogous result holds; we will state it in Section 7.

Theorems 2.3 and 2.4 summarize Poincaré’s approach in the formal category. The idea of this approach is very natural: in order to establish the equivalence of two dynamical systems, we replace the singular partial differential equation (DE) by an infinite sequence of homological equations (HE), which are simply linear equations with respect to the unknown components of the homogenous part $\phi^{[m]}$ of $\phi$.

A much more delicate and difficult issues of constructing $C^\infty$-smooth or real analytic transformations that linearize the equation (in the non-resonant case) or annihilate all nonresonant terms (in the general case) are discussed very briefly in Section 9.

3 Normal forms for single-input systems with controllable linearization

3.1 Introduction

In this section we will be studying nonlinear single-input control-affine systems of the form

$$\Sigma : \dot{\xi} = f(\xi) + g(\xi)u,$$

where $\xi \in X$, an open subset of $\mathbb{R}^n$, $u \in \mathbb{R}$, and $f$ and $g$ are $C^\infty$-smooth vector fields on $X$.

Throughout this section we will study the system $\Sigma$ around a point $\xi_0$ at which $f(\xi_0) = 0$ and $g(\xi_0) \neq 0$. Without loss of generality, we will assume that $\xi_0 = 0$. We will also assume throughout this section that the linear part $(F, G)$ of the system is controllable, where $F = \frac{\partial f}{\partial \xi}(0)$ and $G = g(0)$.

The goal of this section is to obtain a normal form of $\Sigma$ under the action of the feedback group consisting of feedback transformation of the form

$$\Gamma : x = \phi(\xi)$$
$$u = \alpha(\xi) + \beta(\xi)v.$$

Together with the system $\Sigma$ and the feedback transformation $\Gamma$, we will consider their Taylor series expansions $\Sigma^\infty$ and $\Gamma^\infty$, respectively, and we will study the action of $\Gamma^\infty$ on $\Sigma^\infty$ step-by-step, that is, the action of the homogeneous part $\Gamma^m$ of $\Gamma^\infty$ on the homogeneous part $\Sigma^{[m]}$ of $\Sigma^\infty$. In other words, we will generalize to control systems the approach that Poincaré has developed for dynamical systems (and which we recalled in Section 2). It was Kang and Krener [52], [48], [49] who proposed that approach in the context of control systems and who have obtained fundamental results. Their pioneering work has inspired the authors who have obtained further results and all of them form now-a-days a relatively complete theory of formal feedback classification of nonlinear control systems. The first results of Kang and Krener were devoted to obtaining a normal form for single-input control-affine systems with controllable linear approximation and we will also start our systematic presentation in this section by discussing that case. A generalization to non-affine systems will be given at the end of this section while further developments (uncontrollable linear approximation and the problem of canonical forms) will be discussed in next sections.
The section is organized as follows. In Section 3.2 we will say a few words about the notation used in the whole section as well as in Sections 4, 5, and 6. Main results are given in Section 3.3: a normal forms for homogeneous systems, explicit transformations bringing to it, $m$-invariants, and normal form under formal feedback. Finally, in Section 3.4 we will generalize the normal form to non-affine systems.

### 3.2 Notations

We will denote by $P^{m}_\xi$ the space of homogeneous polynomials of degree $m$ of the variables $\xi_1, \ldots, \xi_n$, by $P^{\leq m}_\xi$ the space of polynomials of degree $m$ of the variables $\xi_1, \ldots, \xi_n$, and by $P^{\geq m}_\xi$ the space of formal power series of the variables $\xi_1, \ldots, \xi_n$ starting from terms of degree $m$.

Analogously, we will denote by $V^{m}_\xi(\xi, u)$ the space of homogeneous vector fields whose components are in $P^{m}_\xi(\xi, u)$, by $V^{\leq m}_\xi(\xi, u)$ the space of polynomial vector fields whose components are in $P^{\leq m}_\xi(\xi, u)$, and by $V^{\geq m}_\xi(\xi, u)$ the space of vector fields formal power series whose components are in $P^{\geq m}_\xi(\xi, u)$.

Similar notations $P^{m}_\xi(\xi, u)$, $V^{m}_\xi(\xi, u)$ etc. will also appear and they stand, respectively, for homogeneous polynomials and homogeneous polynomial vector fields depending on the state variables $\xi = (\xi_1, \ldots, \xi_n)$ and control variable $u$, with homogeneity being understood with respect to the all variables $(\xi, u)$.

Because of various normal forms and various transformations that are used throughout the paper, we will keep the following notation. Together with $\Sigma$, we will also consider its Taylor series expansion $\Sigma^{\infty}$ and its homogeneous part $\Sigma^{[m]}$ of degree $m$ given, respectively, by the following systems

$$\Sigma^{[m]} : \dot{\xi} = A\xi + Bu + f^{[m]}(\xi) + g^{[m-1]}(\xi)u,$$

$$\Sigma^{\infty} : \dot{\xi} = A\xi + Bu + \sum_{k=2}^{\infty} (f^{[k]}(\xi) + g^{[k-1]}(\xi)u).$$

The systems $\Sigma$, $\Sigma^{[m]}$, and $\Sigma^{\infty}$ will stand for the systems under consideration. Their state vector will be denoted by $\xi$ and their control by $u$ ($x$ and $v$ being reserved, respectively, for the state and control of various normal forms). The system $\Sigma^{[m]}$ (resp. the system $\Sigma^{\infty}$) transformed via feedback will be denoted by $\Sigma^{[m]}_\Gamma$ (resp. by $\Sigma^{\infty}_\Gamma$). Its state vector will be denoted by $x$, its control by $v$, and the vector fields, defining its dynamics, by $f^{[k]}$ and $g^{[k-1]}$. Feedback equivalence of homogeneous systems $\Sigma^{[m]}$ and $\Sigma^{[m]}_\Gamma$ will be established via a smooth feedback, that is precisely, via a homogeneous feedback $\Gamma^{m}$. On the other hand, feedback equivalence of systems $\Sigma^{\infty}$ and $\Sigma^{\infty}_\Gamma$ will be established via a formal feedback $\Gamma^{\infty}$.

We will introduce two kinds of normal forms, Kang normal forms and dual normal forms (Sections 3 and 5), as well as canonical form and dual canonical form (Sections 4 and 6). The symbol “bar” will correspond to the vector field $\bar{f}^{[m]}$ defining the Kang normal forms $\Sigma^{[m]}_NF$ and $\Sigma^{\infty}_NF$ and the canonical form $\Sigma^{\infty}_CF$ as well as to the vector field $\bar{g}^{[m-1]}$ defining the dual normal forms $\Sigma^{[m]}_{DNF}$ and $\Sigma^{\infty}_{DNF}$ and the dual canonical form $\Sigma^{\infty}_{DCF}$. Analogously, the $m$-invariants (resp. dual $m$-invariants) of the system $\Sigma^{[m]}$ will be denoted by $a^{[m,j,i+2]}$ (resp. by $b^{[m-1,j]}$) and the $m$-invariants (resp. dual $m$-invariants) of the normal form $\Sigma^{[m]}_NF$ (resp. dual normal form $\Sigma^{[m]}_{DNF}$) by $\bar{a}^{[m,j,i+2]}$ (resp. by $\bar{b}^{[m-1,j]}$). Other normal forms will appear in Section 12.
3.3 Normal form and $m$-invariants

All objects, that is, functions, maps, vector fields, control systems, etc., are considered in a neighborhood of $0 \in \mathbb{R}^n$ and assumed to be $C^\infty$-smooth. Let $h$ be a smooth $\mathbb{R}$-valued function. By

$$h(\xi) = h^{[0]}(\xi) + h^{[1]}(\xi) + h^{[2]}(\xi) + \cdots = \sum_{m=0}^{\infty} h^{[m]}(\xi)$$

we denote its Taylor series expansion at $0 \in \mathbb{R}^n$, where $h^{[m]}(\xi)$ stands for a homogeneous polynomial of degree $m$.

Similarly, for a map $\phi$ of an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$ (resp. for a vector field $f$ on an open subset of $\mathbb{R}^n$) we will denote by $\phi^{[m]}$ (resp. by $f^{[m]}$) the homogeneous term of degree $m$ of its Taylor series expansion at $0 \in \mathbb{R}^n$, that is, each component $\phi_j^{[m]}$ of $\phi^{[m]}$ (resp. $f_j^{[m]}$ of $f^{[m]}$) is a homogeneous polynomial of degree $m$ in $\xi$.

Consider the Taylor series expansion of the system $\Sigma$ given by

$$\Sigma^\infty : \dot{\xi} = F\xi + Gu + \sum_{m=2}^{\infty} (f^{[m]}(\xi) + g^{[m-1]}(\xi)u),$$

(3.1)

where $F = \frac{\partial L}{\partial \xi}(0)$ and $G = g(0)$. Recall that we assume in this section that $f(0) = 0$ and $g(0) \neq 0$.

Consider also the Taylor series expansion $\Gamma^\infty$ of the feedback transformation $\Gamma$ given by

$$\Gamma^\infty : \begin{align*}
  x &= T\xi + \sum_{m=2}^{\infty} \phi_j^{[m]}(\xi) \\
  u &= K\xi + Lv + \sum_{m=2}^{\infty} (\alpha_j^{[m]}(\xi) + \beta_j^{[m-1]}(\xi)v)
\end{align*}$$

(3.2)

where $T$ is an invertible matrix and $L \neq 0$. Analogously to the Poincaré’s approach presented in Section 2, we analyze the action of $\Gamma^\infty$ on the system $\Sigma^\infty$ step by step.

To start with, consider the linear system

$$\dot{\xi} = F\xi + Gu.$$ 

Throughout the section we will assume that it is controllable. It can be thus transformed by a linear feedback transformation of the form

$$\Gamma^1 : \begin{align*}
  x &= T\xi \\
  u &= K\xi + Lv
\end{align*}$$

into the Brunovský canonical form $(A, B)$, see e.g. [47] and Example 1.2 in Section 1:

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$
Assuming that the linear part \((F, G)\), of the system \(\Sigma^\infty\) given by (3.1), has been transformed to the Brunovský canonical form \((A, B)\), we follow an idea of Kang and Krener [52], [48] and apply successively a series of transformations

\[
\Gamma^m : \begin{align*}
x &= \xi + \phi^m[1](\xi) \\
u &= v + \alpha^m[1](\xi) + \beta^{m-1}[1](\xi)v,
\end{align*}
\tag{3.3}
\]

for \(m = 2, 3, \ldots\). A feedback transformation defined as an infinite series of successive compositions of \(\Gamma^m\), \(m = 1, 2, \ldots\) is also denoted by \(\Gamma^\infty\) (that is, \(\Gamma^\infty = \cdots \Gamma^m \circ \Gamma^{m-1} \circ \cdots \circ \Gamma^1\)) because, as a formal power series, it is of the form (3.2). We will not address the problem of convergence in general (see Sections 9 and 13 for some comments on this issue, and for a convergent class of analytic systems) and we will call such a series of successive compositions a formal feedback transformation.

Observe that each transformation \(\Gamma^m\), for \(m \geq 2\), leaves invariant all homogeneous terms of degree smaller than \(m\) of the system \(\Sigma^\infty\) and we will call \(\Gamma^m\) a homogeneous feedback transformation of degree \(m\). We will study the action of \(\Gamma^m\) on the following homogeneous system

\[
\Sigma^m : \dot{\xi} = A\xi + Bu + f^m[1](\xi) + g^{[m-1]}(\xi)v.
\tag{3.4}
\]

Consider another homogeneous system \(\tilde{\Sigma}^m\) given by

\[
\tilde{\Sigma}^m : \dot{x} = Ax + Bu + f^m[1](x) + g^{[m-1]}[1](x)v.
\tag{3.5}
\]

We will say that the homogeneous system \(\Sigma^m\) is feedback equivalent to the homogeneous system \(\tilde{\Sigma}^m\) if there exists a homogeneous feedback transformation of the form (3.3), which brings \(\Sigma^m\) into \(\tilde{\Sigma}^m\) modulo terms in \(V_{\geq m+1}(x, v)\).

The starting point for formal classification of single-input control systems is the following result, proved by Kang [48].

**Proposition 3.1** The homogeneous feedback transformation \(\Gamma^m\), defined by (3.3), brings the system \(\Sigma^m\), given by (3.4), into \(\tilde{\Sigma}^m\), given by (3.5), if and only if the following relations

\[
\left\{ \begin{array}{l}
L_A\phi^m_{j+1} = f^m_{j+1}(\xi) - f^m_{j}(\xi) \\
L_B\phi^m_{j}(\xi) = g^{[m-1]}_{j-1}(\xi) - g^{[m-1]}_{j}(\xi) \\
L_A\phi^m_{n} + \alpha[1](\xi) = f^{[m]}_{n}(\xi) - f^{[m]}_{n}(\xi) \\
L_B\phi^m_{n}(\xi) + \beta^{[m-1]}(\xi) = g^{[m-1]}_{n}(\xi) - g^{[m-1]}_{n}(\xi)
\end{array} \right. \tag{3.6}
\]

hold for any \(1 \leq j \leq n - 1\), where \(\phi^m_{j}\) are the components of \(\phi^m\).

This proposition represents the essence of the method developed by Kang and Krener and used for many results in this survey paper. The problem of studying the feedback equivalence of two control-affine systems \(\Sigma\) and \(\tilde{\Sigma}\) requires, in general, solving the system (CDE) of 1-st order partial differential equations (as we have already explained in Section 1). On the other hand, if we perform the analysis step by step, then the problem of establishing the feedback equivalence
of two systems $\Sigma^m$ and $\tilde{\Sigma}^m$ reduces to solving the algebraic system (3.6), called sometimes the control homological equation by its analogy with Poincaré’s homological equation (HE) of Section 2. Indeed, (3.6) can be re-written in the following compact form

\[
\text{(CHE)}
\]

\[
\begin{align*}
\text{ad}_{A\xi}\phi^m(\xi) &= \tilde{f}^m(\xi) - f(\xi) - B\alpha^m(\xi) \\
\text{ad}_{B}\phi^m(\xi) &= \tilde{g}^{m-1}(\xi) - g^{m-1}(\xi) - B\beta^{m-1}(\xi),
\end{align*}
\]

which reduces to (HE) if the control vector field $B + g^{m-1}(\xi)$ is not present, with $A$ playing the role of $J$. Therefore for control systems, solving the differential equation (CDE) is replaced by an infinite sequence of algebraic homological equations (CHE) exactly like for dynamical systems, where the differential equation (CDE) is replaced by an infinite sequence of homological equations (HE) (compare Section 2).

Using the above proposition, Kang [48] proved the following result:

**Theorem 3.2** The homogeneous system $\Sigma^m$ can be transformed, via a homogeneous feedback transformation $\Gamma^m$, into the following normal form

\[
\Sigma^m_{NF} : \left\{ \begin{array}{l}
\dot{x}_1 = x_2 + \sum_{i=3}^n x_i^2 P_{1,i}^{m-2}(x_1, \ldots, x_i) \\
\vdots \\
\dot{x}_j = x_{j+1} + \sum_{i=j+2}^n x_i^2 P_{j,i}^{m-2}(x_1, \ldots, x_i) \\
\vdots \\
\dot{x}_{n-2} = x_{n-1} + x_n^2 P_{n-2,n}^{m-2}(x_1, \ldots, x_n) \\
\dot{x}_{n-1} = x_n \\
\dot{x}_n = v,
\end{array} \right.
\]

(3.7)

where $P_{j,i}^{m-2}(x_1, \ldots, x_i)$ are homogeneous polynomials of degree $m - 2$ depending on the indicated variables.

In order to illustrate this result, consider the case $m = 2$, which actually was for Kang and Krener [52] the starting point for the formal approach to feedback equivalence. Applying the above theorem to $m = 2$ yields that the homogeneous system $\Sigma^{[2]}$ can be transformed, via a homogeneous feedback transformation $\Gamma^2$, into the following normal form

\[
\Sigma^{[2]}_{NF} : \left\{ \begin{array}{l}
\dot{x}_1 = x_2 + a_{1,3} x_3^2 + a_{1,4} x_4^2 + \cdots + a_{1,n} x_n^2 \\
\dot{x}_2 = x_3 + a_{2,4} x_4^2 + \cdots + a_{2,n} x_n^2 \\
\vdots \\
\dot{x}_{n-2} = x_{n-1} + a_{n-2,n} x_n^2 \\
\dot{x}_{n-1} = x_n \\
\dot{x}_n = u,
\end{array} \right.
\]

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where $a_{j,i} \in \mathbb{R}$. Notice that the general normal form $\Sigma_{NF}^{[m]}$ exhibits the same triangular structure as $\Sigma_{NF}^{[2]}$, the only difference being the replacement of the constants $a_{j,i}$ by the polynomials $P_{j,i}^{[m-2]}(x_1, \ldots, x_i)$.

Now we will make a few comments on the number of constants $a_{j,i}$ (for $m = 2$) and that of the polynomials (in the general case) present in the normal forms. Compare the analysis given below (performed for the homogeneous system $\Sigma^{[m]}$) with a similar analysis given for general multi-input systems $\Pi$ and control-affine systems $\Sigma$ in Section 1. Recall that the dimension of the space of polynomials $P^{[m]}$ of degree $m$ of $n$ variables and of the space $V^{[m]}$ of polynomial vector fields on $\mathbb{R}^n$, whose all components belong to $P^{[m]}$, are, respectively

$$(n + m - 1)! \cdot m!/(n-1)! \quad \text{and} \quad n\cdot n!(n-1)!.$$

Homogeneous systems $\Sigma^{[m]}$ are given by two vector fields $f^{[m]} \in V^{[m]}$ and $g^{[m-1]} \in V^{[m-1]}$. So the dimension of the space of single-input systems, homogenous of degree $m$, is

$$d_{\Sigma^{[m]}} = n \cdot n!(n-1)! + \frac{(n + m - 1)!}{m!(n-1)!} + \frac{(n + m - 2)!}{(n-1)!(n-1)!}.$$

The feedback group $\Gamma^m$ is given by $n$ components of the diffeomorphism $\phi^{[m]}$, each in $P^{[m]}$, and two functions $\alpha^{[m]} \in P^{[m]}$ and $\beta^{[m-1]} \in P^{[m-1]}$. Hence the dimension of $\Gamma^m$ is

$$d_{\Gamma^{m}} = n \cdot n!(n-1)! + \frac{(n + m - 1)!}{m!(n-1)!} + \frac{(n + m - 2)!}{(n-1)!(n-1)!}.$$

Both dimensions are polynomials of degree $n-1$ of $m$ and their difference is thus also a polynomial of degree $n-1$ of $m$ starting with

$$d_{\Sigma^{[m]}} - d_{\Gamma^{m}} = \frac{n-2}{(n-1)!} m^{n-1} + \cdots,$$

where dots stand for lower order terms. Now observe that the dimension of the space of $n-2$ functions, each belonging to $P^{[m]}$, is also a polynomial of degree $n-1$ of $m$ starting with

$$(n-2) \cdot n!(n-1)! = \frac{n-2}{(n-1)!} m^{n-1} + \cdots,$$

which explains why in the normal form $\Sigma_{NF}^{[m]}$ we have $n-2$ polynomials of $n$ variables. Since $d_{\Sigma^{[m]}} - d_{\Gamma^{m}} < (n-2)\cdot(n+m-1)!/m!(n-1)!$, it follows that polynomials of fewer variables show up in the normal form $\Sigma_{NF}^{[m]}$. Analogous argument applied to $m$ tending to infinity explains the appearance of $n-2$ functions of $n$ variables in the normal form $\Sigma_{NF}^{[\infty]}$ (see Theorem 3.6 below). In order to calculate the exact number of invariants in the form $\Sigma^{[m]}$ (which is bounded from below by $d_{\Sigma^{[m]}} - d_{\Gamma^{m}}$), we have to study the action of $\Gamma^m$ on the space of homogeneous systems of degree $m$. This action is not free, the isotropy group being of dimension 1 (see [48], [81] and Proposition 4.4 for a detailed calculation). This can be illustrated by the homogeneous system $\Sigma^{[2]}$ of degree 2, for which $d_{\Sigma^{[2]}} = n\cdot\frac{n(n+1)}{2} + mn$ (we have $n$ components of $f^{[2]}$ and $n$ components of $g^{[1]}$) and $d_{\Gamma^{2}} = n\cdot\frac{n(n+1)}{2} + \frac{n(n+1)}{2} + n$ (we have $n$ components of $\phi^{[2]}$ and the function $\alpha^{[2]}$ and $\beta^{[1]}$). It
follows that $d_{\Sigma^2} - d_{\Gamma^2} = \frac{n^2 - 3n}{2}$ while the number of parameters $a_{j,i}$ (which is actually the number of invariants of $\Sigma^2$, see the next section) is $\frac{(n-1)(n-2)}{2}$. The difference $\frac{(n-1)(n-2)}{2} - \frac{n^2 - 3n}{2} = 1$ is actually the dimension of the isotropy subgroup of $\Gamma^2$, which is the dimension of the group of symmetries of any $\Sigma^2$ (see Section 11).

The two following questions concerning the normal form $\Sigma^{[m]}_{N_F}$ are important and arise naturally.

(Q1) Are the polynomials $P_{j,i}^{[m-2]}$ invariant, that is, unique under feedback $\Gamma^m$?

(Q2) How to bring a given system $\Sigma^{[m]}$ into its normal form $\Sigma^{[m]}_{N_F}$?

The answer to question (Q1) is positive, and to construct invariants under homogeneous feedback transformations, let us define the vector fields

$$X_i^{m-1}(\xi) = (-1)^i ad_{\xi}^{i} f^{[m]}(\xi) \left( B + g^{[m-1]}(\xi) \right)$$

and let $X_i^{m-1}$ be its homogeneous part of degree $m - 1$. By $\pi_i$ we will denote the projection on the subspace

$$W_i = \{ \xi = (\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n : \xi_{i+1} = \cdots = \xi_n = 0 \}$$

that is

$$\pi_i(\xi) = (\xi_1, \ldots, \xi_i, 0, \ldots, 0).$$

Following Kang [48], we denote by $a^{[m],j,i+2}(\xi)$ the homogeneous part of degree $m - 2$ of the polynomials

$$CA^{j-1} [X_i^{m-1}, X_{i+1}^{m-1}] (\pi_{n-i}(\xi)) = CA^{j-1} \left( ad_{A'B} X_i^{m-1} - ad^{A+i+1B} X_i^{m-1} \right) (\pi_{n-i}(\xi)),$$

where $C = (1, 0, \ldots, 0)^T \in \mathbb{R}^n$, and $(j,i) \in \Delta \subset \mathbb{N} \times \mathbb{N}$, defined by

$$\Delta = \{ (j,i) \in \mathbb{N} \times \mathbb{N} : 1 \leq j \leq n - 2, 0 \leq i \leq n - j - 2 \}.$$

The homogeneous polynomials $a^{[m],j,i+2}$, for $(j,i) \in \Delta$, will be called $m$-invariants of $\Sigma^{[m]}$, under the action of $\Gamma^m$.

The following result of Kang [48] asserts that $m$-invariants $a^{[m],j,i+2}$, for $(j,i) \in \Delta$, are complete invariants of homogeneous feedback and, moreover, illustrates their meaning for the homogeneous normal form $\Sigma^{[m]}_{N_F}$.

Consider two homogeneous systems $\Sigma^{[m]}$ and $\Sigma^{[m]}$ and let

$$\{ a^{[m],j,i+2} : (j,i) \in \Delta \}, \quad \text{and} \quad \{ \tilde{a}^{[m],j,i+2} : (j,i) \in \Delta \}$$

denote, respectively, their $m$-invariants. The following result was proved by Kang [48]:

**Theorem 3.3** The $m$-invariants have the following properties:

(i) Two homogeneous systems $\Sigma^{[m]}$ and $\Sigma^{[m]}$ are equivalent via a homogeneous feedback transformation $\Gamma^m$ if and only if

$$a^{[m],j,i+2} = \tilde{a}^{[m],j,i+2}, \quad \text{for any } (j,i) \in \Delta.$$
The $m$-invariants $\tilde{a}^{[m]j,i+2}$ of the normal form $\Sigma^{[m]}_{NF}$, defined by (3.7), are given by

$$\tilde{a}^{[m]j,i+2}(x) = \frac{\partial^2}{\partial x_{n-i}^2} x_{n-i}^2 P^{[m-2]}_{j,n-i} (x_1, \ldots, x_{n-i}), \quad \text{for any } (j, i) \in \Delta.$$  (3.8)

In order to answer the second question (Q2) we will construct an explicit feedback transformation that brings the homogeneous system $\Sigma^{[m]}$ to its normal form $\Sigma^{[m]}_{NF}$. Define the homogeneous polynomials $\psi^{[m-1]}_{j,i}(\xi)$ by setting $\psi^{[m-1]}_{j,0}(\xi) = \psi^{[m-1]}_{1,1}(\xi) = 0$, 

$$\psi^{[m-1]}_{j,i}(\xi) = -CA^{j-1} \left( ad_{A^j}^m g^{[m-1]} + \sum_{i=1}^{n-i} (-1)^i ad_{A^j}^{m-1} ad_{A^{n-i}} B f^{[m]} \right),$$

if $1 \leq j < i \leq n$ and

$$\psi^{[m-1]}_{j,i}(\xi) = L_{A^{n-i}} B f^{[m]}_{j-1} (\pi_i(\xi)) + L_{A^j} \psi^{[m-1]}_{j-1,i} (\pi_i(\xi)) + \psi^{[m-1]}_{j-1,i-1} (\pi_i(\xi)) + \int_0^{\xi_i} L_{A^{n-i+1}} B \psi^{[m-1]}_{j-1,i} (\pi_i(\xi)) d\xi_i,$$  (3.9)

if $1 \leq i \leq j$, where $\psi^{[m-1]}_{j,i}(\pi_i(\xi))$ is the restriction of $\psi^{[m-1]}_{j,i}(\xi)$ to the subspace $W_i$. Define the components $\phi^{[m]}_j$ of $\phi^{[m]}$, for $1 \leq j \leq n$, and the feedback $(\alpha^{[m]}, \beta^{[m-1]})$ by

$$\phi^{[m]}_j(\xi) = \sum_{i=1}^{n} \int_0^{\xi_i} \psi^{[m-1]}_{j,i} (\pi_i(\xi)) d\xi_i, \quad 1 \leq j \leq n - 1,$$

$$\phi^{[m]}_n(\xi) = f^{[m]}_{n-1}(\xi) + L_{A^j} \phi^{[m]}_{n-1}(\xi),$$

$$\alpha^{[m]}(\xi) = - \left( f^{[m]}_n(\xi) + L_{A^j} \phi^{[m]}_n(\xi) \right),$$

$$\beta^{[m-1]}(\xi) = - \left( g^{[m-1]}_n(\xi) + L_B \phi^{[m]}_n(\xi) \right).$$  (3.10)

We have the following result of the authors [81]:

**Theorem 3.4** The homogeneous feedback transformation

$$\Gamma^m : x = \xi + \phi^{[m]}(\xi)$$

$$u = v + \alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v,$$

where $\alpha^{[m]}$, $\beta^{[m-1]}$, and the components $\phi^{[m]}_j$ of $\phi^{[m]}$ are defined by (3.10), brings the homogeneous system $\Sigma^{[m]}$ into its normal form $\Sigma^{[m]}_{NF}$ given by (3.7).

**Example 3.5** To illustrate results of this section, we consider the system $\Sigma^{[m]}$, given by (3.4) on $\mathbb{R}^3$. Theorem 3.2 implies that the system $\Sigma^{[m]}$ is equivalent, via a homogeneous feedback transformation $\Gamma^m$ defined by (3.10), to its normal form $\Sigma^{[m]}_{NF}$ (see (3.7))

$$\dot{x}_1 = x_2 + x_3^2 P^{[m-2]}(x_1, x_2, x_3)$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = v.$$
where $P^{[m-2]}(x_1,x_2,x_3)$ is a homogeneous polynomial of degree $m - 2$ of the variables $x_1,x_2,x_3$.

We would like now to discuss the interest of Theorem 3.4. As we have already mentioned, Poincaré’s method allows to replace the partial differential equation (CDE) (given in Section 1) by solving successively linear algebraic equations defined by the control homological equation (CHE), see [52] and [48], and Proposition 3.1. The solvability of this equation was proved in [52] and [48] while Theorem 3.4 provides an explicit solution (in the form of the transformations (3.10), that are easily computable via differentiation and integration of homogeneous polynomials) to the control homological equation. As a consequence, for any given control system, Theorem 3.4 gives transformations bringing the homogeneous part of the system into its normal form. For example, if the system is feedback linearizable, up to order $m_0 - 1$ (see [54]), then a diffeomorphism and a feedback compensating all nonlinearities of degree lower than $m_0$ can be calculated explicitly without solving partial differential equations (compare Section 9). More generally, by a successive application of transformations given by (3.10) we can bring the system, without solving partial differential equations, to its normal form given in Theorem 3.6 below.

Consider the system $\Sigma^\infty$ of the form (3.1) and recall that we assume the linear part $(F,G)$ to be controllable. Apply successively to $\Sigma^\infty$, a series of transformations $\Gamma_m$, $m = 1, 2, 3, \ldots$, such that each $\Gamma_m$ brings $\Sigma^{[m]}$ to its normal form $\Sigma^{NF}_{[m]}$. More precisely, bring $(F,G)$ into the Brunovský canonical form $(A,B)$ via a linear feedback $\Gamma_1$ and denote $\Sigma^\infty,1 = \Gamma_1^*(\Sigma^\infty)$. Assume that a system $\Sigma^{\infty,m-1}$ has been defined. Let $\Gamma_m$ be a homogeneous feedback transformation transforming $\Sigma^{[m]}$, which is the homogeneous part of degree $m$ of $\Sigma^{\infty,m-1}$, to the normal form $\Sigma^{NF}_{[m]}$ $(\Gamma_m$ can be taken, for instance, as the transformations defined by (3.10)). Define $\Sigma^{\infty,m} = \Gamma_m^*(\Sigma^{\infty,m-1})$. Notice that we apply $\Gamma_m$ to the whole system $\Sigma^{\infty,m-1}$ (and not only to its homogeneous part $\Sigma^{[m]}$). Successive repeating of Theorem 3.2 gives the following result of Kang [48].

**Theorem 3.6** There exists a formal feedback transformation $\Gamma^\infty$ which brings the system $\Sigma^\infty$ to a normal form $\Sigma^{\infty}_{NF}$ given by

$$
\begin{align*}
\dot{x}_1 &= x_2 + \sum_{i=3}^{n} x_i^2 P_{1,i}(x_1, \ldots, x_i) \\
&\vdots \\
\dot{x}_j &= x_{j+1} + \sum_{i=j+2}^{n} x_i^2 P_{j,i}(x_1, \ldots, x_i) \\
&\vdots \\
\dot{x}_{n-2} &= x_{n-1} + x_n^2 P_{n-2,n}(x_1, \ldots, x_n) \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= v,
\end{align*}
$$

(3.11)

where $P_{j,i}(x_1, \ldots, x_i)$ are formal power series depending on the indicated variables.

**Example 3.7** Consider a system $\Sigma$ defined on $\mathbb{R}^3$ whose linear part is controllable (compare Example 3.5). Theorem 3.6 implies that the system $\Sigma$ is equivalent, via a formal feedback
transformation $\Gamma^\infty$, to its normal form $\Sigma^\infty_{NF}$

$$
\begin{align*}
\dot{x}_1 &= x_2 + x_3^2 P(x_1, x_2, x_3) \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= v,
\end{align*}
$$

where $P(x_1, x_2, x_3)$ is a formal power series of the variables $x_1, x_2, x_3$. □

### 3.4 Normal form for non-affine systems

In this section we will generalize normal forms $\Sigma^{|m|}_{NF}$ and $\Sigma^\infty_{NF}$ to general control systems. As we explained in Section 1, such a generalization can be performed using Proposition 1.1.

Consider a general control system of the form

$$
\Pi : \dot{\xi} = F(\xi, u),
$$

around an equilibrium point $(\xi_0, u_0)$, that is $F(\xi_0, u_0) = 0$. Without loss of generality, we can assume that $(\xi_0, u_0) = (0, 0)$. Together with $\Pi$ we will consider its Taylor series expansion

$$
\Pi^\infty : \dot{\xi} = F\xi + Gu + \sum_{m=2}^{\infty} F^{[m]}(\xi, u),
$$

where $F^{[m]}(\xi, u)$ stands for homogeneous terms of degree $m$ and homogeneity is understood in this section with respect to the state and control variables together.

Consider the feedback transformation $\Upsilon$ (compare Section 1)

$$
\Upsilon : x = \phi(\xi) \\
v = \psi(\xi, u)
$$

and its Taylor series expansion $\Upsilon^\infty$ given by

$$
\Upsilon^\infty : x = T\xi + \sum_{m=2}^{\infty} \phi^{[m]}(\xi) \\
v = K\xi + Lu + \sum_{m=2}^{\infty} \psi^{[m]}(\xi, u),
$$

where $T$ is an invertible matrix and $L \neq 0$.

We will assume throughout this section that the pair $(F, G)$ is controllable and so we can suppose that it is in the Brunovský canonical form $(A, B)$. Like in the control-affine case, we will consider the action of the homogenous part $\Upsilon^m$ of $\Upsilon^\infty$ given by

$$
\Upsilon^m : x = \xi + \phi^{[m]}(\xi) \\
v = u + \psi^{[m]}(\xi, u)
$$

on the homogeneous part $\Pi^{[m]}$ of $\Pi^\infty$ given by

$$
\Pi^{[m]} : \dot{\xi} = A\xi + Bu + F^{[m]}(\xi, u).
$$

Combining Theorem 3.2 with Proposition 1.1 leads to the following result:
Theorem 3.8 The general homogeneous system $\Pi^{[m]}$ can be transformed, via a homogeneous feedback transformation $\Upsilon^m$, into the following normal form

$$\Pi^{[m]}_{NF} : \begin{cases} 
\dot{x}_1 &= x_2 + \sum_{i=3}^{n} x_i^2 P^{[m-2]}_{1,i}(x_1, \ldots, x_i) + v^2 P^{[m-2]}_1(x_1, \ldots, x_n, v) \\
 & \vdots \\
\dot{x}_j &= x_{j+1} + \sum_{i=j+2}^{n} x_i^2 P^{[m-2]}_{j,i}(x_1, \ldots, x_i) + v^2 P^{[m-2]}_j(x_1, \ldots, x_n, v) \\
 & \vdots \\
\dot{x}_{n-2} &= x_{n-1} + x_n^2 P^{[m-2]}_{n-2,n}(x_1, \ldots, x_n) + v^2 P^{[m-2]}_{n-1}(x_1, \ldots, x_n, v) \\
\dot{x}_{n-1} &= x_n + v^2 P^{[m-2]}_n(x_1, \ldots, x_n, v) \\
\dot{x}_n &= v,
\end{cases} \quad (3.12)$$

where $P^{[m-2]}_{j,i}(x_1, \ldots, x_i)$ and $P^{[m-2]}_j(x_1, \ldots, x_n, v)$ are homogeneous polynomials of degree $m - 2$ depending on the indicated variables.

Notice that formally the above normal form can be obtained as follows. Consider the affine normal form $\Sigma^{[m]}_{NF}$ and apply the reduction defined as the inverse of the extension (preintegration) described just before Proposition 1.1. More precisely, assume that $\Sigma^{[m]}_{NF}$ is controlled by $x_n$, so skip the last equation $\dot{x}_n = v$ and denote it by $v$. What we obtain is an $(n - 1)$-dimensional system, nonlinear with respect to $v$, which gives actually the $(n - 1)$-dimensional form $\Pi^{[m]}_{NF}$.

Like in the control-affine case, a successive application of Theorem 3.8 gives a formal normal form for $\Pi^{[m]}_{NF}$ under $\Upsilon^\infty$. It has the same structure as $\Pi^{[m]}_{NF}$, the only difference being that the polynomials $P^{[m-2]}_{j,i}(x_1, \ldots, x_i)$ and $P^{[m-2]}_j(x_1, \ldots, x_n, v)$ are replaced by formal power series of the same variables.

We will end up with a simple example, which, actually, is a non-affine version of Examples 3.5 and 3.7.

Example 3.9 Consider the general system $\Pi^{[m]}$ on $\mathbb{R}^2$. Theorem 3.6 implies that the system $\Pi^{[m]}$ is equivalent, via a homogeneous feedback transformation $\Upsilon^m$ to its normal form $\Pi^{[m]}_{NF}$, see (3.12):

$$\begin{align*}
\dot{x}_1 &= x_2 + v^2 P^{[m-2]}_1(x_1, x_2, v) \\
\dot{x}_2 &= v,
\end{align*}$$

where $P^{[m-2]}_1(x_1, x_2, v)$ is a homogeneous polynomial of degree $m - 2$ of the variables $x_1, x_2$ and $v$.

As a consequence, the general system $\Pi^\infty$ on $\mathbb{R}^2$ is equivalent, via a formal feedback transformation $\Upsilon^\infty$ to its normal form $\Pi^\infty_{NF}$:

$$\begin{align*}
\dot{x}_1 &= x_2 + v^2 P_1(x_1, x_2, v) \\
\dot{x}_2 &= v,
\end{align*}$$

where $P_1(x_1, x_2, v)$ is a formal power series of the variables $x_1, x_2$ and $v$. \qed
4 Canonical form for single-input systems with controllable linearization

As proved by Kang and recalled in Theorem 3.3, the normal form $\Sigma_{NF}^{[m]}$ is unique under homogeneous feedback transformation $\Gamma^m$. The normal form $\Sigma_{NF}^{\infty}$ is constructed by a successive application of homogeneous transformations $\Gamma^m$, for $m \geq 1$, which bring the corresponding homogeneous systems $\Sigma^{[m]}$ into their normal forms $\Sigma_{NF}^{[m]}$. Therefore a natural and fundamental question which arises is whether the system $\Sigma_{NF}^{\infty}$ can admit two different normal forms, that is, whether the normal forms given by Theorem 3.6 are in fact canonical forms under a general formal feedback transformations of the form $\Gamma^{\infty}$. It turns out that a given system can admit different normal forms, as shows the following example of Kang [48].

The main reason for the nonuniqueness of the normal form $\Sigma_{NF}^{\infty}$ is that, although the normal form $\Sigma_{NF}^{[m]}$ is unique, homogeneous feedback transformation $\Gamma^m$ bringing $\Sigma^{[m]}$ into $\Sigma_{NF}^{[m]}$ is not. It is this small group of homogeneous feedback transformations of order $m$ that preserve $\Sigma_{NF}^{[m]}$ (described by Proposition 4.4), which causes the nonuniqueness of $\Sigma_{NF}^{\infty}$.

Example 4.1 Consider the following system

$$
\begin{align*}
\dot{\xi}_1 &= \xi_2 + \xi_3^2 - 2\xi_1\xi_3^2 \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= u,
\end{align*}
$$

on $\mathbb{R}^3$. Clearly, this system is in Kang normal form (compare with Theorem 3.6), say $\Sigma_{1,NF}^{\infty}$. The feedback transformation

$$
\Gamma^{\leq 3} : 
\begin{align*}
x_1 &= \xi_1 - \xi_1^2 - \frac{4}{3}\xi_3^2 \\
x_2 &= \xi_2 - 2\xi_1\xi_2 \\
x_3 &= \xi_3 - 2(\xi_3^2 + \xi_1\xi_3) - 2\xi_2\xi_3 \\
u &= v + 6\xi_2\xi_3 + 12\xi_1\xi_2\xi_3 - 4\xi_3^3 + 2(\xi_1 + 2\xi_1^2 + 2\xi_2\xi_3) v
\end{align*}
$$

brings the system (4.1) into the form

$$
\begin{align*}
\dot{x}_1 &= x_2 + x_3^2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= v
\end{align*}
$$

modulo terms in $V^{\geq 4}(x, v)$. Applying successively homogeneous feedback transformations $\Gamma^m$ given, for any $m \geq 4$, by (3.10), we transform the above system into the following normal form $\Sigma_{2,NF}^{\infty}$:

$$
\begin{align*}
\dot{x}_1 &= x_2 + x_3^2 + x_3^2 P(x) \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= v,
\end{align*}
$$

where $P$ is a formal power series whose 1-jet at $0 \in \mathbb{R}^3$ vanishes. The systems (4.1) and (4.2) are in their normal forms ($\Sigma_{1,NF}^{\infty}$ and $\Sigma_{2,NF}^{\infty}$, respectively) and, moreover, the systems are feedback equivalent, but the system (4.2) does not contain any term of degree 3. As a consequence, the normal form $\Sigma_{NF}^{\infty}$ is not unique under feedback transformations.

□
A natural and important problem is thus to construct a canonical form and the aim of this section is indeed to construct a canonical form for $\Sigma^\infty$ under feedback transformation $\Gamma^\infty$.

Consider the system $\Sigma^\infty$ of the form

$$\Sigma^\infty : \dot{\xi} = F\xi + Gu + \sum_{m=2}^\infty (f^{|m|}(\xi) + g^{[m-1]}(\xi)u).$$

(4.3)

Since its linear part $(F, G)$ is assumed to be controllable, we bring it, via a linear transformation and linear feedback, to the Brunovsky canonical form $(A, B)$. Let the first homogeneous term of $\Sigma^\infty$ which cannot be annihilated by a feedback transformation be of degree $m_0$. As proved by Krener [54], the degree $m_0$ is given by the largest integer such that all distributions $D^k = \text{span} \{g, \ldots, a^d g^{k-1}\}$, for $1 \leq k \leq n - 1$, are involutive modulo terms of order $m_0 - 2$. We can thus, due to Theorems 3.2 and 3.3, assume that, after applying a suitable feedback $\Gamma \leq m_0$, the system $\Sigma^\infty$ takes the form

$$\dot{\xi} = A\xi + Bu + \bar{f}^{[m_0]}(\xi) + \sum_{m=m_0+1}^\infty (f^{|m|}(\xi) + g^{[m-1]}(\xi)u),$$

where $(A, B)$ is in Brunovsky canonical form and the first nonvanishing homogeneous vector field $\bar{f}^{[m_0]}$ is in the normal form (by Theorem 3.2) with components given by

$$\bar{f}^{[m_0]}_j(\xi) = \begin{cases} \sum_{i=j+2}^n \xi_i^2 P_{j,i}^{[m_0-2]}(\xi_1, \ldots, \xi_i), & 1 \leq j \leq n - 2, \\ 0, & n - 1 \leq j \leq n. \end{cases}$$

(4.4)

Let $(i_1, \ldots, i_{n-s})$, where $i_1 + \cdots + i_{n-s} = m_0$ and $i_{n-s} \geq 2$, be the largest, in the lexicographic ordering, $(n-s)$-tuple of nonnegative integers such that for some $1 \leq j \leq n - 2$, we have

$$\frac{\partial^{m_0} \bar{f}^{[m_0]}_j}{\partial \xi_1^{i_1} \cdots \partial \xi_{n-s}^{i_{n-s}}} \neq 0.$$

Define

$$j^* = \sup \left\{ j = 1, \ldots, n - 2 : \frac{\partial^{m_0} \bar{f}^{[m_0]}_j}{\partial \xi_1^{i_1} \cdots \partial \xi_{n-s}^{i_{n-s}}} \neq 0 \right\}.$$

The following results, whose proofs are detailed in [81], describe the canonical form obtained by the authors.

**Theorem 4.2** The system $\Sigma^\infty$ given by (4.3) is equivalent by a formal feedback $\Gamma^\infty$ to a system of the form

$$\Sigma^\infty_{CF} : \dot{x} = Ax + Bu + \sum_{m=m_0}^\infty \bar{f}^{[m]}(x),$$

where, for any $m \geq m_0$, the components $\bar{f}^{[m]}_j(x)$ of $\bar{f}^{[m]}(x)$ are given by

$$\bar{f}^{[m]}_j(x) = \begin{cases} \sum_{i=j+2}^n x_i^2 P_{j,i}^{[m-2]}(x_1, \ldots, x_i), & 1 \leq j \leq n - 2, \\ 0, & n - 1 \leq j \leq n, \end{cases}$$

(4.4)
additionally, we have
\[ \frac{\partial^{m_0} f^{[m_0]}_j}{\partial x_1^{i_1} \cdots \partial x_{n-s}^{i_{n-s}}} = \pm 1 \] (4.5)

and, moreover, for any \( m \geq m_0 + 1 \),
\[ \frac{\partial^{m_0} f^{[m]}_j}{\partial x_1^{i_1} \cdots \partial x_{n-s}^{i_{n-s}}} (x_1, 0, \ldots, 0) = 0. \] (4.6)

The form \( \Sigma_{\infty}^{CF} \) satisfying (4.4), (4.5) and (4.6) will be called the canonical form of \( \Sigma^\infty \). The name is justified by the following

**Theorem 4.3** Two systems \( \Sigma_1^\infty \) and \( \Sigma_2^\infty \) are formally feedback equivalent if and only if their canonical forms \( \Sigma_{1,CF}^\infty \) and \( \Sigma_{2,CF}^\infty \) coincide.

Kang [48], generalizing [52], proved that any system \( \Sigma^\infty \) can be brought by a formal feedback into the normal form \( \Sigma_{NF}^\infty \), for which (4.4) is satisfied. He also observed that his normal forms are not unique, see Example 4.1. Our results, Theorems 4.2 and 4.3, complete his study. We show that for each degree \( m \) of homogeneity we can use a 1-dimensional subgroup of feedback transformations which preserves the “triangular” structure of (4.4) and at the same time allows us to normalize one higher order term. The form of (4.5) and (4.6) is a result of this normalization. These 1-dimensional subgroups of feedback transformations are given by the following proposition.

**Proposition 4.4** The transformation \( \Gamma^m \) given by (3.3) leaves invariant the system \( \Sigma^{[m]} \) defined by (3.4) if and only if
\[ \begin{align*}
\phi^{[m]}_j &= a_m L^{-1} A_\xi \xi^m, \\
\alpha^{[m]} &= -a_m L^n A_\xi \xi^m, \\
\beta^{[m-1]} &= -a_m L^B L^{-1} A_\xi \xi^m,
\end{align*} \] (4.7)
where \( a_m \) is an arbitrary real parameter.

Theorem 4.2 establishes an equivalence of the system \( \Sigma^\infty \) with its canonical form \( \Sigma_{CF}^\infty \) via a formal feedback. Its direct corollary yields the following result for equivalence under a smooth feedback of the form
\[ \begin{align*}
\Gamma : & \quad x = \phi(\xi) \\
u = & \quad \alpha(\xi) + \beta(\xi)v,
\end{align*} \]
up to an arbitrary order. Indeed, we have the following:

**Corollary 4.5** Consider a smooth control system
\[ \Sigma : \dot{\xi} = f(\xi) + g(\xi)u. \]

For any positive integer \( k \) we have:
(i) There exists a smooth feedback $\Gamma$ transforming $\Sigma$, locally around $0 \in \mathbb{R}^n$, into its canonical form $\Sigma_{CF}^{\leq k}$ given by

$$\Sigma_{CF}^{\leq k} : \dot{x} = Ax + Bv + \sum_{m=m_0}^{k} \dot{f}[m](x),$$

modulo terms in $V^{\geq k+1}(x,v)$, where the components $\dot{f}[m](x)$ of $\dot{f}[m](x)$, for any $m_0 \leq m \leq k$, satisfy (4.4),(4.5), (4.6).

(ii) Feedback equivalence of $\Sigma$ and $\Sigma_{CF}^{\leq k}$, modulo terms in $V^{\geq k+1}(x,v)$, can be established via a polynomial feedback transformation $\Gamma_{\leq k}$ of degree $k$.

(iii) Two smooth systems $\Sigma_1$ and $\Sigma_2$ are feedback equivalent modulo terms in $V^{\geq k+1}(x,v)$ if and only if their canonical forms $\Sigma_{1,CF}^{\leq k}$ and $\Sigma_{2,CF}^{\leq k}$ coincide.

This Corollary follows directly from Theorems 4.2 and 4.3.

To end this section will illustrate our results by two examples.

**Example 4.6** Let us reconsider the system $\Sigma$ given by Example 3.7. It is equivalent, via a formal feedback, to the normal form

$$\begin{align*}
\dot{x}_1 &= x_2 + x_3^2 P(x_1, x_2, x_3) \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= v,
\end{align*}$$

where $P(x_1, x_2, x_3)$ is a formal power series. Assume, for simplicity, that $m_0 = 2$, which is equivalent to the following generic condition: $g, ad_f g,$ and $[g, ad_f g]$ are linearly independent at $0 \in \mathbb{R}^3$. This implies that we can express $P = P(x_1, x_2, x_3)$ as

$$P = c + P_1(x_1) + x_2 P_2(x_1, x_2) + x_3 P_3(x_1, x_2, x_3),$$

where $c \neq 0$ and $P_1(0) = 0$. Observe that any $P(x_1, x_2, x_3)$, of the above form, gives a normal form $\Sigma_{NF}^\infty$. In order to get the canonical form $\Sigma_{CF}^\infty$, we use Theorem 4.2 which assures the existence of a feedback transformation $\Gamma^\infty$ of the form

$$\begin{align*}
\dot{x} &= \phi(x) \\
v &= \alpha(x) + \beta(x)\tilde{v},
\end{align*}$$

which normalizes the constant $c$ and annihilates the formal power series $P_1(x_1)$. More precisely, $\Gamma^\infty$ transforms $\Sigma$ into its canonical form $\Sigma_{CF}^\infty$

$$\begin{align*}
\dot{x}_1 &= \tilde{x}_2 + \tilde{x}_3^2 \tilde{P}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \\
\dot{x}_2 &= \tilde{x}_3 \\
\dot{x}_3 &= \tilde{v},
\end{align*}$$

where the formal power series $\tilde{P}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is of the form

$$\tilde{P}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = 1 + \tilde{x}_2 \tilde{P}_2(\tilde{x}_1, \tilde{x}_2) + \tilde{x}_3 \tilde{P}_3(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3),$$

showing clearly a difference between the normal and canonical form: in the latter, the free term is normalized and the term depending on $x_1$ is annihilated.
Now, we give an example of constructing the canonical form for a physical model of variable length pendulum.

**Example 4.7** Consider the variable length pendulum of Bressan and Rampazzo [8] (see also [17]). We denote by $\xi_1$ the length of the pendulum, by $\xi_2$ its velocity, by $\xi_3$ the angle with respect to the horizontal, and by $\xi_4$ the angular velocity. The control $u = \dot{\xi}_4 = \ddot{\xi}_3$ is the angular acceleration. The mass is normalized to 1. The equations are (compare [8] and [17])

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= -g \sin \xi_3 + \xi_1 \xi_4 \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= u,
\end{align*}
\]

where $g$ denotes the gravity. Notice that if we suppose to control the angular velocity $\xi_4 = \dot{\xi}_3$, which is the case of [8] and [17], then the system is three-dimensional but the control enters nonlinearly.

At any equilibrium point $\xi_0 = (\xi_{10}, \xi_{20}, \xi_{30}, \xi_{40})^T = (\xi_{10}, 0, 0, 0)^T$, the linear part of the system is controllable. Our goal is to produce, for variable length pendulum, a normal form and the canonical form as well as to answer the question whether the systems corresponding to various values of the gravity constant $g$ are feedback equivalent. In order to get a normal form, put

\[
\begin{align*}
x_1 &= \xi_1 \\
x_2 &= \xi_2 \\
x_3 &= -g \sin \xi_3 \\
x_4 &= -g \xi_4 \cos \xi_3 \\
v &= g \xi_4^2 \sin \xi_3 - ug \cos \xi_3.
\end{align*}
\]

The system becomes

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 + x_4^2 \frac{x_1}{g^2 - x_3^2} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= v,
\end{align*}
\]

which gives a normal form. Indeed, we rediscover $\Sigma_{\infty}^{NF}$, given by (3.11), with $P_{1,3} = 0$, $P_{1,4} = 0$, and

\[
P_{2,4} = \frac{x_1}{g^2 - x_3^2}.
\]

To bring the system to its canonical form $\Sigma_{\infty}^{CF}$, firstly observe that $m_0 = 3$. Indeed, the function $x_4^2 \frac{x_1}{g^2 - x_3^2}$ starts with third order terms, which corresponds to the fact that the invariants $a^{[2],i+j+2}$ vanish for any $1 \leq j \leq 2$ and any $0 \leq i \leq 2 - j$. The only non zero component of $\vec{f}^{[3]}$ is $f_2^{[3]} = x_4^2 P_{2,4}^{[1]}$. Hence $j^* = 2$ and the only, and thus the largest, quadruplet $(i_1, i_2, i_3, i_4)$ of nonnegative integers, satisfying $i_1 + i_2 + i_3 + i_4 = 3$ and such that

\[
\frac{\partial^3 f_2^{[3]}}{\partial x_1^{i_1} \ldots \partial x_4^{i_4}} \neq 0,
\]

is $(i_1, i_2, i_3, i_4) = (1, 0, 0, 2)$. In order to normalize $f_2^{[3]}$, put

\[
\begin{align*}
\hat{x}_i &= a_1 x_i, \quad 1 \leq i \leq 4, \\
\tilde{v} &= a_1 v,
\end{align*}
\]
where \( a_1 = 1/g \). We get the following canonical form for the variable length pendulum

\[
\begin{align*}
\dot{x}_1 &= \dot{x}_2 \\
\dot{x}_2 &= \dot{x}_3 + \dot{x}_4 \frac{\dot{x}_1}{1 - \dot{x}_3^2} \\
\dot{x}_3 &= \dot{x}_4 \\
\dot{x}_4 &= \ddot{v}.
\end{align*}
\]

Independently of the value of the gravity constant \( g \), all systems are feedback equivalent to each other. \( \square \)

## 5 Dual normal form and dual \( m \)-invariants

In the normal form \( \Sigma^{[m]}_{NF} \) given by (3.7), all the components of the control vector field \( g^{[m-1]} \) are annihilated and all non removable nonlinearities are grouped in \( f^{[m]} \). Kang and Krener in their pioneering paper [52] have shown that it is possible to transform, via a homogeneous transformation \( \Gamma^2 \) of degree 2, the homogeneous system

\[ \Sigma^{[2]} : \dot{\xi} = A\xi + Bu + f^{[2]}(\xi) + g^{[1]}(\xi)u \]

to a dual normal form. In that form the components of the drift \( f^{[2]} \) are annihilated while all non removable nonlinearities are, this time, present in \( g^{[1]} \). The aim of this section is to propose, for an arbitrary \( m \), a dual normal form for the system \( \Sigma^{[m]} \) and a dual normal form for the system \( \Sigma^{\infty} \).

Our dual normal form, on the one hand, generalizes, for higher order terms, that given in [52] for second order terms, and, on the other hand, dualizes, the normal form \( \Sigma^{[m]}_{NF} \). The structure of this section will follow that of Section 3: we will give the dual normal form, then we define and study dual \( m \)-invariants, and, finally, we give an explicit construction of transformations bringing the system into its dual normal form. For the proofs of all results contained in this section the reader is sent to [81].

Our first result asserts that we can always bring the homogeneous system \( \Sigma^{[m]} \), given by (3.4), into a dual normal form.

**Theorem 5.1** The homogeneous system \( \Sigma^{[m]} \) is equivalent, via a homogeneous feedback transformations \( \Gamma^m \), to the dual normal form \( \Sigma^{[m]}_{DNF} \) given by

\[
\Sigma^{[m]}_{DNF} : \begin{cases} 
\dot{x}_1 = x_2 \\
\dot{x}_2 = x_3 + vx_n Q^{[m-2]}_{2,n}(x_1, \ldots, x_n) \\
\vdots \\
\dot{x}_j = x_{j+1} + v \sum_{i=n-j+2}^{n} x_i Q^{[m-2]}_{j,i}(x_1, \ldots, x_i) \\
\vdots \\
\dot{x}_{n-1} = x_n + v \sum_{i=3}^{n} x_i Q^{[m-2]}_{j,i}(x_1, \ldots, x_i) \\
\dot{x}_n = v,
\end{cases}
\]

where \( Q^{[m-2]}_{j,i}(x_1, \ldots, x_i) \) are homogeneous polynomials of degree \( m-2 \) depending on the indicated variables.
We will give the dual normal form \( \Sigma^{[2]}_{DNF} \) for homogeneous systems of degree two:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 + v x_n q_{2,n} \\
&\vdots \\
\dot{x}_{n-1} &= x_n + v x_3 q_{n-1,3} + \cdots + v x_n q_{n-1,n} \\
\dot{x}_n &= v,
\end{align*}
\]

where \( q_{j,i} \in \mathbb{R} \).

The following example is a particular case of the system above and help illustrate Theorem 5.1.

**Example 5.2** Consider the system \( \Sigma^{[2]} \) defined in \( \mathbb{R}^3 \) by

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 + \xi_3^2 \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= u.
\end{align*}
\]

It is easy to check that the change of coordinates \( x_1 = \xi_1, \ x_2 = \xi_2 + \xi_3^2, \ x_3 = \xi_3, \) and \( x_4 = \xi_4 \) yields the dual normal form \( (n = 3, q_{2,3} = 2, \ and \ v = u) \)

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 + 2x_3v \\
\dot{x}_3 &= v.
\end{align*}
\]

Now we will define dual \( m \)-invariants. To start with, recall that the homogeneous vector field \( X_i^{[m-1]} \) is defined by taking the homogeneous part of degree \( m - 1 \) of the vector field

\[
X_i^{[m-1]} = (-1)^i ad_{A_{x_i}^+ f^{[m]}}(B + g^{[m-1]}).
\]

By \( X_i^{[m-1]}(\pi_i(\xi)) \) we will denote \( X_i^{[m-1]} \) evaluated at the point \( \pi_i(\xi) = (\xi_1, \ldots, \xi_i, 0, \ldots, 0)^T \) of the subspace

\[
W_i = \{ \xi = (\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n : \xi_{i+1} = \cdots = \xi_n = 0 \}.
\]

Consider the system \( \Sigma^{[m]} \) and for any \( j \), such that \( 2 \leq j \leq n - 1 \), define the polynomial \( b_j^{[m-1]} \) by setting

\[
b_j^{[m-1]} = g_j^{[m-1]} + \sum_{k=1}^{j-1} L_B L_{A_{x_i}^+ f^{[m]}} j \sum_{i=1}^n L_B L_{A_{x_i}^+ f^{[m]}} \int_0^{\xi_i} CX_{n-i}^{[m-1]}(\pi_i(\xi)) d\xi_i.
\]

The homogeneous polynomials \( b_j^{[m-1]} \), for \( 2 \leq j \leq n - 1 \), will be called the dual \( m \)-invariants of the homogeneous system \( \Sigma^{[m]} \).

Consider two systems \( \Sigma^{[m]} \) and \( \tilde{\Sigma}^{[m]} \) of the form (3.4) and (3.5). Let

\[
\{ b_j^{[m-1]} : 2 \leq j \leq n - 1 \}, \quad \text{and} \quad \{ \tilde{b}_j^{[m-1]} : 2 \leq j \leq n - 1 \}
\]

denote, respectively, their dual \( m \)-invariants. The following result dualizes that of Theorem 3.3.

**Theorem 5.3** The dual \( m \)-invariants have the following properties:
(i) two systems $\Sigma^{[m]}$ and $\Sigma^{[\infty]}$ are equivalent via a homogeneous feedback transformation $\Gamma^m$ if and only if

$$b^{[m-1]}_j = b^{[m-1]}_j,$$

for any $2 \leq j \leq n - 1$;

(ii) the dual $m$-invariants $\bar{b}_j^{[m-1]}$ of the dual normal form $\Sigma^{[m]}_{DNF}$, defined by (5.1), are given by

$$\bar{b}_j^{[m-1]}(x) = \sum_{i=n-j+2}^{n} x_i Q^{[m-2]}_{j,i}(x_1, \ldots, x_i),$$

for any $2 \leq j \leq n - 1$.

The above result asserts that the dual $m$-invariants, similarly like $m$-invariants, form a set of complete invariants of the homogeneous feedback transformation. Notice, however, that the same information is encoded in both sets of invariants in different ways.

Like Theorem 3.3 for normal form $\Sigma^{[m]}_{NF}$, Theorem 5.3 shows that the polynomial functions $Q^{[m-2]}_{j,j'}$ defining the dual normal form $\Sigma^{[m]}_{DNF}$ are unique under feedback transformation $\Gamma^m$. The remaining question is how to bring a given system into its dual normal form $\Sigma^{[m]}_{DNF}$. To this end, define the following homogeneous polynomials

$$\phi_1^{[m]} = -\sum_{i=1}^{n} \int_{0}^{\xi} CX^{[m-1]}_{n-i}(\pi_i(\xi))d\xi,$$

$$\phi_j^{[m]} = f_j^{[m]} + L_A\phi_j^{[m]}, \quad 1 \leq j \leq n - 1,$$

$$\alpha^{[m]} = -\left(f_n^{[m]} + L_A\phi_n^{[m]}\right),$$

$$\beta^{[m-1]} = -\left(g_n^{[m-1]} + L_B\phi_n^{[m]}\right).$$

Theorem 5.4 The feedback transformation

$$\Gamma^m : \begin{align*}
x &= \xi + \phi^{[m]}(\xi) \\
u &= v + \alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v,
\end{align*}$$

where $\alpha^{[m]}$, $\beta^{[m-1]}$, and the components $\phi_j^{[m]}$ of $\phi^{[m]}$ are defined by (5.2), brings the system $\Sigma^{[m]}$ into its dual normal form $\Sigma^{[m]}_{DNF}$ given by (5.1).

Now our aim is to dualize the normal form $\Sigma^{[\infty]}_{NF}$. Consider the system $\Sigma^{\infty}$ of the form (4.3) and assume that its linear part $(F,G)$ is controllable.

Consider the system $\Sigma^{\infty}$ of the form (3.1) and recall that we assume the linear part $(F,G)$ to be controllable. Apply successively to $\Sigma^{\infty}$ a series of transformations $\Gamma^m$, $m = 1, 2, 3, \ldots$, such that each $\Gamma^m$ brings $\Sigma^{[m]}$ to its normal form $\Sigma^{[m]}_{DNF}$. More precisely, bring $(F,G)$ into the Brunovský canonical form $(A, B)$ via a linear feedback $\Gamma^1$ and denote $\Sigma^{\infty,1} = \Gamma^1(\Sigma^{\infty})$. Assume that a system $\Sigma^{\infty,m-1}$ has been defined. Let $\Gamma^m$ be a homogeneous feedback transformation transforming $\Sigma^{[m]}$, which is the homogeneous part of degree $m$ of $\Sigma^{\infty,m-1}$, to the dual normal form $\Sigma^{[m]}_{DNF}$. $\Gamma^m$ can be taken, for instance, as the transformations defined by (5.2)). Define $\Sigma^{\infty,m} = \Gamma^m(\Sigma^{\infty,m-1})$. Notice that we apply $\Gamma^m$ to the whole system $\Sigma^{\infty,m-1}$ (and not only to its homogeneous part $\Sigma^{[m]}$). Successive repeating of Theorem 5.4 gives the following dual normal form.
Theorem 5.5 The system $\Sigma^\infty$ can be transformed via a formal feedback transformation $\Gamma^\infty$, into the dual normal form $\Sigma_{DNF}^\infty$ given by

$$\Sigma_{DNF}^\infty : \begin{cases} 
\dot{x}_1 = x_2 \\
\dot{x}_2 = x_3 + vx_nQ_{2,n}(x_1, \ldots, x_n) \\
\vdots \\
\dot{x}_j = x_{j+1} + v \sum_{i=n-j+2}^{n} x_iQ_{j,i}(x_1, \ldots, x_i) \\
\vdots \\
\dot{x}_{n-1} = x_n + v \sum_{i=3}^{n} x_iQ_{j,i}(x_1, \ldots, x_i) \\
\dot{x}_n = v,
\end{cases}$$

(5.3)

where $Q_{j,i}(x_1, \ldots, x_i)$ are formal power series depending on the indicated variables.

6 Dual canonical form

Naturally, similarly to normal forms, a given system can admit different dual normal forms. We are thus interested in constructing a dual canonical form (which would dualize the canonical form $\Sigma_{CF}^\infty$ in the same way as $\Sigma_{DNF}^\infty$ dualizes $\Sigma_{NF}^\infty$). Assuming that the linear part $(F, G)$ of the system $\Sigma^\infty$, of the form (4.3), is controllable, we denote by $m_0$ the degree of the first homogeneous term of the system $\Sigma^\infty$ which cannot be annihilated by a feedback transformation. Thus by Theorems 5.3 and 5.4 (using transformations (5.2)), we can assume, after applying a suitable feedback, that $\Sigma^\infty$ takes the form

$$\Sigma^\infty : \dot{\xi} = A\xi + Bu + \sum_{m=m_0+1}^{\infty} \left( f^{[m]}(\xi) + g^{[m-1]}(\xi)u \right),$$

where $(A, B)$ is in Brunovský canonical form and the first nonvanishing homogeneous vector field $g^{[m_0-1]}$ is in the dual normal form, compare (5.1) with components given by

$$g_j^{[m_0-1]}(\xi) = \begin{cases} 
\sum_{i=n-j+2}^{n} \xi_iQ_{j,i}^{[m_0-2]}(\xi_1, \ldots, \xi_i), & 2 \leq j \leq n-1 \\
0, & j = 1 \text{ or } j = n.
\end{cases}$$

Define

$$j_* = \inf \left\{ j = 2, \ldots, n-1 : g_j^{[m_0-1]}(\xi) \neq 0 \right\}$$

and let $(i_1, \ldots, i_n)$, such that $i_1 + \cdots + i_n = m_0 - 1$, be the largest, in the lexicographic ordering, $n$-tuple of nonnegative integers such that

$$\frac{\partial^{m_0-1} g_j^{[m_0-1]}}{\partial \xi_1^{i_1} \cdots \partial \xi_n^{i_n}} \neq 0.$$

We have the following result:
Theorem 6.1 There exists a formal feedback transformation $\Gamma^\infty$ which brings the system $\Sigma^\infty$ into the following form

$$\Sigma^\infty_{DCF}: \dot{x} = Ax + Bv + \sum_{m=m_0}^{\infty} \bar{g}^{[m-1]}(x)v,$$

where for any $m \geq m_0$, the components $\bar{g}^{[m-1]}_j$ of $\bar{g}^{[m-1]}$ are given by

$$\bar{g}^{[m-1]}_j = \begin{cases} \sum_{i=n-j+2}^{n} x_i Q^{[m-2]}_{j,i}(x_1, \ldots, x_i), & 2 \leq j \leq n - 1 \\ 0, & j = 1 \text{ or } j = n. \end{cases}$$

(6.1)

Moreover,

$$\frac{\partial^{m_0-1} \bar{g}^{[m_0-1]}_j}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} = \pm 1$$

(6.2)

and for any $m \geq m_0 + 1$

$$\frac{\partial^{m_0-1} \bar{g}^{[m_0-1]}_j}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}(x_1, 0, \ldots, 0) = 0.$$  

(6.3)

The form $\Sigma^\infty_{DCF}$, which satisfies (6.1), (6.2) and (6.3), will be called dual canonical form of $\Sigma^\infty$. The name is justified by the following.

Theorem 6.2 Two systems $\Sigma_1^\infty$ and $\Sigma_2^\infty$ are formally feedback equivalent if and only if their dual canonical forms $\Sigma_{1,DCF}^\infty$ and $\Sigma_{2,DCF}^\infty$ coincide.

Example 6.3 Let us consider the system

$$\Sigma : \dot{\xi} = f(\xi) + g(\xi)u, \quad \xi \in \mathbb{R}^3, \quad u \in \mathbb{R},$$

whose linear part is assumed to be controllable. Theorem 5.5 assures that the system $\Sigma$ is formally feedback equivalent to the dual normal form $\Sigma^\infty_{DNF}$ given by

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = x_3 + vx_3Q(x_1, x_2, x_3),$$

$$\dot{x}_3 = v,$$

where $Q(x_1, x_2, x_3)$ is a formal power series of the variables $x_1, x_2, x_3$.

Assume for simplicity that $m_0 = 2$, which is equivalent to the condition: $g$, $ad fg$, and $[g, ad fg]$ linearly independent at $0 \in \mathbb{R}^3$. This implies that we can represent $Q = Q(x_1, x_2, x_3)$, as

$$Q = c + x_1Q_1(x_1) + x_2Q_2(x_1, x_2) + x_3Q_3(x_1, x_2, x_3),$$

where $c \in \mathbb{R}$, $c \neq 0$.

Observe that any $Q$ of the above form gives a dual normal form $\Sigma^\infty_{DNF}$. In order to get the dual canonical form we use Theorem 6.1, which assures that the system $\Sigma$ is formally feedback equivalent to its dual canonical form $\Sigma^\infty_{DCF}$ defined by

$$\dot{\tilde{x}}_1 = \tilde{x}_2,$$

$$\dot{\tilde{x}}_2 = \tilde{x}_3 + \tilde{v}\tilde{x}_3\tilde{Q}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3),$$

$$\dot{\tilde{x}}_3 = \tilde{v},$$
where \( \tilde{Q}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \) is a formal power series such that
\[
\tilde{Q}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = 1 + \tilde{x}_2 \tilde{Q}_2(\tilde{x}_1) + \tilde{x}_3 \tilde{Q}_3(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3).
\]

This ends the example. \( \Box \)

7 Normal forms for single-input systems with uncontrollable linearization

7.1 Introduction

In Section 3 we presented the normal forms \( \Sigma^{[m]}_{NF} \) and \( \Sigma^\infty_{NF} \) of the system \( \Sigma \) whose linearization (that is, linear approximation) is controllable. In this section we will deal with system with uncontrollable linearization. A normal form for homogeneous systems of degree 2, with uncontrollable linearization, was proposed by Kang [49]. The normal form presented in this section was obtained by the authors ([79] and [83]) and it generalizes, on the one hand, the normal form of Kang [49] (uncontrollable linearization) and, on the other hand, the normal form \( \Sigma^{[m]}_{NF} \) (controllable linearization) also obtained by Kang and presented in Section 3. Another normal form for single-input systems with uncontrollable linearization has also been proposed by Krener, Kang, and Chang [56], [57] and [58], and by the authors [76], which differ from ours by another definition of homogeneity (we consider the latter with respect to the linearly controllable variables while theirs is respect to all variables); see Example 7.5 illustrating this.

7.2 Taylor series expansions

All objects, that is, functions, maps, vector fields, control systems, etc., are considered in a neighborhood of \( 0 \in \mathbb{R}^n \) and assumed to be \( C^\infty \)-smooth. Consider the single-input system
\[
\Sigma : \dot{\xi} = f(\xi) + g(\xi)u, \quad \xi \in \mathbb{R}^n, \quad u \in \mathbb{R}.
\]

We assume throughout this section that \( f(0) = 0 \) and \( g(0) \neq 0 \). Let
\[
\Sigma^{[1]} : \dot{\xi} = F\xi + Gu,
\]
where \( F = \frac{\partial f}{\partial \xi}(0) \) and \( G = g(0) \), be the linear approximation of the system around the equilibrium point \( 0 \in \mathbb{R}^n \). If the linear approximation is not controllable, which is the case studied in this section, then there exists a positive integer \( r \in \mathbb{N} \) such that
\[
\text{rank} (G, FG, \ldots, F^{n-1}G) = n - r.
\]

Moreover, there exist coordinates \( \xi = (\xi_1, \ldots, \xi_r, \xi_{r+1}, \ldots, \xi_n)^T \) of \( \mathbb{R}^r \times \mathbb{R}^{n-r} \) in which the pair \((F, G)\) admits the following Kalman decomposition
\[
A = \begin{pmatrix} F_1 & 0 \\ F_3 & F_2 \end{pmatrix}_{n \times n} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ G_2 \end{pmatrix}_{n \times 1},
\]
where the pair \((F_2, G_2)\) is controllable. Throughout this section \( r \) will stand for the dimension of the uncontrollable part of the linear approximation of the system and \( \xi = (\xi_1, \ldots, \xi_r, \xi_{r+1}, \ldots, \xi_n)^T \) will denote coordinates defining the Kalman decomposition.
We will use the notation \( C_0^\infty(\mathbb{R}^r) \) for the space of germs at \( 0 \in \mathbb{R}^r \) of smooth \( \mathbb{R} \)-valued functions of \( \xi = (\xi_1, \ldots, \xi_r)^T \in \mathbb{R}^r \). By \( \mathbb{R}[[\xi_1, \ldots, \xi_r]] \) we will denote the space of formal power series of \( \xi_1, \ldots, \xi_r \) with coefficients in \( \mathbb{R} \).

Let \( h \) be a smooth \( \mathbb{R} \)-valued function defined in a neighborhood of \( 0 \times 0 \in \mathbb{R}^r \times \mathbb{R}^{n-r} \). By

\[
h(\xi) = h^{[0]}(\xi) + h^{[1]}(\xi) + h^{[2]}(\xi) + \cdots = \sum_{m=0}^{\infty} h^{[m]}(\xi)
\]

we denote its Taylor series expansion with respect to \( (\xi_{r+1}, \ldots, \xi_n)^T \) at \( 0 \in \mathbb{R}^r \times \mathbb{R}^{n-r} \), where \( h^{[m]}(\xi) \) stands for a homogeneous polynomial of degree \( m \) of the variables \( \xi_{r+1}, \ldots, \xi_n \) whose coefficients are in \( C_0^\infty(\mathbb{R}^r) \).

Similarly, throughout this section, for a map \( \phi \) of an open subset of \( \mathbb{R}^r \times \mathbb{R}^{n-r} \) to \( \mathbb{R}^r \times \mathbb{R}^{n-r} \) (resp. for a vector field \( f \) on an open subset of \( \mathbb{R}^r \times \mathbb{R}^{n-r} \)) we will denote by \( \phi^{[m]} \) (resp. by \( f^{[m]} \)) the term of degree \( m \) of its Taylor series expansion with respect to \( (\xi_{r+1}, \ldots, \xi_n)^T \) at \( 0 \in \mathbb{R}^r \times \mathbb{R}^{n-r} \), that is, each component \( \phi_j^{[m]} \) of \( \phi^{[m]} \) (resp. \( f_j^{[m]} \) of \( f^{[m]} \)) is a homogeneous polynomial of degree \( m \) of the variables \( \xi_{r+1}, \ldots, \xi_n \) whose coefficients are in \( C_0^\infty(\mathbb{R}^r) \).

Consider the Taylor series expansion of the system \( \Sigma \) given by

\[
\Sigma^\infty : \dot{\xi} = F\xi + Gu + f^{[0]}(\xi) + \sum_{n=1}^{\infty} (f^{[m]}(\xi) + g^{[m-1]}u).
\]

Notice that, although we assume \( f(0) = 0 \), the term \( f^{[0]}(\xi) \) is present because the degree is computed with respect to the variables \( \xi_{r+1}, \ldots, \xi_n \) only and thus \( f^{[0]} \) is, in general, a function of \( \xi_1, \ldots, \xi_r \).

Consider also the Taylor series expansion \( \Gamma^\infty \) of the feedback transformation \( \Gamma \) given by

\[
\begin{align*}
x &= T\xi + \sum_{m=0}^{\infty} \phi^{[m]}(\xi) \\
\Gamma^\infty : \quad u &= K\xi + Lv + \alpha^{[0]}(\xi) + \sum_{m=1}^{\infty} (\alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v),
\end{align*}
\]

where \( T \) is an invertible matrix and \( L \neq 0 \).

The method proposed by Kang and Krener is to study the action of \( \Gamma^\infty \) on the system \( \Sigma^\infty \) step by step, that is, to analyze successively the action of the homogeneous parts of \( \Gamma^\infty \) on the homogeneous parts, of the same degree, of \( \Sigma^\infty \). Notice, however, that in their approach, Kang and Krener consider Taylor series expansions with respect to all state variables \( \xi_1, \ldots, \xi_n \) of the system and, as a consequence, homogeneity is considered with respect to all variables \( \xi_1, \ldots, \xi_n \). Following our approach [79], [83] we propose a slight modification of that homogeneity. In view of a different nature of the controllable and uncontrollable parts of the linear approximation, we consider Taylor series expansions with respect to the linearly controllable variables \( \xi_{r+1}, \ldots, \xi_n \) only. This leads to considering as homogeneous parts of the system and of the feedback transformations, according to our definition, terms that are polynomial with respect to \( \xi_{r+1}, \ldots, \xi_n \) with smooth coefficients depending on \( \xi_1, \ldots, \xi_r \). When analyzing the action of a homogeneous transformation \( \Gamma^m \) (understood as homogeneity with respect to the controllable variables) on the system \( \Sigma^\infty \), we can notice three undesirable phenomena (see Section 7.5) that are not present in the action of homogeneous transformations in the controllable case (where homogeneity is
considered with respect to all variables). In order to deal with those problems caused by the presence of the uncontrollable linear part, we will introduce, in Section 7.5, different weights for the components corresponding to the controllable and uncontrollable parts.

7.3 Linear part and resonances

Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r \) be the set of eigenvalues associated to the uncontrollable part of the linear system

\[
\dot{\xi} = F\xi + Gu.
\]

By a linear feedback transformation

\[
\Gamma^1 : \quad x = T\xi \quad u = K\xi + Lv
\]

it is always possible to bring the linear system into the following Jordan-Brunovský canonical form

\[
A = \begin{pmatrix} J & 0 \\ 0 & A_2 \end{pmatrix}_{n \times n} \text{ and } B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}_{n \times 1},
\]

where \( J \) is the Jordan canonical form of dimension \( r \) and \( (A_2, B_2) \) the Brunovský canonical form of dimension \( n - r \). In the case when all eigenvalues \( \lambda_i \) are real, we have

\[
J = \begin{pmatrix} \lambda_1 & \sigma_2 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \sigma_r \\ 0 & \cdots & 0 & \lambda_r \end{pmatrix}_{r \times r}, \quad \sigma_i \in \{0, 1\}.
\]

In the case of complex eigenvalues, we replace in \( J \) the eigenvalue \( \lambda_j \) by the matrix

\[
\Lambda_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}_{2 \times 2},
\]

where \( \lambda_j = \alpha_j + i\beta_j \), and we replace the integer \( \sigma_j \in \{0, 1\} \) either by the matrix

\[
\Xi_j = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{2 \times 2} \text{ or } \Xi_j = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2 \times 2}.
\]

Recall from Section 2 the notion of a resonant eigenvalue and the resonant set associated with it.

**Definition 7.1** An eigenvalue \( \lambda_j \) is called resonant if there exists a \( r \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r \) of nonnegative integers, satisfying \( |\alpha| = \alpha_1 + \cdots + \alpha_r \geq 2 \), such that

\[
\lambda_j = \lambda_1\alpha_1 + \cdots + \lambda_r\alpha_r.
\]

For each eigenvalue \( \lambda_j \), where \( 1 \leq j \leq r \), we define

\[
\mathcal{R}_j = \{ \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r : |\alpha| \geq 2 \text{ and } \lambda_j = \lambda_1\alpha_1 + \cdots + \lambda_r\alpha_r \},
\]

which is called the resonant set associated to \( \lambda_j \).
7.4 Notations and definitions

The method described in Section 3 (proposed by Kang and Krener [52], and then followed by Kang [48], [49] and by the authors [77], [81]) is to analyze step by step the action of the transformation $\Gamma^\infty$ on the system $\Sigma^\infty$. In the controllable case, it consists of bringing the linear part of the system into the Brunovský canonical form and then applying, step by step, homogeneous feedback transformations of the form

$$\Gamma^m : \begin{align*}
x &= \xi + \phi^{[m]}(\xi) \\
u &= v + \alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)u
\end{align*}$$

in order to normalize the homogeneous part of degree $m$ of the system. The advantage of this method, in the controllable case, follows from the fact that homogeneous transformations $\Gamma^m$ leave invariant all terms of degree smaller than $m$ of the system. The situation gets very different in the uncontrollable case with the modified notion of homogeneity. In order to see the difference, we analyze the action of $\Gamma^m$ on the system $\Sigma^\infty$, given by (7.1). Let

$$\tilde{\Sigma}^\infty : \dot{x} = Fx + Gv + \tilde{f}^{[0]}(x) + \sum_{m=1}^{\infty} (\tilde{f}^{[m]}(x) + \tilde{g}^{[m-1]}(x)v)$$

be the system $\Sigma^\infty$ transformed by $\Gamma^m$. Recall that for both systems, the first $r$ components of the state correspond to the uncontrollable part and that the degree of homogeneity (for all homogeneous feedback transformations of the form $\Gamma^m$) is computed with respect to the controllable variables, that is, the last $n - r$ variables only. We can observe the three following undesirable phenomena.

Firstly, notice that the homogeneous transformation $\Gamma^m$ does not preserve homogeneous (with respect to linearly controllable variables) terms of degree smaller than $m$. It only preserves terms of degree smaller than $m - 1$, that is, $\tilde{f}^{[k]} = f^{[k]}$ and $\tilde{g}^{[k-1]} = g^{[k-1]}$, for any $0 \leq k \leq m - 2$, while transforms those of degree $m - 1$ as follows

$$\tilde{g}^{[m-2]} = g^{[m-2]} \quad \text{and} \quad \tilde{f}^{[m-1]} = f^{[m-1]} + \sum_{i=r+1}^{n} \partial \phi^{[m]} \partial \xi_i \tilde{f}^{[0]}.$$

Notice that, when comparing the homogeneous parts of the same degree $k$ of two systems, we have to compare the homogeneous parts of degree $k$ of the drifts and the homogeneous parts of degree $k - 1$ of control vector fields. Indeed, since the homogeneity is considered with respect to the state and the control, the homogeneous part of degree $k$ of the system is represented by the homogeneous part $f^{[k]}$, of degree $k$, of the drift and the homogeneous part $g^{[k-1]}$, of degree $k - 1$, of the control vector field multiplied by the control $u$.

Secondly, $\Gamma^m$ transforms the homogeneous part $(f^{[m]}, g^{[m-1]})$, of degree $m$, not according to a homological equation, but in such a way that

$$\begin{align*}
\tilde{f}^{[m]} &= f^{[m]} + [A\xi, \phi^{[m]}] + B\alpha^{[m]} + \sum_{i=r+1}^{n} \left( \partial \phi^{[m]} \partial \xi_i f^{[1]} - \partial f^{[1]} \partial \xi_i \phi^{[m]} \right) \\
&\quad + \sum_{i=1}^{r} \left( \partial \phi^{[m]} \partial \xi_i \tilde{f}^{[0]} - \partial \tilde{f}^{[0]} \partial \xi_i \phi^{[m]} \right) + g^{[0]} \alpha^{[m]} \\
\tilde{g}^{[m-1]} &= g^{[m-1]} + ad_B \phi^{[m]} + B\beta^{[m-1]} + g^{[0]} \beta^{[m-1]} + \sum_{i=r+1}^{n} \partial \phi^{[m]} \partial \xi_i \tilde{g}^{[0]}.
\end{align*}$$

$$37$$
Thirdly, the Lie bracket of two homogeneous vector fields \( f^{[m]} \) and \( g^{[k]} \) is given by
\[
[f^{[m]}, g^{[k]}](\xi) = \sum_{i=1}^{n} \left( \frac{\partial g^{[k]}}{\partial \xi_i} f^{[m]}_i(\xi) - \frac{\partial f^{[m]}}{\partial \xi_i} g^{[k]}_i(\xi) \right)
\]
and thus fails, in general, to be homogeneous of degree \( m + k - 1 \) because the terms \( \frac{\partial g^{[k]}}{\partial \xi_i} f^{[m]}_i(\xi) \) and \( \frac{\partial f^{[m]}}{\partial \xi_i} g^{[k]}_i(\xi) \), for \( 1 \leq i \leq r \), are, in general, homogeneous of degree \( m + k \).

Those three inconveniences are caused only by the fact that differentiating with respect to the variables \( \xi_1, \ldots, \xi_r \) does not decrease the degree (in particular, by the presence of terms of degree 0 with respect to the variables \( \xi_{r+1}, \ldots, \xi_n \)). To overcome this, we define, for any \( m \geq 0 \),
\[
f^{(m)} = \left( f_1^{[m-1]}, \ldots, f_r^{[m-1]}, f_{r+1}^{[m]}, \ldots, f_n^{[m]} \right)^T,
\]
\[
g^{(m)} = \left( g_1^{[m-1]}, \ldots, g_r^{[m-1]}, g_{r+1}^{[m]}, \ldots, g_n^{[m]} \right)^T,
\]
\[
\phi^{(m)} = \left( \phi_1^{[m-1]}, \ldots, \phi_r^{[m-1]}, \phi_{r+1}^{[m]}, \ldots, \phi_n^{[m]} \right)^T,
\]
where, for any \( 1 \leq i \leq r \), we set \( f_i^{[-1]} = g_i^{[-1]} = \phi_i^{[-1]} = 0 \).

Control systems, vector fields, feedback transformations, etc., that are homogeneous with respect to the above defined weights, will be called \textit{weighted homogeneous}. Notice that just defined weighted homogeneity is related with the decomposition of the state-space \( \mathbb{R}^n \) into uncontrollable and controllable parts and therefore it does not apply to applications with values in \( \mathbb{R} \). In particular, for real-valued homogeneous polynomials \( h^{[m]} \) we will write \( h^{(m)} = h^{[m]} \).

We will denote by \( P^{[m]}(\xi) \) the space of homogeneous polynomials of degree \( m \) of the variables \( \xi_{r+1}, \ldots, \xi_n \) (with coefficients depending on \( \xi_1, \ldots, \xi_r \)) and by \( P^{\geq m}(\xi) \) the space of formal power series of the variables \( \xi_{r+1}, \ldots, \xi_n \) (with coefficients depending on \( \xi_1, \ldots, \xi_r \)) starting from terms of degree \( m \).

We will denote by \( V^{(m)}(\xi) \) the space of weighted \( (m) \)-homogeneous vector fields, that is, the space of vector fields whose first \( r \) components are in \( P^{[m-1]}(\xi) \) and the last \( n-r \) components are in \( P^{[m]}(\xi) \). Moreover, \( V^{(\geq m)}(\xi) \) will denote the space of vector fields formal power series whose first \( r \) components are in \( P^{\geq m-1}(\xi) \) and the last \( n-r \) components are in \( P^{\geq m}(\xi) \).

### 7.5 Weighted homogeneous systems

Applying a linear feedback transformation, we can bring the linear approximation \((F, G)\) of the system into Jordan-Brunovský canonical form \((A, B)\), that is, the uncontrollable part, of dimension \( r \), is in the Jordan form and the controllable part, of dimension \( n-r \), in the Brunovský form. Notice, however, that contrary to the controllable case (where there are no zero degree terms while the terms of degree one are just linear terms that we bring to the Brunovský canonical form), after having normalized linear terms of an uncontrollable system, we are still left with weighted homogeneous terms of degree zero and one. We can normalize them as follows.

**Proposition 7.2** Consider the system
\[
\Sigma^{(\leq 1)} : \dot{\xi} = A\xi + Bu + f^{(0)}(\xi) + f^{(1)}(\xi) + g^{(0)}(\xi)u,
\]
where \((A, B)\) is in the Jordan-Brunovský canonical form.
(i) There exists a smooth feedback transformation of the form
\[
\Gamma^{(\leq 1)} : \begin{align*}
x &= \xi + \phi^{(0)}(\xi) + \phi^{(1)}(\xi) \\
u &= v + \alpha^{(0)}(\xi) + \alpha^{(1)}(\xi) + \beta^{(0)}(\xi)v,
\end{align*}
\] (7.3)
which takes the system \(\Sigma^{(\leq 1)}\) into the system
\[
\hat{\Sigma}^{(\leq 1)} : \dot{x} = Ax + Bv + \tilde{f}^{(1)}(x),
\] (7.4)
modulo terms in \(V^{(\geq 2)}\), where the vector field \(\tilde{f}^{(1)}\) satisfies \(\tilde{f}^{(1)}_{j} = 0\) for \(r + 1 \leq j \leq n\).

(ii) Assume that all eigenvalues \(\lambda_{1}, \ldots, \lambda_{r}\) of the Jordan-Brunovský canonical form \((A, B)\) are real and distinct (in particular all \(\sigma_{j} = 0\)). Then a formal feedback transformation \(\Gamma^{(1)}\) of the form (7.3), which takes the system \(\Sigma^{(\leq 1)}\) into the system \(\hat{\Sigma}^{(\leq 1)}\) defined by (7.4), can be chosen in such a way that the vector field \(\tilde{f}^{(1)}\) is replaced by \(\hat{f}^{(1)}_{j}(x)\) whose components are given by
\[
\hat{f}^{(1)}_{j}(x) = \begin{cases} 
\sum_{m \in \mathbb{N}^{r}} \gamma_{j}^{m}x_{1}^{\alpha_{1}} \cdots x_{r}^{\alpha_{r}} & \text{if } 1 \leq j \leq r \\
0 & \text{if } r + 1 \leq j \leq n.
\end{cases}
\] (7.5)
The existence of a transformation \(x = \xi + \phi^{(1)}(\xi)\) yielding the form of \(\hat{f}^{(1)}_{j}(x)\) in (ii) is an immediate consequence of Theorem 2.3.

Now we will study the action of the weighted homogeneous feedback
\[
\Gamma^{(m)} : \begin{align*}
x &= \xi + \phi^{(m)}(\xi) \\
u &= v + \alpha^{(m)}(\xi) + \beta^{(m-1)}(\xi)v
\end{align*}
\] (7.6)
on the following weighted homogeneous system
\[
\Sigma^{(m)} : \dot{\xi} = A\xi + Bu + \tilde{f}^{(1)}(\xi) + f^{(m)}(\xi) + g^{(m-1)}(\xi)u,
\] (7.7)
where, because of Proposition 7.2, we assume that \(\tilde{f}^{(1)}\) is of the form (7.5). Notice that a weighted homogeneous feedback is a smooth feedback, actually, it depends polynomially on the variables \(\xi_{r+1}, \ldots, \xi_{n}\) and smoothly on the variables \(\xi_{1}, \ldots, \xi_{r}\).

Consider another weighted homogeneous system \(\hat{\Sigma}^{(m)}\) given by
\[
\hat{\Sigma}^{(m)} : \dot{x} = Ax + Bv + \tilde{f}^{(1)}(x) + \hat{f}^{(m)}(x) + \hat{g}^{(m-1)}(x)v,
\] (7.8)
where \(\tilde{f}^{(1)} = \tilde{f}^{(1)}\). We will say that \(\Gamma^{(m)}\) transforms \(\Sigma^{(m)}\) into \(\hat{\Sigma}^{(m)}\), and we will denote it by \(\Gamma^{(m)}(\Sigma^{(m)}) = \hat{\Sigma}^{(m)}\), if \(\Gamma^{(m)}\) transforms \(\Sigma^{(m)}\) into
\[
\dot{x} = Ax + Bv + \tilde{f}^{(1)}(x) + \hat{f}^{(m)}(x) + \hat{g}^{(m-1)}(x)v + R^{(\geq m+1)}(x, v),
\]
where \(R^{(\geq m+1)}(x, v) \in V^{(\geq m+1)}(x, v)\).

Recall that \(\lambda_{j}\), for \(1 \leq j \leq r\), are the eigenvalues of the uncontrollable part of the linear approximation and that \(\sigma_{j}\) for \(2 \leq j \leq r\), define the corresponding Jordan form (see Section 7.3). We define additionally \(\sigma_{r+1} = 0\) and for any \(r + 1 \leq j \leq n - 1\), we put \(\lambda_{j} = 0\) and \(\sigma_{j+1} = 1\).

Analysis of weighted homogeneous systems is based on the following result which generalizes, to the uncontrollable case, that proved by Kang [48] (and recalled in Proposition 3.1):
Proposition 7.3 For any $m \geq 2$, the feedback transformation $\Gamma^{(m)}$, of the form (7.6), brings the system $\Sigma^{(m)}$, given by (7.7), into $\tilde{\Sigma}^{(m)}$, given by (7.8), if and only if the following relations hold for any $1 \leq j \leq n - 1$:

\[
\begin{align*}
L_{A\xi + f^{(1)}}\phi_j^{(m)} - \lambda_j\phi_j^{(m)} - \sigma_{j+1}\phi_{j+1}^{(m)} &= \tilde{f}_j^{(m)} - f_j^{(m)}, \\
L_B\phi_j^{(m)} &= \tilde{g}_j^{(m-1)} - g_j^{(m-1)}, \\
L_{A\xi + f^{(1)}}\phi_n^{(m)} + \alpha^{(m)} &= \tilde{f}_n^{(m)} - f_n^{(m)}, \\
L_B\phi_n^{(m)} + \beta^{(m-1)} &= \tilde{g}_n^{(m-1)} - g_n^{(m-1)}
\end{align*}
\] (7.9)

This proposition can be viewed as the *weighted control homological equation* for systems with uncontrollable linearization (its proof follows the same line as that of Kang [48] for the standard control homological equation (CHE)). Once again, the solving of a system of 1-st order partial differential equations may be avoided if we perform the analysis step by step, and thus reduce the feedback equivalence of two systems $\Sigma^{(m)}$ and $\tilde{\Sigma}^{(m)}$ (with uncontrollable linearization) to solving the algebraic system (7.9).

The following result gives our normal form for weighted homogeneous systems with uncontrollable linearization. Recall the notation $\pi_i(x) = (x_1, \ldots, x_i)$.

Theorem 7.4 For any $m \geq 2$, there exists a feedback transformation $\Gamma^{(m)}$, that transforms the weighted homogeneous system $\Sigma^{(m)}$, given by (7.7), into its weighted homogeneous normal form

\[
\Sigma^{(m)}_{NF} : \dot{x} = Ax + Bv + \bar{f}^{(1)}(x) + \bar{f}^{(m)}(x),
\] (7.10)

where $\bar{f}^{(1)}(x)$ is given by Proposition 7.3 (ii) and the vector field $\bar{f}^{(m)}(x)$ satisfies

\[
\bar{f}_j^{(m)}(x) = \begin{cases} 
 x_r^{m-1}S_{j,m}(\pi_r(x)) + \sum_{i=r+2}^n x_i^2Q_{j,i}^{(m-3)}(\pi_i(x)) & \text{if } 1 \leq j \leq r, \\
 \sum_{i=j+2}^n x_i^2P_{j,i}^{(m-2)}(\pi_i(x)) & \text{if } r + 1 \leq j \leq n - 2, \\
 0 & \text{if } n - 1 \leq j \leq n,
\end{cases}
\] (7.11)

where $S_{j,m}(\pi_r(x)) \in C_0^\infty(\mathbb{R}_r)$ are $C^\infty$-functions of the variables $x_1, \ldots, x_r$ and the functions $P_{j,i}^{(m-2)}$ and $Q_{j,i}^{(m-3)}$ are homogeneous polynomials, respectively of degree $m-2$ and $m-3$, of the variables $x_{r+1}, \ldots, x_i$, with coefficients in $C_0^\infty(\mathbb{R}_i)$.

The proof of Theorem 7.4 is based on Theorem 7.7, stated in Section 7.7, which explicitly gives transformations bringing $\Sigma^{(m)}$ into its normal form $\Sigma_{NF}^{(m)}$. The above normal form generalizes to the uncontrollable case the normal form $\Sigma_{NF}^{(m)}$ of Kang [48] for systems with controllable linearization, which we stated in Theorem 3.2. It can also be viewed as a generalization of the normal form obtained by Kang [49] in the uncontrollable case for second order terms. Other normal forms, for systems with uncontrollable linearization, have been obtained in [76], and in [56], [57] and [58] (for third order terms). Notice, however, that the latters coincide neither with our normal form $\Sigma_{NF}^{(2)}$ nor $\Sigma_{NF}^{(3)}$ because of different weights we use, as we explain in the following example.
Example 7.5 Consider the system $\dot{\xi} = f(\xi) + g(\xi) u$ on $\mathbb{R}^3$ and assume that the dimension of the linearly controllable subsystem is 2 and that the linear part is in the Jordan-Brunovský canonical form. The homogenous system $\Sigma^{[2]}$ (homogeneity being calculated with respect to all variables $\xi_1, \xi_2, \xi_3$)

\[
\begin{align*}
\dot{\xi}_1 &= \lambda \xi_1 + f_1^{[1]}(\xi) + g_1^{[1]}(\xi) u \\
\dot{\xi}_2 &= \xi_3 + f_2^{[1]}(\xi) + g_2^{[1]}(\xi) u \\
\dot{\xi}_3 &= u + f_3^{[2]}(\xi) + g_3^{[1]}(\xi) u.
\end{align*}
\]

The linearly uncontrollable subsystem is of dimension one (with $\xi_1$ being the linearly uncontrollable variable and $(\xi_2, \xi_3)^T$ being the linearly controllable variables) and the resonant set associated with the eigenvalue $\lambda$ is empty if and only if $\lambda \neq 0$. Kang [49] proved that $\Sigma^{[2]}$ is equivalent via a homogeneous feedback $\Gamma$ to the following normal form $\Sigma^{[2]}_{NF}$:

\[
\begin{align*}
\dot{x}_1 &= \lambda x_1 + \gamma_2 x_1^2 + x_2 s_{1,2}(x_1) + x_3^2 q_{1,3} \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u
\end{align*}
\]

where $\gamma_2 = 0$ if $\lambda \neq 0$ (no resonances) and $\gamma_2 \in \mathbb{R}$ if $\lambda = 0$. Moreover, $s_{1,2}$ is a linear function of $x_1$ and $q_{1,3}$ is a constant.

Now we will compare this normal form with the normal forms $\Sigma^{[3]}_{NF}$ and $\Sigma^{[3]}_{NF}$. To this end, we start with

\[
\begin{align*}
\Sigma^{(1)}: \quad &\dot{\xi}_1 = \lambda \xi_1 + f_1^{(1)}(\xi) \\
&\dot{\xi}_2 = \xi_3 + f_2^{(1)}(\xi) \\
&\dot{\xi}_3 = u + f_3^{(1)}(\xi),
\end{align*}
\]

where $f_2^{(1)}(\xi)$ and $f_3^{(1)}(\xi)$ are linear functions with respect to $\xi_2$ and $\xi_3$ with coefficients that are arbitrary functions of $\xi_1$ while $f_1^{(1)}(\xi)$ is an arbitrary function of $\xi_1$ whose development at zero starts with quadratic terms. Now Proposition 7.2 (i) implies that we can annihilate $f_2^{(1)}$ and $f_3^{(1)}$. If $\lambda \neq 0$, then there are no resonances so we can also annihilate $f_1^{(1)}$ and without loss of generality we can assume that $f_1^{(1)}(\xi)$ is in the normal form $\tilde{f}_1^{(1)}(\xi) = 0$ assured by Proposition 7.2. Otherwise, that is, if $\lambda = 0$ then all terms of $f_1^{(1)}$ are resonant so we can assume that $f_1^{(1)}(\xi)$ is in the normal form $\tilde{f}_1^{(1)}(\xi) = (\sum_{i=2}^{\infty} \gamma_i \xi_1^i) \frac{\partial}{\partial \xi_1}$, where $\gamma_i \in \mathbb{R}$. Actually, the vector field $\left(\sum_{i=2}^{\infty} \gamma_i \xi_1^i\right) \frac{\partial}{\partial \xi_1}$ can be normalized by a local diffeomorphism around $\xi_1 = 0$ into the form $(\gamma_p x_1^{p} + \gamma_{2p-1} x_1^{2p-1}) \frac{\partial}{\partial x_1}$, but we will not use it.

Consider the system

\[
\Sigma^{(2)}: \quad \dot{\xi} = A\xi + B v + \tilde{f}_1^{(1)}(\xi) + f_2^{(2)}(\xi) + g_1^{(1)}(\xi) v,
\]

where $(A, B)$ is in the Jordan-Brunovský canonical form and $\tilde{f}_1^{(1)}(\xi)$ is in the normal form described above, that is $A\xi + \tilde{f}_1^{(1)}(\xi) + B u = (\lambda \xi_1 + \tilde{f}_1^{(1)}(\xi)) \frac{\partial}{\partial \xi_1} + \xi_3 \frac{\partial}{\partial \xi_2} + u \frac{\partial}{\partial \xi_3}$, where $f_1^{(1)}(\xi)$ equals 0 or $\sum_{i=2}^{\infty} \gamma_i \xi_1^i$ (depending on $\lambda$). Moreover, the components $f_2^{(2)}(\xi)$ and $f_3^{(2)}(\xi)$ are quadratic functions of $\xi_2, \xi_3$, the components $f_1^{(2)}(\xi)$, $g_2^{(1)}(\xi)$ and $g_3^{(1)}(\xi)$ are linear functions of $\xi_2, \xi_3$ (all coefficients depending on $\xi_1$) and $g_1^{(1)}(\xi)$ depends on $\xi_1$ only. By a weighted homogenous (of
degree 2 with respect to \( \xi_2, \xi_3 \) feedback transformation \( \Gamma^{(2)} \) we can bring \( \Sigma^{(2)} \) into the normal form

\[
\begin{align*}
\dot{x}_1 &= \lambda x_1 + \bar{f}_1^{(1)}(x) + x_2 S_{1,2}(x_1) \\
\Sigma^{(2)}_{NF} : \quad &\dot{x}_2 = x_3 \\
&\dot{x}_3 = u,
\end{align*}
\]

where \( S_{1,2} \) is an arbitrary function of \( x_1 \).

Now we will consider

\[
\Sigma^{(3)} : \quad \dot{x}_1 = A \xi + Bu + \bar{f}_1^{(1)}(\xi) + f^{(3)}(\xi) + g^{(2)}(\xi)u,
\]

where \((A, B)\) is in the Jordan-Brunovský canonical form and \( \bar{f}_1^{(1)} \) is in the normal form described above. Moreover, the components \( f_2^{(3)}(\xi) \) and \( f_3^{(3)}(\xi) \) are cubic functions of \( \xi_2, \xi_3 \), the components \( f_1^{(3)}(\xi), g_2^{(2)}(\xi) \) and \( g_3^{(2)}(\xi) \) are quadratic functions of \( \xi_2, \xi_3 \), and \( g_1^{(2)}(\xi) \) is a linear function of \( \xi_2, \xi_3 \) (all coefficients depending on \( \xi_1 \)). By a weighted homogeneous (of degree 3 with respect to \( \xi_2, \xi_3 \)) feedback transformation \( \Gamma^{(3)} \) we can bring \( \Sigma^{(3)} \) into the normal form

\[
\begin{align*}
\dot{x}_1 &= \lambda x_1 + \bar{f}_1^{(1)}(x) + x_2^2 S_{1,3}(x_1) + x_3^2 Q_{1,3}(x_1) \\
\Sigma^{(3)}_{NF} : \quad &\dot{x}_2 = x_3 \\
&\dot{x}_3 = v,
\end{align*}
\]

where \( S_{1,3} \) and \( Q_{1,3} \) are arbitrary functions of \( x_1 \).

Observe that the term \( x_2 S_{1,2}(x_1) \) of the Kang normal form \( \Sigma^{[2]}_{NF} \), where \( s_{1,2}(x_1) \) is linear, shows up as the first term in the development of \( x_2 S_{1,2}(x_1) \) of \( \Sigma^{(2)}_{NF} \), while the term \( x_3^2 q_{1,3}(x_1) \), where \( q_{1,3} \) is constant, shows up as the first term in the development of \( x_3^2 Q_{1,3}(x_1) \) but both \( S_{1,2}(x_1) \) and \( Q_{1,3}(x_1) \) contain, in general, terms of arbitrary degrees (except for constant terms in \( S_{1,2} \)). This example illustrate mutual differences between the form \( \Sigma^{[2]}_{NF} \) of Kang and of ours \( \Sigma^{(2)}_{NF} \) and \( \Sigma^{(3)}_{NF} \).

### 7.6 Weighted homogeneous invariants

In this section, we will define weighted homogeneous invariants \( a^{(m),i+2} \) of weighted homogeneous systems \( \Sigma^{(m)} \) under weighted homogeneous feedback transformations \( \Gamma^{(m)} \) and we will state for them a result of the authors [79], [83] generalizing, to the uncontrollable case, a result established by Kang [48] in the controllable case.

Consider the weighted homogeneous system (7.7). For any \( i \geq 0 \), let us define the vector field

\[
X_i^{m-1}(\xi) = (-1)^i \lambda a^{(m)_i+1}(\xi) + g^{(m-1)}(\xi)
\]

and let \( X_i^{(m-1)} \) be the homogeneous part of degree \( m - 1 \) of \( X_i^{m-1} \). It means that the first \( r \) components are homogeneous of degree \( m - 2 \) and the last \( n - r \) components homogeneous of degree \( m - 1 \) with respect to the variables \( \xi_{r+1}, \ldots, \xi_n \). One can easily check that

\[
X_i^{(m-1)}(\xi) = (-1)^i \left( a^{(m)_i+1} g^{(m-1)} + \sum_{k=1}^{i} (-1)^k a^{(m-k)} A_{m-1} B f^{(m)} \right).
\]
Define the set of indices \( \Delta_r = \Delta^1_r \cup \Delta^2_r \subset \mathbb{N} \times \mathbb{N} \) by taking
\[
\Delta^1_r = \{ (j,i) \in \mathbb{N} \times \mathbb{N} : \ 1 \leq j \leq r \text{ and } 0 \leq i \leq n - r - 1 \}, \\
\Delta^2_r = \{ (j,i) \in \mathbb{N} \times \mathbb{N} : \ r + 1 \leq j \leq n - 2 \text{ and } 0 \leq i \leq n - j - 2 \}.
\]
For any \( r + 1 \leq k \leq n \) define the following subspaces
\[
W_k = \{ (\xi_1, \ldots, \xi_r, \xi_{r+1}, \ldots, \xi_n)^T \in \mathbb{R}^r \times \mathbb{R}^{n-r} : \xi_{k+1} = \cdots = \xi_n = 0 \},
\]
and let \( \pi_k(\xi) \) denote the projection on \( W_k \), that is, \( \pi_k(\xi) = (\xi_1, \ldots, \xi_r, \xi_{r+1}, \ldots, \xi_k, 0, \ldots, 0)^T \).

For any \( 1 \leq j \leq n \), we denote by
\[
C_j = (0, \ldots, 0, 1, 0, \ldots, 0)
\]
the row vector in \( \mathbb{R}^n \) whose all components are zero except the \( j^{th} \) component which equals 1.

For any \( (j,i) \in \Delta^1_r \) (resp. \( (j,i) \in \Delta^2_r \) ), we define \( a^{(m)j,i+2}(\xi) \) as the homogeneous part of degree \( (m - 3) \) (resp. of degree \( (m - 2) \)) of the function
\[
C_j [X_i^{m-1}, X_{i+1}^{m-1}] (\pi_{n-i}(\xi)).
\]
One can easily establish that
\[
a^{(m)j,i+2}(\xi) = C_j \left( ad_{A'B}X_{i+1}^{(m-1)} - ad_{A'iB}X_{i}^{(m-1)} \right) (\pi_{n-i}(\xi)).
\]
The functions \( a^{(m)j,i+2} \) thus defined will be called \textit{weighted homogeneous \( (m) \)-invariants} of the weighted homogeneous system \( \Sigma^{(m)} \).

Consider, along with \( \Sigma^{(m)} \) defined by (7.7), the system \( \tilde{\Sigma}^{(m)} \) given by (7.8) and denote by
\( \tilde{a}^{(m)j,i+2} \) the weighted homogeneous \( \langle m \rangle \)-invariants of the latter system.

The following result, which generalizes that obtained by Kang [48] for systems with controllable linearization, asserts that the weighted homogeneous \( \langle m \rangle \)-invariants \( a^{(m)j,i+2} \) are complete invariants of weighted homogeneous feedback and, moreover, illustrates their meaning for the normal form \( \Sigma^{(m)}_{NF} \).

\textbf{Theorem 7.6} For any \( m \geq 2 \) we have the following properties.

(i) Two weighted homogeneous systems \( \Sigma^{(m)} \), given by (7.7), and \( \tilde{\Sigma}^{(m)} \), given by (7.8), are equivalent via a weighted homogeneous feedback \( \Gamma^{(m)} \) if and only if, for any \( (j,i) \in \Delta_r \), we have
\[
a^{(m)j,i+2} = \tilde{a}^{(m)j,i+2}.
\]
(ii) The \( \langle m \rangle \)-invariants \( a^{(m)j,i+2} \) of the weighted homogeneous normal form \( \Sigma^{(m)}_{NF} \), defined by (7.10)-(7.11), are given by
\[
a^{(m)j,i+2}(\xi) = \frac{\partial^2 \tilde{f}^{(m)}_j}{\partial x_i^2}(\pi_i(x)).
\]
7.7 Explicit normalizing transformations

We will give in this section explicit weighted homogeneous transformations bringing the system \( \Sigma^{(m)} \) into its normal form \( \Sigma^{(m)}_{NF} \). Their interest is two-fold. Firstly, they are easily computable (via differentiation and integration of polynomials). Secondly, the proof of Theorem 7.4 giving the normal form \( \Sigma^{(m)}_{NF} \) is based on them.

For any \( r+1 \leq i \leq n \) and any \( 1 \leq j < i \leq n \), the homogeneous polynomial \( \psi^{(m-1)}_{j,i} \) is defined by
\[
\psi^{(m-1)}_{j,i} = C_{j} X^{(m-1)}_{n-i}.
\]

For any \( r+1 \leq i \leq j \leq n \), we define recursively the polynomials \( \psi^{(m-1)}_{j,i} \) by setting
\[
\psi^{(m-1)}_{j,r} = \psi^{(m-1)}_{r+1,r+1} = 0
\]
and by taking
\[
\psi^{(m-1)}_{j,i} = \frac{\partial f^{(m)}_{j-1}}{\partial \xi} + L_{A} f^{(1)} + \psi^{(m-1)}_{j-1,i} + \int_{0}^{\xi} \frac{\partial \psi^{(m-1)}_{j-1,i}}{\partial \xi} d\xi.
\]

Notice that the degree of the homogeneous polynomial \( \psi^{(m-1)}_{j,i} \) is either \( m-2 \) if \( 1 \leq j \leq r \) or \( m-1 \) if \( r+1 \leq j \leq n \).

Consider the weighted homogeneous feedback transformation
\[
\Gamma^{(m)} : \begin{aligned}
x &= \xi + \phi^{(m)}_{j}(\xi) \\
u &= v + \alpha^{(m)}(\xi) + \beta^{(m-1)}(\xi)v
\end{aligned}
\]
defined, for any \( 1 \leq j \leq n \), by
\[
\phi^{(m)}_{j}(\xi) = \sum_{i=r+1}^{n} \int_{0}^{\xi} \psi^{(m-1)}_{j,i}(\tilde{\xi}) d\tilde{\xi},
\]
\[
\alpha^{(m)}(\xi) = -f^{(m)}_{n}(\xi) - L_{A} f^{(1)}(\phi^{(m)}_{n})(\xi),
\]
\[
\beta^{(m-1)}(\xi) = -g^{(m-1)}_{n}(\xi) - L_{B} \phi^{(m)}_{n}(\xi).
\]

We have the following result:

**Theorem 7.7** For any \( m \geq 2 \), the weighted homogeneous feedback transformation \( \Gamma^{(m)} \), defined by (7.12), brings the weighted homogeneous system \( \Sigma^{(m)} \), given by (7.7), into its weighted homogeneous normal form \( \Sigma^{(m)}_{NF} \), defined by (7.10).

7.8 Weighted normal form for single-input systems with uncontrollable linearization

We will present in this section our main result giving a normal form under a formal feedback transformation \( \Gamma^{\infty} \) (see Section 3 for some comments on formal feedback) of any single-input control system (with controllable or uncontrollable linearization). For any \( 1 \leq i \leq n \) we denote \( \pi_{i}(x) = (x_{1}, \ldots, x_{i})^{T} \).
Consider the system $\Sigma^\infty$, given by (7.1) and assume that all eigenvalues of the uncontrollable part of the linear approximation are real. There exists a formal feedback transformation $\Gamma^\infty$ of the form (7.2), which brings the system $\Sigma^\infty$, given by (7.1), into its normal form

$$\Sigma^\infty_{NF} : \dot{x} = Ax + Bv + \bar{f}(x),$$

given by

$$\dot{x}_j = \begin{cases} 
\lambda_j x_j + \sigma_j x_{j+1} + \bar{f}_j(x) & \text{if } 1 \leq j \leq r, \\
x_{j+1} + \bar{f}_j(x) & \text{if } r + 1 \leq j \leq n - 1, \\
v & \text{if } j = n, 
\end{cases} \quad (7.13)$$

where

$$\bar{f}_j(x) = \begin{cases} 
\sum_{\alpha \in R_j} \gamma_{j}^{\alpha} x_1^{\alpha_1} \cdots x_r^{\alpha_r} + x_{r+1} S_j(\pi(x)_{r+1}) + \sum_{i=r+2}^{n} x_i^2 Q_{j,i}(\pi_i(x)), & \text{if } 1 \leq j \leq r, \\
\sum_{i=j+2}^{n} x_i^2 P_{j,i}(\pi_i(x)), & \text{if } r + 1 \leq j \leq n - 2, \\
0, & \text{if } n - 1 \leq j \leq n, 
\end{cases} \quad (7.14)$$

where $P_{j,i}, Q_{j,i},$ and $S_j$ are formal power series of the indicated variables and $\gamma_j^{\alpha} \in \mathbb{R}$.

**Remark 7.9** In the general case of complex eigenvalues of the uncontrollable part, for each complex eigenvalue $\lambda_j = \alpha_j + i\beta_j$, we replace the expression for $\dot{x}_j$ by

$$\begin{pmatrix} \dot{x}_{j,1} \\ \dot{x}_{j,2} \end{pmatrix} = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix} \begin{pmatrix} x_{j,1} \\ x_{j,2} \end{pmatrix} + \Xi_j \begin{pmatrix} x_{j,1} \\ x_{j,2} \end{pmatrix} + \bar{f}_j,$$

where $\bar{f}_j = (\bar{f}_{j,1}, \bar{f}_{j,2})^T$, for $1 \leq j \leq r$, being defined by formula (7.14), with $\gamma_j^\alpha = (\gamma_j^{\alpha_1}, \gamma_j^{\alpha_2})^T \in \mathbb{R}^2$, and $S_j$ and $Q_{j,i}$ being $\mathbb{R}^2$-valued formal power series of the indicated variables. Of course, the resonant set of a complex eigenvalue $\lambda_j$ is the same as that corresponding to its conjugate $\bar{\lambda}_j$, which explains why we have the same $R_j$ for both components $x_{j,1}$ and $x_{j,2}$. \hfill \square

Notice that the above normal form is a natural combination of the two extreme cases: that of dynamical systems and that of systems with controllable linearization. Indeed, if $r = n$, that is we deal with a dynamical system, then the coordinates $(x_{r+1}, \ldots, x_n)$ are not present and the normal form $\Sigma^\infty_{NF}$ reduces to a dynamical system $\dot{x} = Jx + \bar{f}(x)$ containing resonant terms only, that is $\bar{f}_j(x) = \sum_{\alpha \in R_j} \gamma_j^{\alpha} x_1^{\alpha_1} \cdots x_r^{\alpha_r}$, for $1 \leq j \leq n$. This is, of course, Poincaré normal form of a vector field under a formal diffeomorphism (see, e.g., [1] and Theorem 2.4). On the other hand, if $r = 0$, that is the linear approximation is controllable, the coordinates $(x_1, \ldots, x_r)$ are not present and our normal form reduces to $\Sigma^m_{NF}$ of Kang [48] (see Section 3), for which $\bar{f}_j(x) = \sum_{i=j+2}^{n} x_i^2 P_{j,i}(\pi_i(x))$, for $1 \leq j \leq n - 2$ and $\bar{f}_j(x) = 0$ otherwise.

Another normal form for nonlinear single-input systems with uncontrollable linearization was obtained by the authors in [76] (see also [75]) and by Krener et al in [56], [57] and [58]. Notice, however, that those normal forms differ substantially from the one proposed in this paper and in [79]. Indeed, in the presented approach the homogeneity is calculated with respect to the
linearly controllable variables while it is calculated with respect to all variables in [76], [56], [57] and [58].

7.9 Example

Example 7.10 (Kapitsa Pendulum) We consider in this example the Kapitsa Pendulum whose equations (see [5] et [17]) are given by

\[ \begin{align*}
\dot{\alpha} &= p + \frac{w}{l} \sin \alpha \\
\dot{p} &= (gL - \frac{w^2}{2l} \cos \alpha) \sin \alpha - \frac{w}{l} p \cos \alpha \\
\dot{z} &= w,
\end{align*} \]

where \( \alpha \) denotes the angle of the pendulum with the vertical \( z \)-axis, \( w \) the velocity of the suspension point \( z \), and \( p \) is proportional to the generalized impulsion; \( g \) is the gravity constant and \( l \) the length of the pendulum.

We assume to control the acceleration \( a = \dot{w} \). Introduce the coordinate system \((\xi_1, \xi_2, \xi_3, \xi_4) = (\alpha, p, z/l, w/l)\) and take \( u = a/l \) as the control.

The system (7.15) considered around an equilibrium point \((\alpha_0, p_0, z_0, u_0) = (k\pi, 0, 0, 0)\), where \( k \in \mathbb{Z} \), rewrites as

\[ \begin{align*}
\dot{\xi}_1 &= \xi_2 + \xi_1 \xi_4 + \xi_4 T_1(\xi_1) \\
\dot{\xi}_2 &= \epsilon g_0 \xi_1 - \xi_2 \xi_4 + \xi_2 \xi_4 T_2(\xi_1) + \xi_2^2 Q_2(\xi_1) + R_2(\xi_1) \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= u,
\end{align*} \]

where \( g_0 = g/l \), \( \epsilon = \pm 1 \) and \( T_1, T_2, R_2 \) are analytic functions whose 1-jets at \((k\pi, 0, 0, 0)\) vanish and \( Q_2 \) is an analytic function vanishing at \((k\pi, 0, 0, 0)\). Above, the case \( \epsilon = 1 \) corresponds to \( \alpha_0 = 2n\pi \) and the case \( \epsilon = -1 \) to \( \alpha_0 = (2n + 1)\pi \). One can easily check that the quadratic feedback transformation

\[ \Gamma^2 : \begin{align*}
y_1 &= \xi_1 - \xi_1 \xi_3 \\
y_2 &= \xi_2 + \xi_2 \xi_3 \\
y_3 &= \xi_3 \\
y_4 &= \xi_4
\end{align*} \]

brings the system (7.16) into the system

\[ \begin{align*}
\dot{y}_1 &= y_2 + y_2 y_3 \tilde{S}_1(y_3) + y_4 \tilde{T}_1(y_1, y_3) \\
\dot{y}_2 &= \epsilon g_0 y_1 + y_1 y_3 \tilde{S}_2(y_1, y_3) + y_4 \tilde{T}_2(y_1, y_2, y_3) + y_2^2 \tilde{Q}_2(y_1, y_3) + \tilde{R}_2(y_1) \\
\dot{y}_3 &= y_4 \\
\dot{y}_4 &= u,
\end{align*} \]

where \( \tilde{Q}_1, \tilde{Q}_2, \tilde{R}_2, \tilde{S}_2, \tilde{T}_1, \) and \( \tilde{T}_2 \) are analytic functions.

Since the vector field defined in \( \mathbb{R}^3 \) by

\[ f = \tilde{T}_1(y_1, y_3) \partial / \partial y_1 + \tilde{T}_2(y_1, y_2, y_3) \partial / \partial y_2 + \partial / \partial y_3 \]
does not vanish at \((0,0,0) \in \mathbb{R}^3\) (resp. at \((\pi,0,0) \in \mathbb{R}^3\)), there exists, in a neighborhood of \((0,0,0) \in \mathbb{R}^3\) (resp. of \((\pi,0,0) \in \mathbb{R}^3\)), an analytic transformation \(x = \phi(y)\) of the form
\[
\begin{align*}
x_1 &= \phi_1(y_1, y_3) \\
x_2 &= \phi_2(y_1, y_2, y_3) \\
x_3 &= y_3
\end{align*}
\]
such that
\[
(\phi_x f)(x) = \partial/\partial x_3.
\]
This latter transformation, completed with \(x_4 = y_4\) and \(u = v\), brings the system (7.17) into the normal form (compare with Theorem 7.8)
\[
\begin{align*}
\dot{x}_1 &= x_2 + \tilde{R}_1(x_1, x_2) + x_3 \tilde{S}_1(\pi_3(x)) \\
\dot{x}_2 &= \epsilon g_0 x_1 + \tilde{R}_2(x_1, x_2) + x_3 \tilde{S}_2(\pi_3(x)) + x_4^2 \tilde{Q}_2(\pi_3(x)) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= v,
\end{align*}
\]
where \(\pi_3(x) = (x_1, x_2, x_3)\).

Clearly, the dimension of the linearly controllable part of (7.16), that is that of (7.18), equals 2, which means that \(r = 2\).

In the case \(\epsilon = 1\), the eigenvalues of the uncontrollable linear part are \(\lambda_1 = -\lambda_2 = \sqrt{g_0}\) while in the case \(\epsilon = -1\), they are given by \(\lambda_1 = -\lambda_2 = i \sqrt{g_0}\).

In both cases, those eigenvalues are resonant and satisfy, for any \(m \geq 2\), the relations
\[
\lambda_1 = m\lambda_1 + (m-1)\lambda_2 \quad \text{and} \quad \lambda_2 = (m-1)\lambda_1 + m\lambda_2.
\]
Applying Poincaré’s method (see [1]), we get rid, by a formal diffeomorphism in the space \((x_1, x_2)\), of all nonresonant terms of the dynamical system
\[
\begin{align*}
\dot{x}_1 &= x_2 + \tilde{R}_1(x_1, x_2) \\
\dot{x}_2 &= \epsilon g_0 x_1 + \tilde{R}_2(x_1, x_2)
\end{align*}
\]
and thus we transform the system, for \(\epsilon = 1\) and \(\epsilon = -1\), into one of the following normal forms.

Set \(\lambda = \sqrt{g_0}\). For \(\epsilon = 1\), which is the case of real eigenvalues, the normal form is given by (compare with Theorem 7.8)
\[
\begin{align*}
\dot{x}_1 &= \lambda x_1 + \sum_{m=2}^{\infty} a_m x_1(x_1 x_2)^{m-1} + x_3 \tilde{S}_1(\pi_3(x)) + x_4^2 \tilde{Q}_1(\pi_3(x)) \\
\dot{x}_2 &= -\lambda x_2 + \sum_{m=2}^{\infty} b_m x_2(x_1 x_2)^{m-1} + x_3 \tilde{S}_2(\pi_3(x)) + x_4^2 \tilde{Q}_2(\pi_3(x)) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= v.
\end{align*}
\]
For \(\epsilon = -1\), which corresponds to the case of complex eigenvalues (see Remark following Theorem 7.8), the normal form is given by
\[
\begin{align*}
\dot{x}_1 &= \lambda x_2 + \sum_{m=2}^{\infty} (c_m x_1 + d_m x_2)(x_1^2 + x_2^2)^{m-1} + x_3 \tilde{S}_1(\pi_3(x)) + x_4^2 \tilde{Q}_1(\pi_3(x)) \\
\dot{x}_2 &= -\lambda x_1 + \sum_{m=2}^{\infty} (-d_m x_1 + c_m x_2)(x_1^2 + x_2^2)^{m-1} + x_3 \tilde{S}_2(\pi_3(x)) + x_4^2 \tilde{Q}_2(\pi_3(x)) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= v.
\end{align*}
\]
Above, the functions $\bar{S}_1, \bar{Q}_1, \bar{S}_2,$ and $\bar{Q}_2$ on the one hand side, and the functions $\tilde{S}_1, \tilde{Q}_1, \tilde{S}_2,$ and $\tilde{Q}_2$ on the other hand side, are formal power series which, in general, are different from the objects denoted earlier by the same symbols.

Notice that we transform the original system into its normal form (7.18) using analytic feedback transformations and that the only passage defined by a formal feedback transformation is that transforming the system (7.18) into (7.19) (resp. (7.20)).

8 Normal forms for multi-input nonlinear control systems

In this section we give a generalization of normal forms obtained in Section 3 for multi-input nonlinear control systems with controllable linearization (see [87] for more details). Normal forms for two-input nonlinear control systems has been obtained previously by the authors [85], and will be derived here as a particular case. The general case of multi-input systems with uncontrollable linearization will appear in [88]. We will illustrate normal forms in this section by considering three physical examples: A model of a crane in Example 8.5, a model of a planar vertical takeoff and landing aircraft as Example 8.6, and finally the Example 8.7 deals with a prototype of a wireless multi-vehicle testbed [13] and [15].

Consider control systems of the form

$$\Pi : \dot{\xi} = F(\xi, u), \quad \xi(\cdot) \in \mathbb{R}^n, \quad u(\cdot) = (u_1(\cdot), \ldots, u_p(\cdot))^T \in \mathbb{R}^p$$

around the equilibrium point $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^p$, that is, $f(0, 0) = 0$, and denote by

$$\Pi^{[1]} : \dot{\xi} = F\xi + G u = F\xi + G_1 u_1 + \cdots + G_p u_p,$$

its linearization at this point, where

$$F = \frac{\partial F}{\partial \xi}(0, 0), \quad G_1 = \frac{\partial F}{\partial u_1}(0, 0), \ldots, \quad G_p = \frac{\partial F}{\partial u_p}(0, 0).$$

We will assume for simplicity (see [87], [88] for general case) that $G_1 \land \cdots \land G_p \neq 0$, and the linearization is controllable, that is

$$\operatorname{span} \{ F^i G_k : 0 \leq i \leq n - 1, \ 1 \leq k \leq p \} = \mathbb{R}^n.$$  

Let $(r_1, \ldots, r_p), \ 1 \leq r_1 \leq \cdots \leq r_p = r,$ be the largest, in the lexicographic ordering, $p$-tuple of positive integers, with $r_1 + \cdots + r_p = n$, such that

$$\operatorname{span} \{ F^i G_k : 0 \leq i \leq r_k - 1, \ 1 \leq k \leq p \} = \mathbb{R}^n. \quad (8.1)$$

Without loss of generality we can assume that the linearization is in Brunovský canonical form

$$\Pi^{[1]}_{CF} : \dot{\xi} = A\xi + Bu = A\xi + B_1 u_1 + \cdots + B_p u_p,$$

where $A = \text{diag}(A_1, \ldots, A_p), \ B = (B_1, \ldots, B_p) = \text{diag}(b_1, \ldots, b_p)$, that is,

$$A = \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_p \end{pmatrix}_{n \times n}, \quad B = \begin{pmatrix} b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_p \end{pmatrix}_{n \times p}. \quad (8.2)$$
with \((A_k,b_k)\) in Brunovský single-input canonical forms of dimensions \(r_k\), for any \(1 \leq k \leq p\).

With the \(p\)-tuple \((r_1, \ldots, r_p)\) we associate the \(p\)-tuple \((d_1, \ldots, d_p)\) of nonnegative integers, \(0 = d_p \leq \cdots \leq d_1 \leq r-1\), such that \(r_1 + d_1 = \cdots = r_p + d_p = r\).

Our aim is to give a normal form of feedback classification of such systems under invertible feedback transformations of the form

\[
\Upsilon : \begin{aligned} x &= \phi(\xi) \\
u &= \gamma(\xi, v), \end{aligned}
\]

where \(\phi(0) = 0\) and \(\gamma(0,0) = 0\).

We study, step by step, the action of the Taylor series expansion \(\Upsilon^\infty\) of the feedback transformation \(\Upsilon\), given by

\[
\Upsilon^\infty : \begin{aligned} x &= \phi(\xi) = \xi + \sum_{m=2}^{\infty} \phi^m(\xi) \\
u &= \gamma(\xi, v) = v + \sum_{m=2}^{\infty} \gamma^m(\xi, v). \end{aligned} \tag{8.3}
\]
on the Taylor series expansion \(\Pi^\infty\) of the system \(\Pi\), given by

\[
\Pi^\infty : \dot{\xi} = A\xi + Bu + \sum_{m=2}^{\infty} f^m(\xi, u). \tag{8.4}
\]
Throughout this section, in particular in formulas (8.3) and (8.4), the homogeneity of \(f^m\) and \(\gamma^m\) will be taken with respect to the variables \(\xi, v\) and \(\xi, u\) respectively.

### 8.1 Non-affine normal forms

Let \(1 \leq s \leq t \leq p\). We denote by

\[
x_s = (x_{s,d_s+1}, \ldots, x_{s,r})^T, \quad x_{s,r+1} = v_s
\]

and we set \(\bar{x}_{s,i} = (x_{s,d_s+1}, \ldots, x_{s,i})^T\) for any \(d_s + 1 \leq i \leq r + 1\).

We also define the projections

\[
\pi_{t,i}^s(x) = (\bar{x}_{1,i}, \ldots, \bar{x}_{s,i}, \bar{x}_{s+1,i-1}, \ldots, \bar{x}_{t-1,i-1}, \bar{x}_{t,i}, \bar{x}_{t+1,i-1}, \ldots, \bar{x}_{p,i-1})^T,
\]
where \(\bar{x}_{s,i}\) is empty whenever \(0 \leq i \leq d_s\).

Our main result for multi-input nonlinear control systems with controllable linearization is as following.

**Theorem 8.1** The control system \(\Pi^\infty\), defined by (8.4), is feedback equivalent, by a formal feedback transformation \(\Upsilon^\infty\) of the form (8.3), to the normal form

\[
\Pi^\infty_{NF} : \dot{x} = Ax + Bu + \sum_{m=2}^{\infty} \bar{f}^m(x, v),
\]

where for any \(m \geq 2\), we have

\[
\bar{f}^m(x, v) = \sum_{k=1}^{p} \sum_{j=d_k+1}^{r-1} \bar{f}_j^k(x, v) \frac{\partial}{\partial x_{k,j}}, \tag{8.5}
\]
The control system formal feedback transformation given by
d
for any \( 1 \leq k \leq p \) and any \( d_k + 1 \leq j \leq r - 1 \).

Above, the functions \( P_{j,i,s,t}^{k[m-2]} \) and \( Q_{j,i,s,t}^{k[m-2]} \) stand for homogeneous polynomials of degree \( m - 2 \) of the indicated variables; \( P_{j,i,s,t}^{k[m-2]} \) (resp. \( Q_{j,i,s,t}^{k[m-2]} \)) being equal zero for \( 1 \leq i \leq d_s \) (resp. \( 1 \leq i \leq d_d + 1 \)).

Notice that when \( p = 1 \), that is, if we deal with single-input control systems, then the homogeneous polynomials \( Q_{j,i,s,t}^{k[m-2]} \) are zero, and thus the normal form reduces to Kang normal form [48] given by

\[
\bar{f}_j^{[m]}(x, v) = \sum_{i=j+2}^{p+1} x^{i,j} P_{j,i,s,t}^{k[m-2]}(\pi^{s}_{i,t}(x)) + \sum_{i=j+2}^{p+1} x^{i,j} Q_{j,i,s,t}^{k[m-2]}(\pi^{s}_{i,t-1}(x)) \quad (8.6)
\]

for any \( 1 \leq k \leq p \) and any \( d_k + 1 \leq j \leq r - 1 \).

**Two-input control systems.** The normal form for two-input control systems with controllable linearization was obtained long before by the authors [85] and deduces from Theorem 8.1 as corollary by taking \( p = 2 \). We have

**Corollary 8.2** The control system \( \Pi^\infty \), defined by (8.4) with \( p = 2 \), is feedback equivalent, by a formal feedback transformation \( \Upsilon^\infty \) of the form (8.3), to the normal form

\[
\Pi^\infty_{NF} : \dot{x} = Ax + Bv + \sum_{m=2}^{\infty} \bar{f}_m^{[m]}(x, v),
\]

where for any \( m \geq 2 \), we have

\[
\bar{f}_m^{[m]}(x, v) = \sum_{j=d_1+1}^{r-1} \bar{f}_j^{[m]}(x, v) \frac{\partial}{\partial x_{1,j}} + \sum_{j=d_2+1}^{r-1} \bar{f}_j^{[m]}(x, v) \frac{\partial}{\partial x_{2,j}} ,
\]

with,

\[
\bar{f}_j^{[m]}(x, v) = \sum_{i=j+2}^{p+1} x^{i,j} P_{j,i,s,t}^{k[m-2]}(\bar{x}_{1,i}, \bar{x}_{2,i-1}) + x^{i,j} Q_{j,i,s,t}^{k[m-2]}(\bar{x}_{1,i-1}, \bar{x}_{2,i})
\]

for any \( k = 1, 2 \) and any \( d_k + 1 \leq j \leq r - 1 \).

The homogeneous polynomials \( P_{j,i}^{k[m-2]} \), \( Q_{j,i}^{k[m-2]} \), and \( S_{j,i}^{k[m-2]} \) being equal zero for \( 1 \leq i \leq d_1 \).

**8.2 Affine normal forms**

Here we consider the action of \( \Gamma^\infty \), given by

\[
\Gamma^\infty :\begin{align*}
x &= \phi(\xi) = T\xi + \sum_{m=2}^{\infty} \phi^{[m]}(\xi) \\
u &= \alpha(\xi) + \beta(\xi) v = K\xi + Lv + \sum_{m=2}^{\infty} (\alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)) v
\end{align*}
\]

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on $\Sigma^\infty$, given by

$$\Sigma^\infty: \dot{\xi} = A\xi + Bu + \sum_{m=2}^{\infty} (f^{|m|}(\xi) + g^{|m-1|}(\xi)u),$$

(8.8)

where we assume the linear part to be already in the Brunovsky canonical form $(A, B)$ (see (8.2)).

The main result of [87] in the affine case is as follows:

**Theorem 8.3**

(i) The formal system $\Sigma^\infty$, defined by (8.8), is feedback equivalent, by a formal feedback transformation $\Gamma^\infty$ of the form (8.7), to the normal form

$$\Sigma_{NF}^\infty: \dot{x} = Ax + Bu + \sum_{m=2}^{\infty} \left( f^{|m|}(x) + \sum_{s=1}^{p} g^{|m-1|}_s(x) v_1 + \cdots + g^{|m-1|}_p(x) v_{p-1} \right),$$

where for any $m \geq 2$, the vector field $\bar{f}^{|m|}(x)$ and the vector fields $\bar{g}^{|m-1|}_s(x)$, for $1 \leq s \leq p - 1$, are given by

$$\bar{f}^{|m|}_j(x) = \sum_{k=1}^{p} \sum_{j=d_k+1}^{r-1} \bar{f}^{|m|}_j(x) \frac{\partial}{\partial x_{k,j}}, \quad \bar{g}^{|m-1|}_s(x) = \sum_{k=1}^{p} \sum_{j=d_k+1}^{r-1} \bar{g}^{|m-1|}_s(x) \frac{\partial}{\partial x_{k,j}},$$

with

$$\bar{f}^{|m|}_j(x) = \sum_{1 \leq s < t \leq p} \sum_{i=j+2}^{r} x_{s,i} x_{t,i} \bar{p}^{k|m-2|}_{j,i,s,t}(\pi^s_{t,i}(x)) + \sum_{1 \leq s < t \leq p} \sum_{i=j+2}^{r} x_{s,i} x_{t,i-1} \bar{Q}^{k|m-2|}_{j,i,s,t}(\pi^s_{t,i-1}(x)),$$

and

$$\bar{g}^{|m-1|}_{s,j}(x) = \sum_{t=s+1}^{p} x_{t,r} \bar{Q}^{k|m-2|}_{j,r+1,s,t}(\pi^s_{t,r}(x))$$

for any $1 \leq k \leq p$, and any $d_k + 1 \leq j \leq r - 1$.

(ii) Moreover, if the formal distribution

$$\mathcal{G}^\infty = \text{span} \left\{ B_1 + \sum_{m=2}^{\infty} g^{|m-1|}_1, \ldots, B_p + \sum_{m=2}^{\infty} g^{|m-1|}_p \right\}$$

is involutive, then the homogeneous polynomials $\bar{Q}^{k|m-2|}_{j,r+1,s,t}(\pi^s_{t,r}(x))$ are equal zero, that is, the normal form reduces to

$$\Sigma_{NF}^\infty: \dot{x} = Ax + Bu + \sum_{m=2}^{\infty} \bar{f}^{|m|}(x).$$

**Remark 8.4** We can notice that only $p - 1$ control vector fields are present in the normal form, the control vector field $g_n$ being normalized to $(0, \ldots, 0, 1)^T$. This is what happened in the single-input case. If we take $p = 1$, then all homogeneous control vector fields $\bar{g}^{|m-1|}_n(x)$, as well as the homogeneous polynomials $\bar{Q}^{k|m-2|}_{j,r+1,s,t}(\pi^s_{t,r}(x))$ are not present in the normal form above. Thus, the normal form given by Theorem 8.3 reduces to Kang normal form. In item (ii) we rediscover a well-known result: If a non singular distribution is involutive, there is coordinates that normalize the whole distribution.
8.3 Examples

In this section we will illustrate the theory of normal forms for multi-input control systems by considering three physical examples. We will first treat the case of a model of a crane, then a prototype of a planar vertical takeoff and landing aircraft, and finally we will consider the model of a multi-vehicle wireless test-bed presented by Caltech.

Example 8.5 (Model of a crane) Consider the following model of a crane borrowed from [16] (see also [14]). The state equations are

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -\frac{g \sin z_1}{R} - \frac{2z_2}{\sqrt{R}} \dot{R} - d \cos z_1 \frac{R}{R},
\end{align*}
\]

where \(z_1\) is the angle between the rope and the vertical axis, \(z_2\) the angular velocity, \(R\) the length of the rope, and \(d\) the trolley acceleration. We suppose to control \(\dot{R} = u_1\) and \(\dot{d} = u_2\) (notice that in [16] controls are \(R\) and \(D\), where \(d = \ddot{D}\)). We consider the system around the equilibrium point \(z_{10} = z_{20} = d_0 = 0\), \(R_0 = 1\). The linear approximation is controllable with controllability indices \(r_1 = 1\) and \(r_2 = 3\). Then \(d_1 = 2\) and \(d_2 = 0\). Introduce the coordinates

\[
\begin{align*}
\xi_{1,3} &= R - R_0 = R - 1 \\
\xi_{2,1} &= z_1 \\
\xi_{2,2} &= z_2 \\
\xi_{2,3} &= d
\end{align*}
\]

in which the system takes the form

\[
\begin{align*}
\dot{\xi}_{1,3} &= u_1 \\
\dot{\xi}_{2,1} &= \xi_{2,2} \\
\dot{\xi}_{2,2} &= -\frac{g \sin \xi_{2,1}}{1 + \xi_{1,3}} - \xi_{2,3} \frac{\cos \xi_{2,1}}{1 + \xi_{1,3}} - \frac{2\xi_{2,2}}{1 + \xi_{1,3}} u_1 \\
\dot{\xi}_{2,3} &= u_2.
\end{align*}
\]

In order to bring the above system to its normal form, we rectify the involutive distribution \(\mathcal{G} = \text{span}\{g_1, g_2\}\), with \(g_1 = (1,0,-\frac{2\xi_{2,2}}{1 + \xi_{1,3}},0)^T\) and \(g_2 = (0,0,0,1)^T\), and we normalize the component \(f_{2,2}^2\) by taking

\[
\begin{align*}
x_{1,3} &= \xi_{1,3} \\
x_{2,1} &= \xi_{2,1} \\
x_{2,2} &= \xi_{2,2}(1 + \xi_{1,3})^2 \\
x_{2,3} &= -(1 + \xi_{1,3})(g \sin \xi_{2,1} + \xi_{2,3} \cos \xi_{2,1})
\end{align*}
\]

followed by a suitable feedback. This yields

\[
\begin{align*}
\dot{x}_{1,3} &= u_1 \\
\dot{x}_{2,1} &= x_{2,2} - 2x_{1,3}x_{2,2} + 3x_{1,3}^2 \frac{x_{2,2}}{(1 + x_{1,3})^2} \\
\dot{x}_{2,2} &= x_{2,3} \\
\dot{x}_{2,3} &= u_2,
\end{align*}
\]

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which is in a normal form (compare with Corollary 8.2), where all homogeneous polynomials 
\( P_{j,i}^{m-2}, Q_{j,i}^{m-2}, R_{j,i}^{m-2} \), and 
\( S_{j,i}^{m-2} \) are equal zero except for 
\( S_{1,1}^{2[0]} = -2 \) and the homogeneous polynomials 
\( P_{1,1}^{m-2} \), for \( m \geq 3 \), which are equal to the homogeneous parts of degree \( m - 2 \) of the function \( \frac{3x^2}{(1+x^1)^2} \).

**Example 8.6 (PVTOL aircraft)** In this example we consider the prototype of a Planar Vertical TakeOff and Landing (PVTOL) aircraft. The equations of motion of the PVTOL (see [91]) are given by

\[
\begin{align*}
\ddot{x} &= -\sin \theta u_1 + \varepsilon^2 \cos \theta u_2, \\
\ddot{y} &= \cos \theta u_1 + \varepsilon^2 \sin \theta u_2 - 1, \\
\dot{\theta} &= u_2,
\end{align*}
\]

where \((x, y)\) denotes the position of the center mass of the aircraft, \(\theta\) the angle of the aircraft relative to the \(x\)-axis, \(" -1"\) the gravitational acceleration and \(\varepsilon \neq 0\) the (small) coefficient giving the coupling between the rolling moment and the lateral acceleration of the aircraft. The control inputs \(u_1\) and \(u_2\) are the thrust (directed out the bottom of the aircraft) and the rolling moment.

We introduce the variables

\[
\begin{align*}
\xi_{1,1} &= y, & \xi_{1,2} &= \dot{y}, \\
\xi_{2,1} &= x, & \xi_{2,2} &= \dot{x}, \\
\xi_{2,3} &= \theta, & \xi_{2,4} &= \dot{\theta}, \\
w_1 &= u_1 - 1, & w_2 &= u_2.
\end{align*}
\]

The equations of motion of the PVTOL become

\[
\begin{align*}
\ddot{\xi}_{1,1} &= \xi_{1,2} \\
\ddot{\xi}_{1,2} &= \cos \xi_{2,3} w_1 + \varepsilon^2 \sin \xi_{2,3} w_2 + \cos \xi_{2,3} - 1 \\
\ddot{\xi}_{2,1} &= \xi_{2,2} \\
\ddot{\xi}_{2,2} &= -\sin \xi_{2,3} w_1 + \varepsilon^2 \cos \xi_{2,3} w_2 - \sin \xi_{2,3} \\
\dot{\xi}_{2,3} &= \xi_{2,4} \\
\dot{\xi}_{2,4} &= w_2.
\end{align*}
\]  \(\text{(8.9)}\)

The equilibria of the system is defined by

\[
(\xi_{1,1}^e, \xi_{1,2}^e, \xi_{2,1}^e, \xi_{2,2}^e, \xi_{2,3}^e, \xi_{2,4}^e, w_1^e, w_2^e)^T = (c, 0, 0, 0, 0, 0, 0, 0)^T,
\]

where \(c\) is any constant. The linearization of the system (8.9) around the equilibria is given by

\[
\begin{align*}
\dot{\xi}_{1,1} &= \xi_{1,2} \\
\dot{\xi}_{1,2} &= w_1 \\
\dot{\xi}_{2,1} &= \xi_{2,2} \\
\dot{\xi}_{2,2} &= -\xi_{2,3} + \varepsilon^2 w_2 \\
\dot{\xi}_{2,3} &= \xi_{2,4} \\
\dot{\xi}_{2,4} &= w_2.
\end{align*}
\]

It is easy to see that the linear system is controllable with controllability indices \(r_1 = 2\) and \(r_2 = 4\), and hence \(d_1 = 2\) and \(d_2 = 0\).
The change of coordinates given by

\[ x_{1,3} = \xi_{1,1} - \varepsilon^2 \int_0^{\xi_{2,3}} \frac{dt}{\cos t} \]
\[ x_{1,4} = \xi_{1,2} + \xi_{1,2} \tan \xi_{2,3} - \frac{\varepsilon^2}{\cos \xi_{2,3}} \xi_{2,4} \]
\[ x_{2,1} = \xi_{2,1} \]
\[ x_{2,2} = \xi_{2,2} \]
\[ x_{2,3} = -\tan \xi_{2,3} \]
\[ x_{2,4} = -\xi_{2,4}(1 + \tan^2 \xi_{2,3}) = \dot{x}_{2,3} \]

followed by the feedback

\[ w_1 = \frac{v_1}{\cos \xi_{2,1}} - \varepsilon^2 v_2 \tan \xi_{2,1} + \frac{1}{\cos \xi_{2,1}} - 1 \]
\[ v_2 = \dot{x}_{2,4} = -w_2(1 + \tan^2 \xi_{2,3}) - 2\xi_{2,4} \tan \xi_{2,3}(1 + \tan^2 \xi_{2,3}) \]

takes the system into the following one

\[ \dot{x}_{1,3} = x_{1,4} \]
\[ \dot{x}_{1,4} = v_1 \]
\[ \dot{x}_{2,1} = x_{2,2} + x_{1,4}x_{2,3} \]
\[ \dot{x}_{2,2} = x_{2,3} - x_{1,4}x_{2,4} + \varepsilon^2(1 - x_{2,3}^2)x_{2,4} \]
\[ \dot{x}_{2,3} = x_{2,4} \]
\[ \dot{x}_{2,4} = \phi_2 . \]

This system is in normal form (compare with Corollary 8.2), with

\[ Q_{1,4,1,2}^{2[0]}(x) = 1, \quad P_{2,4,1,2}^{2[0]}(x) = -1, \quad P_{2,4,2,2}^{2[0]}(x) = \varepsilon^2, \quad P_{2,4,2,2}^{2[2]}(x) = -\varepsilon^2 x_{2,3}^2 . \]

**Example 8.7 (Multi-Vehicle Wireless Testbed)** We consider the Caltech Multi-Vehicle Wireless Testbed, presented in [13], [15] and we study the normal form of one vehicle. The equations of motion of a MVWT vehicle (see [13], [15]) are given by

\[ m\ddot{x} = -\eta \dot{x} + (F_s + F_p) \cos \theta, \]
\[ m\ddot{y} = -\eta \dot{y} + (F_s + F_p) \sin \theta, \]
\[ J\ddot{\theta} = -\psi \dot{\theta} + (F_s - F_p)l, \]

where \((x, y)\) denotes the position of the center mass of the vehicle, \(\theta\) the angle of the axis of the vehicle with the \(x\)-axis, \(m\) the mass of the vehicle, \(J\) the rotational inertia, \(F_s\) and \(F_p\) denote respectively the starboard and port fan forces, and \(l\) (\(r\) in [13], [15]) the common moment arm of the forces. The center mass of the vehicle and the center of geometry are assumed to coincide. The constants \(\eta\) and \(\psi\) stand respectively for the coefficients of viscous friction and rotational friction.
Let us introduce the variables 
\[ \xi_{0,1} = y, \quad \xi_{0,2} = \dot{y}, \quad \xi_{1,1} = x, \quad \xi_{1,2} = \dot{x}, \quad \xi_{2,1} = \theta, \quad \xi_{2,2} = \dot{\theta}, \quad u_1 = F_s + F_p, \quad u_2 = F_s - F_p. \]

The equations of motion of a MVWT vehicle rewrites
\[
\begin{align*}
\dot{\xi}_{0,1} &= \xi_{0,2} \\
\dot{\xi}_{0,2} &= -\eta \xi_{0,2} + u_1 \sin \xi_{2,1} \\
\dot{\xi}_{1,1} &= \xi_{1,2} \\
\dot{\xi}_{1,2} &= -\eta \xi_{1,2} + u_1 \cos \xi_{2,1} \\
\dot{\xi}_{2,1} &= \xi_{2,2} \\
\dot{\xi}_{2,2} &= -\psi \xi_{2,2} + u_2 l.
\end{align*}
\tag{8.10}
\]

We can notice that the system is affine and its distribution \( G = \text{span} \{ g_1, g_2 \} \), where
\[
\begin{align*}
g_1 &= (0, \sin \xi_{2,1}, 0, \cos \xi_{2,1}, 0, 0)^T \\
g_2 &= (0, 0, 0, 0, 0, 1)^T.
\end{align*}
\]
is involutive and of constant rank 2. An equilibrium point for the system \( (8.10) \) is any constant position and orientation \((x_c, y_c, \theta_c)\) such that
\[
\begin{align*}
\xi_{e,1} &= (\xi_{0,1}, \xi_{0,2}, \xi_{1,1}, \xi_{1,2}, \xi_{2,1}, \xi_{2,2})^T,
\end{align*}
\]
around an equilibrium (we assume \( \theta_c = 0 \)) is given by
\[
\begin{align*}
\dot{\xi}_{0,1} &= \xi_{0,2}, \\
\dot{\xi}_{0,2} &= -\eta \xi_{0,2}, \\
\dot{\xi}_{1,1} &= \xi_{1,2}, \\
\dot{\xi}_{1,2} &= -\eta \xi_{1,2} + u_1, \\
\dot{\xi}_{2,1} &= \xi_{2,2}, \\
\dot{\xi}_{2,2} &= -\psi \xi_{2,2} + u_2 l.
\end{align*}
\]

It is easy to see that the system is not controllable because
\[
\text{span} \left\{ F^i G_k, 0 \leq i \leq 5, \; 1 \leq k \leq 2 \right\} = \mathbb{R}^4,
\]
where
\[
F = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & -\eta & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -\eta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -\psi
\end{pmatrix}, \quad G_1 = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
1
\end{pmatrix}, \quad \text{and} \quad G_2 = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
1
\end{pmatrix}.
\]

The feedback transformation defined by
\[
\begin{align*}
u_1 &= \frac{1}{\cos \xi_{2,1}} v_1 + \frac{\xi_{1,2}}{\cos \xi_{2,1}} \\
v_2 &= \frac{u_2}{l} + \frac{\psi}{T} \xi_{2,2}
\end{align*}
\]
takes the system into the following one
\[
\begin{align*}
\dot{\xi}_{0,1} &= \xi_{0,2}, \\
\dot{\xi}_{0,2} &= -\eta \xi_{0,2} + \eta \xi_{1,2} \tan \xi_{2,1} + u_1 \tan \xi_{2,1}, \\
\dot{\xi}_{1,1} &= \xi_{1,2}, \\
\dot{\xi}_{1,2} &= v_1, \\
\dot{\xi}_{2,1} &= \xi_{2,2}, \\
\dot{\xi}_{2,2} &= v_2.
\end{align*}
\]

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The change of coordinates given by
\[
\begin{align*}
  x_{0,1} &= \xi_{0,1}, \\
  x_{0,2} &= \xi_{0,2} - \xi_{1,2} \tan \xi_{2,1}, \\
  x_{1,1} &= \xi_{1,1}, \\
  x_{1,2} &= \xi_{1,2}, \\
  x_{2,1} &= \xi_{2,1}, \\
  x_{2,2} &= \xi_{2,2},
\end{align*}
\]

brings the system into the normal form
\[
\begin{align*}
  \dot{x}_{0,1} &= x_{0,2} + x_{1,2} \tan x_{2,1}, \\
  \dot{x}_{0,2} &= -\eta x_{0,2} - x_{1,2} x_{2,2} (1 + \tan^2 x_{2,1}), \\
  \dot{x}_{1,1} &= x_{1,2}, \\
  \dot{x}_{1,2} &= v_1, \\
  \dot{x}_{2,1} &= x_{2,2}, \\
  \dot{x}_{2,2} &= v_2.
\end{align*}
\]

Here the controllable part is linearizable because the indices of controllability are \( r_1 = r_2 = 2 \).

### 9 Feedback linearization

In this section we will discuss the feedback linearization problem. We will recall the well known result characterizing feedback linearizability in term of involutivity of certain distributions and compare it with a condition using the homogeneous \( m \)-invariants. Then we will consider systems with uncontrollable linearization and, similarly, we will compare a geometric condition involving involutivity of suitable distributions with a condition using weighted homogeneous invariants. Finally, we will discuss feedback linearizability of general systems (that is, not necessarily affine in controls).

Consider a \( C^\infty \)-smooth control-affine system of the form
\[
\Sigma: \dot{\xi} = f(\xi) + \sum_{i=1}^{m} g_i(\xi) u_i,
\]
where \( f(\xi_0) = 0 \), which we will assume throughout this section. To state a feedback linearization result for \( \Sigma \), we define the following distributions
\[
\begin{align*}
  D^1(x) &= \text{span} \{ g_i(x), \ 1 \leq i \leq m \}, \\
  D^j(x) &= \text{span} \{ \text{ad}_{f}^{-1} g_i(x), \ 1 \leq q \leq j, \ 1 \leq i \leq m \},
\end{align*}
\]
for \( j \geq 2 \). If the dimensions \( d_j(x) \) of \( D^j(x) \) are constant (see (FL1) and (FL1)’ below) we denote them by \( d_j \) and we define indices \( \rho_j \) as follows. Define \( d_0 = 0 \) and put \( r_j = d_j - d_{j-1} \) for \( 1 \leq j \leq n \). Then we define
\[
\rho_i = \max \{ r_j \mid r_j \geq i \}. \tag{9.1}
\]
Clearly, we have \( \rho_1 \geq \cdots \geq \rho_m \) (and also \( \sum_{i=1}^{m} \rho_i = n \) if the linear part \( (F, G) \) of \( \Sigma \) is controllable).

For the linear controllable system \( \dot{\xi} = F\xi + Gu \), the integers \( \rho_i \)'s form the set of controllability indices (compare Example 1.2).
We will be interested in feedback linearization, that is, in feedback equivalence of $\Sigma$ to a linear system of the form

$$\dot{x} = Ax + Bv = Ax + \sum_{i=1}^{m} b_i v_i$$

under a feedback transformation $x = \phi(\xi), u = \alpha(\xi) + \beta(\xi)v$. The following result (see, e.g., [35], [36], [43], [64]) describes control-affine systems that are locally feedback equivalent to linear controllable systems.

**Theorem 9.1** The following conditions are equivalent:

(i) $\Sigma$ is locally, at $\xi_0 \in \mathbb{R}^n$, feedback equivalent to a linear controllable system;

(ii) $\Sigma$ satisfies in a neighborhood of $\xi_0$

(FL1) $\dim D^j(\xi) = \text{const}$, for $1 \leq j \leq n$,

(FL2) the distributions $D^j$ are involutive, for $1 \leq j \leq n$,

(FL3) $\dim D^n(\xi_0) = n$.

(iii) $\Sigma$ satisfies in a neighborhood of $\xi_0$

(FL1)' $\dim D^j(\xi) = \text{const}$, for $1 \leq j \leq n$,

(FL2)' the distributions $D^{\rho_j-1}$ are involutive, for $1 \leq j \leq m$,

(FL3)' $\dim D^{\rho_1}(\xi_0) = n$, where $\rho_1$ is the largest controllability index.

The conditions (FL1)'-(FL3)' involve the minimal number of distributions whose involutivity has to be checked. On the other hand, the conditions (FL1)-(FL3) are more transparent and do not require calculating controllability indices.

In the single-input case $m = 1$, the condition (FL3) (or, equivalently, (FL3)') states that $g(\xi_0), \ldots, \text{ad}^{n-1}_f g(\xi_0)$ are independent, which implies that all distributions $D^j$, for $1 \leq j \leq n$, are of constant rank. In the single-input case, we have the following corollary of Theorem 9.1:

**Corollary 9.2** A single-input system $\Sigma$ is feedback linearizable if and only if it satisfies

(FL1)$_{SI}$ $g(\xi_0), \ldots, \text{ad}^{n-1}_f g(\xi_0)$ are independent,

(FL2)$_{SI}$ the distribution $D^{n-1}$ is involutive.

Now consider the single input-system $\Sigma$, given by $\dot{\xi} = f(\xi) + g(\xi)u$. Without loss of generality we assume that $\xi_0 = 0$. Consider the infinite Taylor series expansion of $\Sigma$ given at $\xi_0 = 0 \in \mathbb{R}^n$ by

$$\Sigma^\infty : \dot{\xi} = F\xi + Gu + \sum_{m=2}^{\infty} (f^{[m]}(\xi) + g^{[m-1]}(\xi))u.$$

Of course, a necessary condition for feedback equivalence of $\Sigma^\infty$ to a linear controllable system is that the linear part $(F, G)$ of $\Sigma^\infty$ is controllable. So we can put it by a linear feedback transformation $\Gamma^1$ into the Brunovský canonical form $(A, B)$. Now consider the homogeneous system

$$\Sigma^{[m]} : \dot{\xi} = A\xi + Bu + f^{[m]}(\xi) + g^{[m-1]}(\xi)u.$$

Recall that $\Delta = \{(j, i) \in \mathbb{N} \times \mathbb{N} : 1 \leq j \leq n - 2, 0 \leq i \leq n - j - 2\}$ and that, by definition, equivalence of $m$-homogeneous systems means equivalence modulo terms of degree higher than $m$ (see Section 3). Theorems 3.2 and 3.3 imply the following result of Kang [48]:
Proposition 9.3 The homogenous system $\Sigma^{[m]}$ is equivalent, via a homogeneous feedback transformation $\Gamma^m$, to the linear system

$$\dot{x} = Ax + Bv$$

if and only if

$$a^{[m]j,i+2} = 0,$$

$(j, i) \in \Delta$, that is, all homogenous m-invariants vanish.

This observation leads to the following linearization step-by-step procedure. Consider the system $\Sigma^\infty$ and apply a linear feedback transformation $\Gamma^1$ bringing the linear part $(F, G)$ into the Brunovský canonical form $(A, B)$. Denote $\Sigma^\infty, 1 = \Gamma^1(\Sigma^\infty)$. Now bring the homogenous system $\Sigma^{[2]}$ of $\Sigma^\infty, 1$ into its normal form $\Sigma^{[2]}_{NF}$ via a homogeneous transformation $\Gamma^2$. If the 2-invariants $a^{[2]j,i+2}$, $(j, i) \in \Delta$, vanish then the system $\Sigma^\infty, 2 = \Gamma^2(\Sigma^\infty, 1)$ is linear modulo terms in $V^{\geq 2}$. Notice that, although the transformation $\Gamma^2$ is determined by the homogenous part $\Sigma^{[2]}$ of $\Sigma^\infty, 1$ only, we apply $\Gamma^2$ to the whole system $\Sigma^\infty, 1$ and thus we modify, in general, all terms of $\Sigma^\infty, 1$ in order to get $\Sigma^\infty, 2$. Now suppose that a sequence of systems $\Sigma^\infty, 1, \ldots, \Sigma^\infty, m-1$ has been defined, and $\Sigma^\infty, m-1$ is linear modulo terms in $V^{\geq m}$. Bring the homogenous system $\Sigma^{[m]}$ of $\Sigma^\infty, m-1$ into its normal form $\Sigma^{[m]}_{NF}$ via a homogeneous transformation $\Gamma^m$. If the m-invariants $a^{[m]j,i+2}$, $(j, i) \in \Delta$, vanish then the system $\Sigma^\infty, m = \Gamma^m(\Sigma^\infty, m-1)$ is linear modulo terms in $V^{\geq m+1}$.

We thus have the following counterpart of Corollary 9.2:

Proposition 9.4 The system $\Sigma^\infty$, with controllable linearization, is feedback equivalent via a formal feedback $\Gamma^\infty$ to a controllable linear system $\dot{x} = Ax + Bv$ if and only if for any $m \geq 2$

$$a^{[m]j,i+2} = 0,$$

where $(j, i) \in \Delta$ and $a^{[m]j,i+2}$ are m-invariants of the homogeneous system $\Sigma^{[m]}$ of $\Sigma^\infty, m-1 = \Gamma^{m-2}_s \cdots \Gamma^2_1(\Sigma^\infty)$ with $\Gamma^0 = Id$.

Clearly, if a system $\Sigma$ is feedback linearizable (that is, satisfies the conditions (FL1)$_{SI}$-(FL2)$_{SI}$ of Corollary 9.2), then its infinite Taylor series expansion $\Sigma^\infty$ satisfies the conditions of Proposition 9.4. Now we will answer the important question of whether we can reverse this implication.

Proposition 9.5 Consider an analytic system $\Sigma$, with a controllable linearization $(F, G)$. Assume that

$$a^{[m]j,i+2} = 0,$$

for any $m \geq 2$, where $(j, i) \in \Delta$ and $a^{[m]j,i+2}$ are m-invariants of the homogeneous system $\Sigma^{[m]}$ of $\Sigma^\infty, m-1 = \Gamma^{m-2}_s \cdots \Gamma^2_1(\Sigma^\infty)$. Then $\Sigma$ is equivalent, via a local analytic feedback transformation $\Gamma$, to a controllable linear system $\dot{x} = Ax + Bv$. 

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An analogous result does not hold in the $C^\infty$-category. To see it consider, for example, the system
\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 + f_1(\xi_n) \\
\dot{\xi}_2 &= \xi_3 \\
\vdots \\
\dot{\xi}_{n-1} &= \xi_n \\
\dot{\xi}_n &= u,
\end{align*}
\]
where $n \geq 3$ and $f_1(\xi_n)$ is a $C^\infty$-smooth function such that all derivatives of $f_1(\xi_n)$ at $0 \in \mathbb{R}^n$ vanish but the function does not vanish identically in a neighborhood. Clearly, all the invariants $a^{[m],i+2}$ vanish but the system is not feedback linearizable linearizable because the distribution $\mathcal{D}^2 = \text{span} \{ g, ad_{f}g \}$ is not involutive.

Now we will discuss the problem of feedback linearization of systems with uncontrollable linearization. We will start with the following immediate generalization of Theorem 9.1:

**Proposition 9.6** A smooth (analytic) system $\Sigma$, is locally at $\xi^0$, equivalent via a smooth (analytic) feedback to
\[
\begin{align*}
\dot{x}_1 &= f^2(x_1), \\
\dot{x}_2 &= Ax_2 + Bv
\end{align*}
\]
with $(A, B)$ controllable (in Brunovský form), where $(x_1, x_2)^T = (x_{1,1}, \ldots, x_{1,r}, x_{2,1}, \ldots, x_{2,n-r})^T$ if and only if it satisfies around $\xi^0$ the conditions (FL1) and (FL2) of Theorem 9.1. Moreover,\[
\dim x_1 = \dim \mathcal{D}^n(\xi^0) = r.
\]

In the single-input case we get the following:

**Corollary 9.7** A smooth (analytic) single-input system $\Sigma$, is locally at $\xi^0$, equivalent via a smooth (analytic) feedback to
\[
\begin{align*}
\dot{x}_1 &= f^2(x_1), \\
\dot{x}_2 &= Ax_2 + Bv
\end{align*}
\]
with $(A, B)$ controllable, if and only if it satisfies in a neighborhood of $\xi^0$
\[
\begin{align*}
(FUL1)_{SI} \quad &\dim \mathcal{D}^r(\xi) = \dim \mathcal{D}^{r+1}(\xi) = r; \\
(FUL2)_{SI} \quad &\text{the distribution } \mathcal{D}^{r-1} \text{ is involutive.}
\end{align*}
\]
If, additionally,
\[
(FUL3) \quad \text{the eigenvalues of } J = \frac{\partial f^2}{\partial x_1}(x^0_1)^T \text{ are nonresonant, where}
\]
\[
x^0 = (x^0_1, x^0_2)^T = (x^0_{1,1}, \ldots, x^0_{1,r}, x^0_{2,1}, \ldots, x^0_{2,n-r})^T,
\]
then $\Sigma^\infty$ (the infinite series expansion of $\Sigma$) is equivalent to the linear system
\[
\begin{align*}
\dot{x}_1 &= Jx_1 \\
\dot{x}_2 &= Ax_2 + Bv,
\end{align*}
\]
via a formal feedback $\Gamma^\infty$. 

Notice that in order to check (FUL3), we do not need to bring the system into the partially linear form $\dot{x}_1 = f_2(x_1)$, $\dot{x}_2 = Ax_2 + Bv$, (which would, in general, require solving a system of 1st-order PDE’s). Indeed, the condition (FUL1) implies that 

$$[f, D^r] \subset D^r$$

which in turn yields

$$[F\xi, D^r] \subset D^r,$$

where $F\xi$ is the homogenous part of degree 1 of the vector field $f$ and $D^r$ is the homogenous part of degree 0 of the distribution $D^r$. In other words, the linear map $F$ leaves the linear subspace $D^r$ of $\mathbb{R}^n$ invariant. It follows that $F$ passes to the quotient, that is, defines the map $F_{D^r}: \mathbb{R}^n/\sim \rightarrow \mathbb{R}^n/\sim$, where $x \sim \bar{x}$ if and only if $x - \bar{x} \in D^r$. The eigenvalues of $F$ are just the eigenvalues of $F_{D^r}$. To calculate them, find a linear map $x = T\xi$ defining linear coordinates $(x_1, x_2)^T = (x_{1,1}, \ldots, x_{1,r}, x_{2,1}, \ldots, x_{2,n-r})^T$ such that $D^r = \text{span}\{\frac{\partial}{\partial x_{1,1}}, \ldots, \frac{\partial}{\partial x_{1,r}}\}$. Express $TFT^{-1}x = (\tilde{F}^1 x, \tilde{F}^2 x)^T$ then, clearly, $\tilde{F}^1 x = Jx_1$ and the eigenvalues of $J$ are just the eigenvalues of $F_{D^k}$. Of course, the same analysis holds in the multi-input case.

We will end up this section by giving a $C^\infty$-version of Corollary 9.7. Combining it with the linearizability results of Chen and Sternberg, we get the following:

**Corollary 9.8** If a $C^\infty$-smooth system $\Sigma$ satisfies the conditions (FUL1)$_{SI}$, (FUL2)$_{SI}$, (FUL3) of Corollary 9.7, then it is equivalent to the linear system

$$\dot{x}_1 = Jx_1$$
$$\dot{x}_2 = Ax_2 + Bv,$$

via a $C^\infty$-smooth formal feedback $\Gamma$.

An analogous result in the analytic category is more restrictive and much more subtle and requires introducing the notion of Poincaré-Siegel domains (see [1]).

### 10 Normal forms for discrete-time control systems

The method of normal forms has proved to be a useful approach in studying dynamical systems, and control systems as illustrated throughout this survey. The pioneer of this formal approach applied it to both continuous-time dynamical systems (vector fields) and discrete-time dynamical systems (maps), see [65]. This survey may sound incomplete if we omit to mention the work done for discrete-time control systems. Normal forms for discrete-time control systems has been studied using a similar approach to that presented in previous sections. Thus, quadratic and cubic normal forms for discrete-time control systems has been treated in [4, 28, 30, 59]. Those normal forms has been utilized for stabilization of systems with uncontrollable linearization [26, 29, 27, 25, 31, 57, 58, 59]. Recently, a normal form, any any degree $m$, for discrete-time control systems with controllable linearization was given by Hamzi and Tall [32]. We propose to expose briefly those results in this section.

The problem is to study the action of a feedback transformation

$$\Upsilon : \begin{cases} x = \phi(\xi) \\ u = \gamma(x, v) \end{cases}$$
on a discrete-time nonlinear control system

\[ \Pi : \xi^+ = F(\xi, u), \quad \xi(\cdot) \in \mathbb{R}^n \quad u(\cdot) \in \mathbb{R}, \]

where \( \xi^+ = \xi(k+1), \) and \( F(\xi, u) = F(\xi(k), u(k)) \) for any \( k \in \mathbb{N}. \) The transformation \( \Upsilon \) brings \( \Pi \) to the system

\[ \tilde{\Pi} : x^+ = \tilde{F}(x, v), \]

whose dynamics are given by

\[ \tilde{F}(x, v) = F(\phi^{-1}(x), \gamma(x, v)). \]

We suppose that \((0, 0) \in \mathbb{R}^n \times \mathbb{R}\) is an equilibrium point, that is, \( F(0, 0) = 0, \) and we denote by

\[ \Pi^{[1]} : \xi^+ = \mathbb{F}\xi + \mathbb{G}u, \]

its linearization at this point, where

\[ \mathbb{F} = \frac{\partial F}{\partial \xi}(0, 0), \quad \mathbb{G} = \frac{\partial F}{\partial u}(0, 0). \]

We will assume that this linearization is controllable, that is

\[ \text{span}\{ \mathbb{F}^i\mathbb{G} : 0 \leq i \leq n - 1 \} = \mathbb{R}^n. \]

Let us consider the Taylor series expansion \( \Pi^{\infty} \) of the system \( \Pi, \) given by

\[ \Pi^{\infty} : \xi^+ = \mathbb{F}\xi + \mathbb{G}u + \sum_{m=2}^{\infty} \mathbb{F}^{[m]}(\xi, u) \tag{10.1} \]

and the Taylor series expansion \( \Upsilon^{\infty} \) of the feedback transformation \( \Upsilon, \) given by

\[ \Upsilon^{\infty} : \begin{align*}
    x &= \phi(\xi) = T\xi + \sum_{m=2}^{\infty} \phi^{[m]}(\xi) \\
    u &= \gamma(\xi, v) = K\xi + Lv + \sum_{m=2}^{\infty} \gamma^{[m]}(\xi, v). 
\end{align*} \tag{10.2} \]

Throughout this section, in particular in formulas (10.1) and (10.2), the homogeneity of \( f^{[m]} \) and \( \gamma^{[m]} \) will be taken with respect to the variables \((\xi, u)^T\) and \((\xi, v)^T\) respectively.

We first notice that, because of the controllability assumption, there always exists a linear feedback transformation

\[ \Upsilon^{1} : \begin{align*}
    x &= T\xi \\
    u &= K\xi + Lv
\end{align*} \]

bringing the linear part

\[ \Pi^{[1]} : \xi^+ = \mathbb{F}\xi + \mathbb{G}u \]

into the Brunovský canonical form

\[ \Pi^{[1]}_{CF} : x^+ = Ax + Bv. \]
Then we study, successively for \( m \geq 2 \), the action of the homogeneous feedback transformations

\[
\Upsilon^m : \begin{align*}
x &= \xi + \phi^m(\xi) \\
u &= v + \gamma^m(\xi, v)
\end{align*}
\]

(10.3)
on the homogeneous systems

\[
\Pi^m : \begin{align*}
\dot{\xi} &= A\xi + Bu + f^m(\xi, u) .
\end{align*}
\]

(10.4)

Let us consider another homogeneous system

\[
\tilde{\Pi}^m : \begin{align*}
\dot{x} &= Ax + Bv + \tilde{f}^m(x, v) .
\end{align*}
\]

(10.5)

**Definition 10.1** We say that the homogeneous system \( \Pi^m \), given by (10.4), is feedback equivalent to the homogeneous system \( \tilde{\Pi}^m \), given by (10.5), if there exist a homogeneous feedback transformation \( \Upsilon^m \), of the form (10.3), which brings the system \( \Pi^m \) into the system \( \tilde{\Pi}^m \) modulo terms in \( V \geq m + 1 \).

The following proposition is the analog, for discrete-time control systems, of Proposition 3.1, stated for continuous systems. It establishes the conditions of equivalence of two homogeneous systems.

**Proposition 10.2** The homogeneous feedback transformation \( \Upsilon^m \), defined by (10.3), brings the homogeneous system \( \Pi^m \), given by (10.4), into the homogeneous system \( \tilde{\Pi}^m \), given by (10.5), if and only if the following relation

\[
\phi^m_j(A\xi + Bu) - \phi^m_{j+1}(\xi) = \tilde{f}^m_j(\xi, u) - f^m_j(\xi, u)
\]

and

\[
\phi^m_n(A\xi + Bu) + \gamma^m(\xi) = \tilde{f}^m_n(\xi, u) - f^m_n(\xi, u)
\]

hold for all \( 1 \leq j \leq n - 1 \).

The proof of this proposition is straightforward.

**Main Results.** Let us denote the control by \( v = x_{n+1} \), and for any \( 1 \leq i \leq n+1 \),

\[
\pi(x)_i = (x_1, \ldots, x_i)^T.
\]

The main result for discrete-time nonlinear control systems with controllable linearization is as following.

**Theorem 10.3** The homogeneous control system \( \Pi^m \), defined by (10.4), is feedback equivalent, by a homogeneous feedback transformation \( \Upsilon^m \) of the form (10.3), to the normal form

\[
\Pi^m_{NF} : \begin{align*}
\dot{x} &= Ax + Bv + \bar{f}^m(x, v) ,
\end{align*}
\]

where the components of the vector field \( \bar{f}^m(x, v) \) are given by

\[
\bar{f}^m_j(x, v) = \begin{cases} 
\sum_{i=j+2}^{n+1} x_i x_i P^m_{j,i}(\pi(x)_i) & \text{if } 1 \leq j \leq n - 1 \\
0 & \text{if } j = n.
\end{cases}
\]

(10.6)
We can notice the similarity of this normal form with that of continuous systems (3.7) with the notable difference that the polynomials $P_{j,i}^{m-2}(\pi(x)_i)$, instead of being multiplied by $x_i^2$, are multiplied by $x_1x_i$. As the homogeneous feedback transformations $\Upsilon^m$ leave invariant the terms of degree less than $m$, a successive application of Theorem 10.3 gives the following corollary.

**Corollary 10.4** The control system $\Pi^\infty$, defined by (10.1), is feedback equivalent, by a formal feedback transformation $\Upsilon^\infty$ of the form (10.2), to the normal form

$$\Pi_{NF}^\infty : \ x^+ = Ax + Bv + \sum_{m=2}^{\infty} f^{[m]}(x,v),$$

where for any $m \geq 2$, the components of the vector field $f^{[m]}(x,v)$ are given by (10.6).

To illustrate our results we consider the following example of a pendulum described in [81].

### 10.1 Example: Bressan and Rampazzo Pendulum

Consider the Bressan and Rampazzo pendulum (see [8], [81]) described by the equations

$$\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= -g \sin \xi_3 + \xi_1 \xi_4^2 \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= u,
\end{align*}$$

$\xi_1$ denotes the length of the pendulum, $\xi_2$ its velocity, $\xi_3$ the angle of the pendulum with respect to the horizontal, $\xi_4$ its angular velocity, and $g$ the gravity constant.

We discretize the system by taking

$$\begin{align*}
\dot{\xi}_1 &= \xi_1^+ - \xi_1, \\
\dot{\xi}_2 &= \xi_2^+ - \xi_2, \\
\dot{\xi}_3 &= \xi_3^+ - \xi_3, \\
\dot{\xi}_4 &= \xi_4^+ - \xi_4.
\end{align*}$$

The system above rewrites

$$\begin{align*}
\xi_1^+ &= \xi_1 + \xi_2 \\
\xi_2^+ &= \xi_2 - g \sin \xi_3 + \xi_1 \xi_4^2 \\
\xi_3^+ &= \xi_3 + \xi_4 \\
\xi_4^+ &= \xi_4 + u.
\end{align*}$$

Let us consider the change of coordinates

$$\begin{align*}
z_1 &= \xi_1 \\
z_2 &= \xi_2 + \xi_1 \\
z_3 &= -g \sin \xi_3 + 2\xi_2 + \xi_1 \\
z_4 &= -g \sin(\xi_4 + \xi_3) + 3\xi_2 - 2g \sin \xi_3 + 2\xi_1 \xi_4^2 + \xi_1 \\
v &= z_4^+.
\end{align*}$$

whose inverse is such that $\xi_4 = h(z_1, z_2, z_3, z_4)$ is a smooth function. This change of coordinates takes the system into the form

$$\begin{align*}
z_1^+ &= z_2 \\
z_2^+ &= z_3 + z_1 h^2(z_1, z_2, z_3, z_4) \\
z_3^+ &= z_4 \\
z_4^+ &= v.
\end{align*}$$
Actually the function \( h^2(z_1, z_2, z_3, z_4) \) could be decomposed as
\[
h^2(z_1, z_2, z_3, z_4) = h_1(z_1, z_2, z_3) + z_4 h_2(z_1, z_2, z_3, z_4)
\]
where the 1-jet at 0 of \( h_1 \) is zero and \( h_2(0) = 0 \). Put \( H_1(z_1, z_2, z_3) = z_1 h_1(z_1, z_2, z_3) \).

The objective is to show that we can get rid of the terms \( H_1(z_1, z_2, z_3) \). Let us suppose that the \( k \)-jet at 0 of \( H_1 \) is zero.

Consider the change of coordinates \( \tilde{z}_1 = z_1, \tilde{z}_2 = z_2, \tilde{z}_3 = z_3 + H_1(z_1, z_2, z_3), \tilde{z}_4 = \tilde{z}_4^+ \). This change of coordinates, completed by the feedback \( \tilde{z}_4^+ = w \), takes the system into the form
\[
\begin{align*}
\tilde{z}_1^+ &= \tilde{z}_2 \\
\tilde{z}_2^+ &= \tilde{z}_3 + \tilde{H}_1(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) + \tilde{z}_1 \tilde{z}_4 \tilde{H}_2(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4) \\
\tilde{z}_3^+ &= \tilde{z}_4 \\
\tilde{z}_4^+ &= w,
\end{align*}
\]
where \( \tilde{H}_1(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) \) and \( \tilde{H}_2(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4) \) are some smooth functions. It is enough to remark that the \((k+2)\)-jet at 0 of \( H_1(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) \) is zero because the 2-jet of \( z_1 z_4 H_2(z) \) is zero. Then by iteration we can cancel terms \( H_1(z_1, z_2, z_3) \) and put the system into the desired normal form
\[
\begin{align*}
x_1^+ &= x_2 \\
x_2^+ &= x_3 + x_1 x_4 H(x_1, x_2, x_3, x_4) \\
x_3^+ &= x_4 \\
x_4^+ &= v.
\end{align*}
\]

11 Symmetries of control systems

In this section we will discuss relations between the canonical form \( \Sigma_{CF}^{\infty} \) given in Section 4 and symmetries of nonlinear control systems.

Recently there has been a growing interest in symmetries of nonlinear control systems. The structure of control systems possessing symmetries was analyzed e.g. by Grizzle and Marcus in [24] and Gardner, Shadwick, and Wilkens (for linearizable systems) in [20] and [22]. The role of symmetries in the optimal control problems has been studied, among others by Jurdjevic [45], [46] (for systems on Lie groups), van der Schaft [93], and Sussmann [92]. In [41], Jakubczyk gave a complete characterization of symmetries in terms of symbols of control systems.

In this section we study symmetries of single-input nonlinear control affine systems whose linear approximation, at an equilibrium point \( p \), is controllable. We will give two results of the authors devoted, respectively, to stationary symmetries [71] and nonstationary symmetries [70]. The first, given in Section 11.2, states that “almost any” single-input control system, which is truly nonlinear (that is non linearizable via feedback) does not admit any stationary symmetry, that is any symmetry preserving the equilibrium point \( p \). “Almost any” refers to all systems away from a small class of odd systems which admit one nontrivial stationary symmetry, that is conjugated to minus identity by a diffeomorphism bringing the system to its canonical form \( \Sigma_{CF}^{\infty} \) of Section 4. In Section 11.3, for the same class of systems and around an equilibrium point \( p \), we study nonstationary symmetries, that is symmetries which do not preserve \( p \). Our main result states also that for nonstationary symmetries a complete picture can be deduced from the canonical form. We prove that an analytic system, equivalent by an analytic feedback transformation to its canonical form, admits a nonstationary symmetry if and only if the drift vector
field defining the canonical form is periodic with respect to the first variable and that a system admits a 1-parameter family of symmetries if and only if that drift vector field does not depend on the first variable. Moreover, we show that in the latter case the set of all symmetries is given either by exactly one 1-parameter family of symmetries (in the non odd case) or by exactly two 1-parameter families of symmetries (in the odd case). In the case when the feedback transformation, bringing the system to its canonical form, is given by a (not necessarily convergent) formal power series, we prove that an analogous result holds for a formal infinitesimal symmetry. In fact, its existence is equivalent to the fact that the drift of the formal canonical form does not depend on the first variable.

We will also describe all symmetries of feedback linearizable systems (we will follow [68], see also [20] and [22]) in order to show an enormous gap between the group of symmetries of feedback linearizable and nonlinearizable systems.

\section{Symmetries}

In this section we will introduce the notion of symmetries of nonlinear control systems (see also [24], [41], [71], [93]). Let us consider the system

$$\Pi : \dot{x} = F(x, u),$$

where \( x \in X \), a smooth \( n \)-dimensional manifold and \( u \in U \), a smooth \( m \)-dimensional manifold. The map \( F : X \times U \to TX \) is assumed to be smooth with respect to \((x, u)\) and for any value \( u \in U \) of the control parameter, \( F \) defines a smooth vector field \( F_u \) on \( X \), where \( F_u(\cdot) = F(\cdot, u) \).

Consider the field of \textit{admissible velocities} \( \mathcal{F} \) associated to the system \( \Pi \) and defined as (see Section 1)

$$\mathcal{F}(x) = \{ F_u(x) : u \in U \} \subset T_xX.$$  

We say that a diffeomorphism \( \sigma : X \to X \) is a \textit{symmetry} of \( \Pi \) if it preserves the field of admissible velocities \( \mathcal{F} \), that is,

$$\sigma_*\mathcal{F} = \mathcal{F}.$$  

Recall that for any vector field \( f \) on \( X \) and any diffeomorphism \( y = \phi(x) \) of \( X \), we put

$$(\phi_*f)(y) = D\phi(\psi^{-1}(y)) \cdot f(\phi^{-1}(y)).$$

A \textit{local symmetry} at \( p \in X \) is a local diffeomorphism \( \sigma \) of \( X_0 \) onto \( \tilde{X}_0 \), where \( X_0 \) and \( \tilde{X}_0 \) are, respectively, neighborhoods of \( p \) and \( \sigma(p) \), such that

$$(\sigma_*\mathcal{F})(q) = \mathcal{F}(q)$$

for any \( q \in \tilde{X}_0 \).

A local symmetry \( \sigma \) at \( p \) is called a \textit{stationary symmetry} if \( \sigma(p) = p \) and a \textit{nonstationary symmetry} if \( \sigma(p) \neq p \).

Let us consider a single-input control affine system

$$\Sigma : \dot{x} = f(x) + g(x)u,$$

where \( x \in X \), \( u \in U = \mathbb{R} \) and \( f \) and \( g \) are smooth vector fields on \( X \). The field of admissible velocities for the system \( \Sigma \) is the following field of affine lines:

$$\mathcal{A}(x) = \{ f(x) + ug(x) : u \in \mathbb{R} \} \subset T_xX.$$
A specification of the above definition says that a diffeomorphism $\sigma : X \rightarrow X$ is a symmetry of $\Sigma$ if it preserves the affine line field $\mathcal{A}$ (in other words, the affine distribution $\mathcal{A}$ of rank 1), that is, if

$$\sigma_* \mathcal{A} = \mathcal{A}.\)$$

We will call $p \in X$ to be an equilibrium point of $\Sigma$ if $0 \in \mathcal{A}(p)$. For any equilibrium point $p$ there exists a unique $\tilde{u} \in \mathbb{R}$ such that $\tilde{f}(p) = 0$, where $\tilde{f}(p) = f(p) + \tilde{u}g(p)$. By the linear approximation of $\Sigma$ at an equilibrium $p$ we will mean the pair $(F, G)$, where $F = \frac{\partial f}{\partial x}(p)$ and $G = g(p)$.

We will say that $\Sigma$ is an odd system at $p \in X$ if it admits a stationary symmetry at $p$, denoted by $\sigma^-$, such that

$$\frac{\partial \sigma^-}{\partial x}(p) = -\text{Id}.$$ 

### 11.2 Symmetries of single-input nonlinearizable systems

In this section we deal with single-input control affine systems of the form

$$\Sigma : \dot{x} = f(x) + g(x)u,$$

where $x \in X$, $u \in \mathbb{R}$. Our analysis will be local so we can assume that $X = \mathbb{R}^n$. Throughout this section we will assume that the point $p$ around which we work is an equilibrium, that is $f(p) = 0$ and, moreover, that $g(p) \neq 0$. We will prove that if $\Sigma$ is not feedback linearizable (see Section 9), then the group of local symmetries of $\Sigma$ around an equilibrium $p \in \mathbb{R}^n$ is very small. More precisely, the following result of the authors [70], [71] says that if $\Sigma$ is analytic, then it admits at most two 1-parameter families of local symmetries. We will say that $\sigma_c$, where $c \in (-\epsilon, \epsilon) \subset \mathbb{R}$, is a nontrivial 1-parameter analytic family of local symmetries if each $\sigma_c$ is a local analytic symmetry, $\sigma_{c_1} \neq \sigma_{c_2}$ if $c_1 \neq c_2$, and $\sigma_c(x)$ is jointly analytic with respect to $(x, c)$.

Assume that the system $\Sigma$ is analytic. If the feedback transformation $\Gamma = (\phi, \alpha, \beta)$, bringing $\Sigma_\infty$ into its canonical form $\Sigma_{\infty, CF}$, is analytic then we will denote the analytic canonical form of $\Sigma$ by $\Sigma_{CF}$. (that is, the analytic system whose infinite Taylor expansion is given by $\Sigma_{\infty, CF}$).

**Theorem 11.1** Assume that the system $\Sigma$ is analytic, the linear approximation $(F, G)$ of $\Sigma$ at an equilibrium point $p$ is controllable and that $\Sigma$ is not locally feedback linearizable at $p$. Assume, moreover, that the local feedback transformation, bringing $\Sigma$ into its canonical form $\Sigma_{CF}$, is analytic at $p$. Then there exists a local analytic diffeomorphism $\phi : X_0 \rightarrow \mathbb{R}^n$, where $X_0$ is a neighborhood of $p$, with the following properties.

(i) If $\sigma$ is a local analytic stationary symmetry of $\Sigma$ at $p$, then either $\sigma = \text{Id}$ or

$$\phi \circ \sigma \circ \phi^{-1} = -\text{Id}.$$ 

(ii) If $\sigma$ is a local analytic nonstationary symmetry of $\Sigma$ at $p$, then

$$\phi \circ \sigma \circ \phi^{-1} = T_c,$$

where $c \in \mathbb{R}$ and $T_c$ is either the translation $T_c = (x_1 + c, x_2, \ldots, x_n)$ or $T_c$ is replaced by $T_c^- = T_c \circ (-\text{Id}) = (-x_1 + c, -x_2, \ldots, -x_n)$. 

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(iii) If \( \sigma_c, \ c \in (-\epsilon, \epsilon) \) is a nontrivial 1-parameter analytic family of local symmetries of \( \Sigma \) at \( p \), then

\[
\phi \circ \sigma_c \circ \phi^{-1} = T_c,
\]

where \( T_c \) is as above, for \( c \in (-\epsilon, \epsilon) \).

If we drop the assumption that \( \Sigma \) is equivalent to its canonical form \( \Sigma_{CF} \) by an analytic feedback transformation, then items (i) and (iii) remain valid with the local analytic diffeomorphisms \( \phi \) being replaced by a formal diffeomorphism. This implies that the group of stationary symmetries of an analytic single-input control system is very small. Indeed, we have the following:

**Corollary 11.2** Consider an analytic single-input control system \( \Sigma \) and assume that it is not feedback linearizable and its linear approximation at an equilibrium point \( p \) is controllable. Then \( \Sigma \) possesses at most two analytic stationary symmetries at \( p \): identity and, if \( \Sigma \) is odd, a symmetry \( \sigma^- \) satisfying \( \frac{\partial \sigma^-}{\partial x}(p) = -\text{Id} \).

### 11.3 Symmetries of the canonical form

Symmetries take a very simple form if we bring the system into its canonical form. Indeed, we have the following result obtained by the authors (see [70] and [71] for proofs and details):

**Proposition 11.3** Assume that the system \( \Sigma \) is analytic, the linear approximation \((F, G)\) of \( \Sigma \) at an equilibrium point \( p \) is controllable and \( \Sigma \) is not locally feedback linearizable at \( p \). Assume, moreover, that the local feedback transformation, bringing \( \Sigma \) into its canonical form \( \Sigma_{CF} \), is analytic at \( p \).

(i) \( \Sigma \) admits a nontrivial local stationary symmetry if and only if the drift

\[
\bar{f}(x) = Ax + \sum_{m=m_0}^{\infty} \bar{f}^{|m|}(x)
\]

of the canonical form \( \Sigma_{\infty CF} \) satisfies

\[
\bar{f}(x) = -\bar{f}(-x),
\]

that is, the system is odd.

(ii) \( \Sigma \) admits a nontrivial local nonstationary symmetry if and only if the drift \( \bar{f}(x) \) of the canonical form \( \Sigma_{\infty CF} \) satisfies

\[
\bar{f}(x) = \bar{f}(T_c(x)),
\]

that is \( \bar{f} \) is periodic with respect to \( x_1 \).

(iii) \( \Sigma \) admits a nontrivial local 1-parameter family of symmetries if and only if the drift \( \bar{f}(x) \) of the canonical form \( \Sigma_{\infty CF} \) satisfies

\[
\bar{f}(x) = \bar{f}(x_2, \ldots, x_n).
\]

The above result describes all symmetries around an equilibrium of any single-input nonlinear system that is not feedback linearizable and whose first order approximation at the equilibrium is controllable. If we drop the assumption that \( \Sigma \) is equivalent to its canonical form \( \Sigma_{CF} \) by an analytic feedback transformation, then the ”only if” statements in items (i) and (iii) remain valid while in the ”if” statements we have to replace local symmetries by formal symmetries, that is, by formal diffeomorphisms which preserve the field of admissible velocities (see [70] and [?]), which we will do in the next section.
11.4 Formal symmetries

We do not know whether, in general, the feedback transformation $\Gamma^\infty$ bringing the system to its canonical form $\Sigma_{CF}^\infty$ converges. If it does, Theorem 11.1 and Proposition 11.3 describe all local symmetries of $\Sigma$. If it does not, the canonical form $\Sigma_{CF}^\infty$ is considered as a formal power series but even in this case it keeps, as we will show in the sequel, important information about symmetries.

We say that a vector field $v$ on an open subset $X \subset \mathbb{R}^n$ is an infinitesimal symmetry of the system $\Sigma$ if the (local) flow $\gamma^v_t$ of $v$ is a local symmetry of $\Sigma$, for any $t$ for which it exists.

Consider the system $\Sigma$ and denote by $G$ the distribution spanned by the vector field $g$. We have the following characterization of infinitesimal symmetries.

**Proposition 11.4** A vector field $v$ is an infinitesimal symmetry of $\Sigma$ if and only if

$$[v, f] = 0 \text{ mod } G, \quad [v, g] = 0 \text{ mod } G.$$ 

This characterization of infinitesimal symmetries justifies the following notion. We say that a vector field formal series

$$v^\infty(\xi) = \sum_{m=0}^{\infty} v^{[m]}(\xi)$$

is a formal infinitesimal symmetry of the system

$$\Sigma^\infty : \dot{\xi} = f(\xi) + g(\xi)u = \sum_{m=1}^{\infty} (f^{[m]}(\xi) + g^{[m-1]}u)$$

if it satisfies

$$[v^\infty, f] = 0 \text{ mod } G, \quad [v^\infty, g] = 0 \text{ mod } G.$$ 

Here, $[\cdot, \cdot]$ is understood as the Lie bracket of formal power series vector fields.

**Theorem 11.5** Consider the system $\Sigma^\infty$. Assume that its linear approximation $(F, G)$ is controllable and that $\Sigma^\infty$ is not feedback linearizable. The following conditions are equivalent.

(i) $\Sigma^\infty$ admits a formal infinitesimal symmetry.

(ii) The only formal infinitesimal symmetry of $\Sigma^\infty$ is $v^\infty = (\phi^{-1})_* \frac{\partial}{\partial x_1}$, where $\phi$ is the diffeomorphism defining a feedback transformation $\Gamma^\infty$ that brings $\Sigma^\infty$ into its canonical form $\Sigma_{CF}^\infty$.

(iii) The canonical form $\Sigma_{CF}^\infty$ of $\Sigma^\infty$ satisfies $\bar{f}^{[m]}(x) = \bar{f}^{[m]}(x_2, \ldots, x_n)$, for any $m \geq m_0$, where the vector fields $\bar{f}^{[m]}$ are of the form (4.4), (4.5), and (4.6).

(iv) For any $c_1 \in \mathbb{R}$, the translation $T_{c_1}(x) = (x_1 + c_1, x_2, \ldots, x_n)^T$ is a symmetry of the canonical form $\Sigma_{CF}^\infty$.

(v) The vector field $v^\infty_{CF} = \frac{\partial}{\partial x_1}$ is a formal infinitesimal symmetry of the canonical form $\Sigma_{CF}^\infty$.

This result, established in the formal category, provides the following necessary condition for the existence of analytic 1-parameter families of symmetries. Notice that below we do not assume that the feedback transformation $\Gamma^\infty$, bringing $\Sigma$ to its canonical form $\Sigma_{CF}^\infty$, converges.
Proposition 11.6 Consider an analytic system $\Sigma$ and assume that its linear approximation is controllable and the system is not feedback linearizable. If $\Sigma$ admits a nontrivial analytic local 1-parameter group of symmetries $\sigma_{c_1}$, for $c_1 \in (-\epsilon, \epsilon)$, then the drift vector field of the canonical form $\Sigma_{C_F}$ satisfies $\bar{f}^{[m]}(x) = \bar{f}^{[m]}(x_2, \ldots, x_n)$, for any $m \geq m_0$.

We will end this section by giving a necessary condition for the existence of a family of local nonstationary symmetries which does not require to bring the system to its canonical form but only to normalize a finite number of terms.

Let $m_0$ denote the largest nonnegative integer such that for any $1 \leq k \leq n$, the distributions $D_k = (g, \text{ad}_f g, \ldots, \text{ad}^{k-1}_f g)$ have constant rank $k$ and are involutive modulo terms of order $m_0 - 2$. It follows that the system $\Sigma$ is feedback linearizable up to order $m_0 - 1$ (see [54]). We thus can bring the system to the form

$$\hat{\Sigma} : \dot{x} = Ax + Bu + \bar{f}^{[m_0]}(x) + R(x, v),$$

where $R(x, v) \in V(x, v)^{\geq m_0+1}$ and $(A, B)$ is in the Brunovský canonical form and the first nonlinearizable homogeneous vector field $\bar{f}^{[m_0]}$ whose components are given by

$$\bar{f}^{[m_0]}_j(x) = \begin{cases} 
\sum_{i=j+2}^n x_i^2 P^{[m_0-2]}_{j,i}(x_1, \ldots, x_i) & \text{if } 1 \leq j \leq n-2, \\
0 & \text{if } n-1 \leq j \leq n,
\end{cases}$$

is in Kang normal form $\Sigma^{[m_0]}$.

Proposition 11.7 Under the assumptions of Proposition 11.6, if $\bar{f}^{[m_0]}(x)$ depends on $x_1$ then $\Sigma$ does not admit any nontrivial analytic local 1-parameter group of symmetries.

We would like to emphasize that the above condition is checkable via an algebraic calculation. In fact, bringing the terms of degree smaller than $m_0$ of $\Sigma$ to their Kang normal form means simply to annihilate them (compare Section 9). In Section 3 we gave explicit polynomial transformations that bring a homogenous part of any degree of a system to Kang normal form $\Sigma^{[m_0]}$. Therefore a successive use of those polynomial transformations, of degree 2 up to $m_0$, brings $\Sigma$ into $\hat{\Sigma}$ for which we can apply Proposition 11.7.

11.5 Symmetries of feedback linearizable systems

In the previous section we proved that the group of symmetries of feedback nonlinearizable systems around an equilibrium is very small provided that the linear approximation at the equilibrium is controllable. A natural question is thus what are symmetries of feedback linearizable systems? In this section we will show that symmetries of such systems form an infinite dimensional group parameterized by $m$ arbitrary functions of $m$ variables, where $m$ is the number of controls. It is interesting to observe that just one nonlinearity, which is not removable by feedback, destroys this infinite dimensional group leaving, at most, two one-parameter families of symmetries (compare Theorem 11.1).

We will describe symmetries of linear systems in Brunovský canonical form and then of feedback linearizable systems. For simplicity we will deal with systems with all controllability indices equal. Another description of symmetries of linear systems in Brunovský canonical form was given by Gardner et al in [20] and [22].
Consider a linear control system in the Brunovský canonical form with all controllability indices equal, say to \( n \),

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= v,
\end{align*}
\]
on \( \mathbb{R}^N \), where \( \dim v = m \), \( \dim x^j = n \), \( N = nm \).

Put \( \pi^1(x) = x^1 \). For any diffeomorphism \( \mu \) of \( \mathbb{R}^m \) we define \( \mu^1 : \mathbb{R}^N \to \mathbb{R}^m \) by

\[ \mu^1 = \mu \circ \pi^1. \]

**Proposition 11.8** Consider the linear system \( \Lambda \) in Brunovský form.

(i) For any diffeomorphism \( \mu \) of \( \mathbb{R}^m \), the map

\[
\lambda_{\mu} = \left( \begin{array}{c} \mu^1 \\ L_{Ax} \mu^1 \\ \vdots \\ L^n_{Ax} \mu^1 \end{array} \right)
\]

is a symmetry of \( \Lambda \).

(ii) Conversely, if \( \sigma \) is a symmetry of \( \Lambda \), then

\[ \sigma = \lambda_{\mu}, \]

for some diffeomorphism \( \mu \) of \( \mathbb{R}^m \).

Notice that \( \mu^1 \) is a map from \( \mathbb{R}^N \) into \( \mathbb{R}^m \) depending on the variables \( x^1 \) only. The transformation \( \lambda_{\mu} : \mathbb{R}^N \to \mathbb{R}^N \) is defined by successive differentiating this map with respect to the drift \( Ax \). Item (i) claims that such a transformation is always a symmetry of the linear system \( \Lambda \) (in particular, a diffeomorphism) while item (ii) claims that all symmetries of linear systems are always of this form.

**Remark 11.9** Clearly, an analogous result holds for local symmetries, that is, if \( \mu \) is a local diffeomorphism of \( \mathbb{R}^m \), then the corresponding \( \lambda_{\mu} \) is a local symmetry of \( \Lambda \) and, conversely, any local symmetry of \( \Lambda \) is of the form \( \lambda_{\mu} \) for some local diffeomorphism \( \mu \).

This local version of the above result will allow us to describe in the next section all local symmetries of feedback linearizable systems.

Consider a control-affine system of the form

\[ \Sigma : \dot{\xi} = f(\xi) + \sum_{i=1}^{m} g_i(\xi) u_i, \]

where \( \xi \in \Xi \), an \( N \)-dimensional manifold, and \( f \) and \( g_i \) for \( 1 \leq i \leq m \) are \( C^\infty \) vector fields on \( \Xi \). We will say that \( \Sigma \) is feedback equivalent (or feedback linearizable) to a linear system of the form

\[ \Lambda : \dot{x} = Ax + Bv, \]
if there exists a feedback transformation of the form
\[
\Gamma : \quad x = \Phi(\xi) \\
u = \alpha(\xi) + \beta(\xi)v,
\]
with \(\beta(\xi)\) invertible, transforming \(\Sigma\) into \(\Lambda\). We say that \(\Sigma\) is locally feedback linearizable at \(\xi_0\) if \(\Phi\) is a local diffeomorphism at \(\xi_0\) and \(\alpha\) and \(\beta\) are defined locally around \(\xi_0\).

Define the following distributions:
\[
D^0 = \text{span} \{g_1, \ldots, g_m\} \quad \text{and} \quad D^{i+1} = D^i + [f, D^i].
\]
The system \(\Sigma\) is, locally at \(\xi_0\), feedback equivalent to a linear system \(\Lambda\), see Section 9, with all controllability indices equal to \(n\), if and only if the distributions \(D^j\) are involutive and of constant rank \((j + 1)m\) for \(0 \leq j \leq n - 1\).

For any map \(\varphi : \Xi_0 \rightarrow \mathbb{R}^m\), where \(\Xi_0\) is a neighborhood of \(\xi_0\), put
\[
\Phi_\varphi = \begin{pmatrix}
\varphi \\
L_1\varphi \\
\vdots \\
L_m\varphi
\end{pmatrix}.
\]

Notice that \(\Phi_\varphi\) is a map from \(\Xi_0\) in \(\mathbb{R}^N\). If the map \(\varphi = (\varphi_1, \ldots, \varphi_m)\) is chosen such that
\[
(D^{n-1})^\perp = \text{span} \{d\varphi\} = \text{span} \{d\varphi_1, \ldots, d\varphi_m\},
\]
then it is well known (see, e.g., [36], [NS-book]) that \(\Phi_\varphi\) is a local diffeomorphism of an open neighborhood \(\Xi_\varphi\) of \(\xi_0\) onto \(X_\varphi = \Phi_\varphi(\Xi_\varphi)\), an open neighborhood of \(x_0 = \Phi_\varphi(\xi_0)\), and gives local linearizing coordinates for \(\Sigma\) in \(\Xi_\varphi\). To keep the notation coherent, we will denote by \(\xi\), with various indices, points of \(\Xi_\varphi\), by \(x\), with various indices, points of \(X_\varphi = \Phi_\varphi(\Xi_\varphi) \subset \mathbb{R}^N\), and by \(y\), with various indices, points of \(\pi^1(X_\varphi) \subset \mathbb{R}^m\), where \(\pi^1\) is the projection \(\pi^1(x) = x^1\).

Combining this result with Proposition 11.8, we get the following complete description of local symmetries of feedback linearizable systems with equal controllability indices. The notation \(\text{Diff}(\mathbb{R}^m; y_0, \tilde{y}_0)\) will stand for the family of all local diffeomorphisms of \(\mathbb{R}^m\) at \(y_0\) transforming \(y_0\) into \(\tilde{y}_0\) (more precisely, all diffeomorphisms germs with the base point \(y_0\) and its image \(\tilde{y}_0\)).

**Theorem 11.10** Let the system \(\Sigma\) be locally feedback linearizable at \(\xi_0\) with equal controllability indices. Fix \(\varphi : \Xi_0 \rightarrow \mathbb{R}^m\) such that \((D^{n-1})^\perp = \text{span} \{d\varphi\} = \text{span} \{d\varphi_1, \ldots, d\varphi_m\}\).

(i) Let \(\mu \in \text{Diff}(\mathbb{R}^m; y_0, \tilde{y}_0)\), where \(y_0 = \pi^1(x_0)\) and \(\tilde{y}_0 = \pi^1(\lambda_\mu(x_0))\), such that \(\lambda_\mu(x_0) \in X_\varphi\). Then
\[
\sigma_{\mu, \varphi} = \Phi^{-1}_\varphi \circ \lambda_\mu \circ \Phi_\varphi
\]
is a local symmetry of \(\Sigma\) at \(\xi_0\).

(ii) Conversely, if \(\sigma\) is a local symmetry of \(\Sigma\) at \(\xi_0\), such that \(\sigma(\xi_0) \in \Xi_\varphi\), then there exits \(\mu \in \text{Diff}(\mathbb{R}^m; y_0, \tilde{y}_0)\), where \(y_0 = \pi^1(x_0), \tilde{y}_0 = \pi^1(\tilde{x}_0), \tilde{x}_0 = \Phi_\varphi(\sigma(\xi_0))\) such that
\[
\sigma = \sigma_{\mu, \varphi}.
\]
Moreover, \(\sigma_{\mu, \varphi} = \Phi^{-1}_\varphi \circ \lambda_\mu \circ \Phi_\varphi = \Phi^{-1}_\varphi \circ \Phi_{\mu_\varphi}\).

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The structure of symmetries of feedback linearizable systems is thus summarized by the following diagram.

\[
\begin{array}{c}
(\Sigma, \xi_0) \xrightarrow{\sigma_{\mu, \varphi}} (\Sigma, \hat{\xi}_0) \\
\Phi_{\varphi} \downarrow \Phi_{\mu \circ \varphi} \downarrow \Phi_{\varphi} \\
(\Lambda, x_0) \xrightarrow{\lambda_{\mu}} (\Lambda, \hat{x}_0)
\end{array}
\]

Item (i) states that composing a linearizing transformation \( \Phi_{\varphi} \) with a symmetry \( \lambda_{\mu} \) of the linear equivalent \( \Lambda \) of \( \Sigma \) and with the inverse \( \Phi_{\varphi}^{-1} \) we get a symmetry of \( \Sigma \), provided that the image \( \hat{x}_0 = \lambda_{\mu}(x_0) \) belongs to \( X_{\varphi} \) (otherwise the composition is not defined). Item (ii) asserts that any local symmetry of a feedback linearizable system is of this form. Moreover, any local symmetry can be expressed as a composition of one linearizing transformation with the inverse of another linearizing transformation. Indeed, observe that for any fixed \( \varphi \), the map \( \Phi_{\mu \circ \varphi} \), for \( \mu \in Diff(R^m; \tilde{y}_0, \hat{y}_0) \), gives a linearizing diffeomorphism and taking all \( \mu \in Diff(R^m; \tilde{y}_0, \hat{y}_0) \) for all \( \tilde{y}_0 \in \pi^1(X_{\varphi}) \), the corresponding maps \( \Phi_{\mu \circ \varphi} \) provide all linearizing transformations around \( \xi_0 \).

It follows from item (ii) that the group of symmetries of feedback linearizable systems is infinite dimensional and parameterized by \( m \) functions of \( m \) variables. It is interesting to observe that just one nonlinearity, which is not removable by feedback, destroys this infinite dimensional group leaving, at most, two one-parameter families of symmetries (compare Theorem 11.1).

12 Feedforward and strict feedforward forms

In this section, we study the problem of transforming a single-input nonlinear control system to feedforward form and strict feedforward forms via a static state feedback. We provide checkable necessary and sufficient conditions (which involve the homogeneous \( m \)-invariants defined in Section 3) to bring the homogeneous terms of any fixed degree of the system into homogeneous feedforward form. If those conditions are satisfied, this leads to a constructive procedure which transforms the system, step by step, into feedforward or strict feedforward form. We illustrate our solution by analyzing the four-dimensional case. In particular, we compute the codimension of four-dimensional systems that are feedback equivalent to the feedforward form and strict feedforward form.

This section is organized as follows. In Section 12.1 we will define the class of feedforward and strict feedforward systems, in both general and control-affine cases. We will also fix some notations used throughout the whole section. In Section 12.2 we will introduce feedforward and strict feedforward normal forms. Then we will present a step-by-step method transforming a given system to the feedforward or strict feedforward form (whenever it is possible): in Section 12.3 for the first nonlinearizable term and in Section 12.4 for terms of an arbitrary degree. We will illustrate our approach by analyzing feedforward and strict feedforward systems on \( \mathbb{R}^4 \) in Section 12.5. Finally, we will discuss the geometry of feedforward and strict feedforward systems in Section 12.6.
12.1 Introduction and notations

Consider a single-input nonlinear control system of the form

\[ \dot{\xi} = F(\xi, u), \]

where \( \xi \in \mathbb{R}^n \) and \( u \in \mathbb{R} \). We say that the system \( \Pi \) is in feedforward form (resp. in strict feedforward form) if we have

\[
\begin{align*}
\dot{\xi}_1 &= F_1(\xi_1, \ldots, \xi_n, u) \\
\dot{\xi}_2 &= F_2(\xi_2, \ldots, \xi_n, u) \\
&\vdots \\
\dot{\xi}_{n-1} &= F_{n-1}(\xi_{n-1}, \xi_n, u) \\
\dot{\xi}_n &= F_n(u).
\end{align*}
\]

(resp. \[
\begin{align*}
\dot{\xi}_1 &= F_1(\xi_2, \ldots, \xi_n, u) \\
\dot{\xi}_2 &= F_2(\xi_3, \ldots, \xi_n, u) \\
&\vdots \\
\dot{\xi}_{n-1} &= F_{n-1}(\xi_n, u) \\
\dot{\xi}_n &= F_n(u).
\end{align*}
\]

One of the most appealing features of systems in (strict) feedforward form is that we can construct for them a stabilizing feedback. This important result goes back to Teel [89] and has been followed by a growing literature on stabilization and tracking for systems in (strict) feedforward form (see e.g. [37], [62], [74], [90], [3], [63]).

Feedforward systems can be viewed as duals of feedback linearizable systems. To see this, recall that in the single-input case, the class of feedback linearizable systems coincides with that of flat systems. Single-input flat systems are defined as systems for which we can find a function of the state that, together with its derivatives, gives all the states and the control of the system (see [17], [18], [39], [66]). In a dual way, for systems in strict feedforward form, we can find all states via a successive integration starting from a function of the control. Indeed, knowing \( u(t) \) we integrate \( F_n(u(t)) \) to get \( \xi_n(t) \), then we integrate \( F_{n-1}(\xi_n(t), u(t)) \) to get \( \xi_{n-1}(t) \), we keep doing that, and finally we integrate \( F_1(\xi_2(t), \ldots, \xi_n(t), u(t)) \) to get \( \xi_1(t) \). For feedforward systems, solutions can be found by solving scalar differential equations: for each component we have to solve one scalar differential equation.

It is therefore natural to ask which systems are equivalent to one of feedforward forms defined above. In [61], the problem of transforming a system, linear with respect to controls, into (strict) feedforward form via a diffeomorphism, i.e., via a nonlinear change of coordinates, was studied. A geometric description of systems transformable into feedforward form, either via a diffeomorphism or via feedback, has been given in [2]. Similar conditions for the strict feedforward form have recently been proposed by the authors [72], where relations between strict feedforward systems and the notion of symmetries (as defined in Section 11) are studied. The conditions of [2] and of [72] (which we recall in Section 12.6) are intrinsic and explain the geometry of the problem but in most cases are not checkable. In [78], [82], and in [80] we proposed a constructive procedure which allows to verify, step by step, whether a given system is feedback equivalent to the feedforward or strict feedforward form and to bring it to that form whenever it is possible. Our solutions were inspired by and based on formal approach to the feedback equivalence problem described in Section 3 and thus constitute a nice example of the strength of the formal approach.

We will be dealing with control-affine systems of the form

\[ \dot{\xi} = f(\xi) + g(\xi)u, \]

where \( \xi \in \mathbb{R}^n \) and \( u \in \mathbb{R} \). As usual, we assume that \( f(0) = 0 \) and \( g(0) \neq 0 \). A specification of the general definition to the control-affine case implies that \( \Sigma \) is in feedforward form, or that it
is a feedforward system, if we have

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_n
\end{pmatrix} = \begin{pmatrix}
f_1(x_1, \ldots, x_n) \\
f_2(x_2, \ldots, x_n) \\
\vdots \\
f_n(x_n)
\end{pmatrix} = \begin{pmatrix}
g_1(x_1, \ldots, x_n) \\
g_2(x_2, \ldots, x_n) \\
\vdots \\
g_n(x_n)
\end{pmatrix}.
\]

Similarly, \( \Sigma \) is in strict feedforward form (equivalently, it is a strict feedforward system), if we have

\[
\dot{\xi} = \begin{pmatrix}
f_1(\xi_1, \ldots, \xi_n) \\
f_2(\xi_2, \ldots, \xi_n) \\
\vdots \\
f_n(\xi_n)
\end{pmatrix} = \begin{pmatrix}
g_1(\xi_1, \ldots, \xi_n) \\
g_2(\xi_2, \ldots, \xi_n) \\
\vdots \\
g_n(\xi_n)
\end{pmatrix},
\]

where the components \( f_n, g_n \) are constant and satisfy \( f_n = 0 \) (because 0 is assumed to be an equilibrium) and \( g_n \neq 0 \).

In order to present our step-by-step approach, together with the system

\[
\dot{\xi} = f(\xi) + g(\xi)u,
\]

we will consider its Taylor series expansion

\[
\Sigma^\infty : \dot{\xi} = F\xi + Gu + \sum_{m=2}^{\infty} (f^{[m]}(\xi) + g^{[m-1]}(\xi))u,
\]

where \( F = \frac{\partial f}{\partial \xi}(0) \) and \( G = g(0) \).

Consider the Taylor series expansion \( \Gamma^\infty \) of the feedback transformation \( \Gamma \) given by

\[
\dot{x} = \phi(\xi) = T\xi + \sum_{m=2}^{\infty} \phi^{[m]}(\xi)
\]

\[
\Gamma^\infty : u = \alpha(\xi) + \beta(\xi)v = K\xi + Lv + \sum_{m=2}^{\infty} \alpha^{[m]}(\xi) + \beta^{[m-1]}(\xi)v,
\]

where \( T \) is an invertible matrix and \( L \neq 0 \). We will try (whenever it is possible) to bring the system \( \Sigma^\infty \) into the (strict) feedforward form step by step analyzing the action of \( \Gamma^\infty \).

**Notations.** We will use normal forms and transformations which have been already defined in the paper but we will also introduce normal forms specific for this section. The symbols \( \Sigma^{[m]} \), \( \Sigma^{[\leq m]} \), and \( \Sigma^\infty \) will stand for the systems under consideration: homogenous, polynomial, and formal, respectively. Their state vector will be denoted by \( \xi \) and their control by \( u \). The system \( \Sigma^{[m]} \) (resp. \( \Sigma^{[\leq m]} \) and \( \Sigma^\infty \)) transformed via a feedback transformation \( \Gamma^m \) (resp. \( \Gamma^{\leq m} \) and \( \Gamma^\infty \)) will be denoted by \( \bar{\Sigma}^{[m]} \) (resp. \( \bar{\Sigma}^{[\leq m]} \) and \( \bar{\Sigma}^\infty \)). Its state vector will be denoted by \( x \), its control by \( v \), and the vector fields, defining its dynamics, by \( \bar{f}^{[m]} \) and \( \bar{g}^{[m-1]} \). Feedback equivalence of systems \( \Sigma^{[m]} \) and \( \bar{\Sigma}^{[m]} \) and of systems \( \Sigma^{[\leq m]} \) and \( \bar{\Sigma}^{[\leq m]} \) will be established via a smooth feedback. To be more precise, via a homogeneous feedback \( \Gamma^m \) in the former case and via a polynomial feedback \( \Gamma^{\leq m} \) in the latter. On the other hand, feedback equivalence of systems \( \Sigma^\infty \) and \( \bar{\Sigma}^\infty \) will be established via a formal feedback \( \Gamma^\infty \).
We will use three kinds of normal forms for systems: Kang normal forms, feedforward normal forms, and strict feedforward normal forms. The symbol “bar” will correspond to the vector field $\bar{f}^{[m]}$ defining Kang normal forms $\Sigma^{[m]}_{NF}$, $\Sigma_{\leq m}^{NF}$, and $\Sigma_{\infty}^{NF}$. The symbol “hat” will correspond to the vector field $\hat{f}^{[m]}$ defining feedforward normal forms $\Sigma_{SFNF}^{[m]}$, $\Sigma_{\leq m}^{SFNF}$, and $\Sigma_{\infty}^{SFNF}$ and strict feedforward normal forms $\Sigma_{SFNF}^{[m]}$, $\Sigma_{\leq m}^{SFNF}$, and $\Sigma_{\infty}^{SFNF}$. Analogously, the $m$-invariants of the system $\Sigma^{[m]}$ will be denoted by $a^{[m]}_{j,i+2}$, the $m$-invariants of the system $\Sigma^{[m]}_{NF}$ by $\bar{a}^{[m]}_{j,i+2}$, and the $m$-invariants of the systems $\Sigma^{[m]}_{SFNF}$ or $\Sigma_{SFNF}^{[m]}$ by $\hat{a}^{[m]}_{j,i+2}$.

### 12.2 Feedforward and strict feedforward normal forms

We suppose throughout this section that the linear part $(F, G)$ of the system $\Sigma^{\infty}$, given by (12.1), is controllable, and thus we can assume, without loss of generality, that the system is in the form

$$\Sigma^{\infty}: \dot{\xi} = A\xi + Bu + \sum_{m=2}^{\infty} (f^{[m]}(\xi) + g^{[m-1]}(\xi)u),$$

where $(A, B)$ is in the Brunovský canonical form.

Recall, see Section 3, that (as proved by Kang [48], compare also [81]) any nonlinear system of the form (12.1) can be put, via a formal feedback transformation $\Gamma^{\infty}$, to the following Kang normal form:

$$\Sigma^{\infty}_{NF}: \dot{x} = Ax + Bv + \sum_{m=2}^{\infty} \bar{f}^{[m]}(x),$$

where for any $m \geq 2$,

$$\bar{f}^{[m]}_j(x) = \begin{cases} \sum_{i=j+2}^{n} x_i^2 P_{j,i}^{[m-2]}(x_1, \ldots, x_i) & \text{if } 1 \leq j \leq n - 2, \\ 0 & \text{if } n - 1 \leq j \leq n. \end{cases}$$

(12.2)

It is natural to ask whether it is possible to bring a system, that is feedback equivalent to feedforward form (resp. strict feedforward form), to Kang normal form (12.2) which would be simultaneously feedforward (resp. strict feedforward), that is, which would satisfy

$$P_{j,i}^{[m-2]}(x) = P_{j,i}^{[m-2]}(x_1, \ldots, x_i) \quad (\text{resp. } P_{j,i}^{[m-2]}(x) = P_{j,i}^{[m-2]}(x_{j+1}, \ldots, x_i)).$$

Although, this is always possible for the first nonlinearizable term, see Theorem 12.4 below, the answer to the above question is, in general, negative. For this reason we will introduce the following notions.

**Definition 12.1** *Strict feedforward normal form* is the system

$$\Sigma^{\infty}_{SFNF}: \dot{x} = Ax + Bv + \sum_{m=2}^{\infty} \hat{f}^{[m]}(x),$$

such that for any $m \geq 2$,

$$\hat{f}^{[m]}_j(x) = \begin{cases} c_{m,j} x_{j+1}^{m} + \sum_{i=j+2}^{n} x_i^2 P_{j,i}^{[m-2]}(x_{j+1}, \ldots, x_i), & 1 \leq j \leq n - 2, \\ 0 & n - 1 \leq j \leq n, \end{cases}$$

(12.2)
where \( c_{m,j} \in \mathbb{R} \) and \( P_{j,i}^{[m-2]} \) are homogeneous polynomials of degree \( m - 2 \), depending on the indicated variables.

In the feedforward case we introduce similarly:

**Definition 12.2** *Feedforward normal form* is the system

\[
\Sigma_{FNF}^\infty : \dot{x} = Ax + Bv + \sum_{m=2}^{\infty} \hat{f}^m(x),
\]

such that for any \( m \geq 2 \),

\[
\hat{f}^m_j(x) = \begin{cases} 
  x_j k_j^{[m-1]}(x_j, x_{j+1}) + \sum_{i=j+2}^n x_i^2 P_{j,i}^{[m-2]}(x_j, \ldots, x_i), & 1 \leq j \leq n-2, \\
  0 & n-1 \leq j \leq n,
\end{cases}
\]

where \( k_j^{[m-1]} \) and \( P_{j,i}^{[m-2]} \) are homogeneous polynomials, of degree \( m - 1 \) and \( m - 2 \), respectively, depending on the indicated variables.

Usefulness of feedforward and strict feedforward normal forms is justified by the following result:

**Theorem 12.3** The system \( \Sigma^\infty \), given by (12.1), is feedback equivalent to the feedforward form (resp. strict feedforward form) if and only if it is feedback equivalent to the feedforward normal form \( \Sigma_{FNF}^\infty \) (resp. strict feedforward normal form \( \Sigma_{SFNF}^\infty \)).

### 12.3 Feedforward and strict feedforward form: first nonlinearizable term

Consider the system \( \Sigma^\infty \), given by (12.1). Our goal is to study when it is possible to bring \( \Sigma^\infty \) to the feedforward (resp. strict feedforward form) and, if it is possible, to do it step by step. Assume that \( \Sigma^\infty \) is feedback linearizable up to order \( m_0 - 1 \). As proved by Krener [54], \( m_0 \) is the largest integer such that all distributions

\[
\mathcal{D}^k = \text{span} \left\{ g, ad_f g, \ldots, ad_f^{k-1} g \right\},
\]

for \( 1 \leq k \leq n-1 \), are involutive modulo terms of order \( m_0 - 2 \). Since \( \Sigma^\infty \) is feedback linearizable up to order \( m_0 - 1 \), it is also feedback equivalent to the strict feedforward (in particular, to the feedforward) form up to the same order, and we can thus assume without loss of generality that the system \( \Sigma^\infty \) is in feedforward (resp. strict feedforward) normal form up to order \( m_0 - 1 \) (see Theorem 12.3), that is, it takes the form

\[
\Sigma^{\leq m_0} : \dot{\xi} = A\xi + Bu + \sum_{m=2}^{m_0-1} h_j^m(\xi) + f_j^{[m_0]}(\xi) + g_j^{[m_0-1]}(\xi)u, \tag{12.3}
\]

modulo terms in \( V^{\geq m_0+1}(\xi, u) \), where for any \( 2 \leq m \leq m_0 - 1 \) we have

\[
h_j^m(\xi) = \begin{cases} 
  \xi_j k_j^{[m-1]}(\xi_j, \xi_{j+1}) & 1 \leq j \leq n-2, \\
  0 & n-1 \leq j \leq n \tag{12.4}
\end{cases}
\]
\begin{align}
\left( \text{resp. } h_j^{[m]}(\xi) = \begin{cases} c_{m,j}^n & \text{if } 1 \leq j \leq n-2, \\
0 & \text{if } n-1 \leq j \leq n \end{cases} \right). \tag{12.5}
\end{align}

Let us denote by $a^{[m_0],i,j+2}$ the $m_0$-invariants associated to the homogeneous part of degree $m_0$ of the system (12.3)-(12.4) or (12.3)-(12.5). Recall that

$$\Delta = \{ (j, i) \in \mathbb{N} \times \mathbb{N} : 1 \leq j \leq n-2, 0 \leq i \leq n-j-2 \}.$$ 

We have the following result concerning the first nonlinearizable term:

**Theorem 12.4** Consider the system $\Sigma^{[\leq m_0]}$, given by (12.3).

(i) There exists a transformation $\Gamma^{\leq m_0}$ bringing the system (12.3) into the feedforward form, up to order $m_0$ if and only if

$$L_{A^{m_0}} a_{[m_0],i,j+2} = 0 \tag{12.6}$$

for any $(j, i) \in \Delta$ and any $1 \leq q \leq j-1$.

(ii) There exists a transformation $\Gamma^{\leq m_0}$ bringing the system (12.3) into the strict feedforward form, up to order $m_0$ if and only if

$$L_{A^{m_0}B} a_{[m_0],i,j+2} = 0 \tag{12.7}$$

for any $(j, i) \in \Delta$ and any $1 \leq q \leq j$.

**Corollary 12.5** If there exists a transformation $\Gamma^{\infty}$ bringing the system

$$\Sigma^{\infty} : \dot{\xi} = A\xi + Bu + \sum_{m=m_0}^{\infty} (f^{[m]}(\xi) + g^{[m-1]}(\xi)u)$$

to feedforward form (resp. strict feedforward form), then the condition (12.6) (resp. (12.7)) is satisfied for any $(j, i) \in \Delta$ and any $1 \leq q \leq j-1$ (resp. $1 \leq q \leq j$).

In other words, the above result says that if a system is feedback equivalent to the feedforward form (resp. strict feedforward form), then, after having linearized lower order terms, the first nonlinearizable term must be feedforward (resp. strict feedforward) when transformed to the Kang normal form. This is the case if and only if the condition (12.6) (resp. (12.7)) is satisfied. If $m_0 = 2$, then the 2-invariants $a^{[2],i,j+2}$ are constant and the conditions (12.6) and (12.7) are automatically satisfied, which implies that any system is equivalent to the strict feedforward form (in particular, to the feedforward form) up to order 2. Actually, in this case, Kang-Krener normal form $\Sigma^{[2]}_{\Sigma}$ (recalled just after Theorem 3.2) is strict feedforward and can serve as a strict feedforward normal form (we do not have to add the vector field $h^{[2]}$).

**Corollary 12.6** (Kang-Krener) If $m_0 = 2$ then the system (12.3) is always equivalent to the strict feedforward form (in particular, to the feedforward form) up to order 2.
12.4 Feedforward and strict feedforward forms: the general step

According to Theorem 12.4, Kang normal form of the first nonlinearizable term of a system, which is feedback equivalent to feedforward (resp. strict feedforward form), must be feedforward (resp. strict feedforward). We will see in the sequel, that the situation gets different when we proceed to higher order terms. Let us assume that the system $\Sigma^\infty$, given by (12.1), is in feedforward (resp. strict feedforward) normal form up to order $m_0 + l - 1$, that is, $\Sigma^\infty$ takes the form

$$\Sigma^\infty : \dot{\xi} = A\xi + Bu + \sum_{m=0}^{m_0} h_m^{[m]}(\xi) + \sum_{m=m_0}^{m_0+l-1} \tilde{f}_j^{[m]}(\xi) + f^{[m_0+l]}(\xi) + g^{[m_0+l-1]}(\xi)u + R(\xi, u),$$

where $R(\xi, u) \in V^{\geq m_0+l+1}(\xi, u)$ and for any $0 \leq m \leq m_0 + l - 1$,

$$\tilde{f}_j^{[m]}(\xi) = \begin{cases} \sum_{i=j+2}^{\infty} \xi_i^{[m-2]} f_j^{[m-i]}(\xi, \xi, \ldots, \xi) & \text{if} \ 1 \leq j \leq n - 2, \\ 0 & \text{if} \ n - 1 \leq j \leq n, \end{cases}$$

and

$$h_j^{[m]}(\xi) = \begin{cases} \xi_j h_j^{[m-1]}(\xi, \xi, \ldots, \xi) & \text{if} \ 1 \leq j \leq n - 2, \\ 0 & \text{if} \ n - 1 \leq j \leq n. \end{cases}$$

(resp. $h_j^{[m]}(\xi) = \begin{cases} c_{m,j}^{[m]} & \text{if} \ 1 \leq j \leq n - 2, \\ 0 & \text{if} \ n - 1 \leq j \leq n. \end{cases}$)

By the definition of $m_0$, it follows that there exists $1 \leq j \leq n - 2$ such that $\tilde{f}_j^{[m_0]} \neq 0$. Throughout we will assume that $\tilde{f}_j^{[m_0]} \neq 0$, which simplifies the exposition. The analysis of the general case, although more technical, follows the same line (see [78] for the strict feedforward form).

Crucial objects in studying the strict feedforward case are $a_k^{[m_0+l]}$, which are the homogeneous invariants associated to the homogeneous system

$$\Sigma_k^{[m_0+l]} : \dot{\xi} = A\xi + Bu + \left[\tilde{f}_j^{[m_0]}, Y^{[l+1]}_k\right](\xi),$$

where

$$Y^{[l+1]}_k = Y^{[l+1]}_{k,0} = \xi^{l+1}_k + L_{A\xi}^{[l+1]} \frac{\partial}{\partial \xi_k} + L_{A\xi}^{[l+1]} \frac{\partial}{\partial \xi^{l+1}_k} + \cdots + L_{A\xi}^{[l+1]} \frac{\partial}{\partial \xi^{l+1}_k} \frac{\partial}{\partial \xi^{l+1}_n}.$$  

For any $2 \leq k \leq n - 2$ and any $0 \leq s \leq l + 1$, consider the homogeneous vector fields

$$Y_k^{[l+1]} = \xi^{[l+1]}_k + L_{A\xi}^{[l+1]} \frac{\partial}{\partial \xi_k} + L_{A\xi}^{[l+1]} \frac{\partial}{\partial \xi^{l+1}_k} + \cdots + L_{A\xi}^{[l+1]} \frac{\partial}{\partial \xi^{l+1}_k} \frac{\partial}{\partial \xi^{l+1}_s}.$$  

In studying the feedforward case, crucial objects are $a_k^{[m_0+l]}$, which are the $(m_0 + l)$-invariants of the homogeneous system $\Sigma^{[m_0+l]}$, defined by (12.8)-(12.10), and by $a_k^{[m_0+l]}$ the $(m_0 + l)$-invariants of the homogeneous system

$$\Sigma_k^{[m_0+l]} : \dot{\xi} = A\xi + Bu + \left[\tilde{f}_j^{[m_0]}, Y_k^{[l+1]}\right](\xi).$$

Our two main results of this section can be stated as follows. In the strict feedforward case we have (see [82] for the proof, compare also [78]):
Theorem 12.7 The system $\Sigma^\infty$, defined by (12.8), (12.9), (12.11), is feedback equivalent to the strict feedforward form, up to order $m_0+l$, if and only if there exist real constants $\sigma_{2,0}, \sigma_{3,0}, \ldots, \sigma_{n-2,0}$ such that for any $(j, i) \in \Delta$ and any $1 \leq q \leq j$

$$L_{A^n-qB} \left( a^{[m_0+l]j,i+2} - \sum_{k=2}^{n-2} \sigma_{k,0} a^{[m_0+l]j,i+2}_{k,0} \right) = 0. \quad (12.12)$$

In the feedforward case we have (see [80] for the proof and comments):

Theorem 12.8 The system $\Sigma^\infty$, defined by (12.8), (12.9), (12.10), is feedback equivalent, up to order $m_0+l$, to the feedforward form if and only if there exist real constants $\sigma_{k,s}$ for $2 \leq k \leq n-2$ and $1 \leq s \leq l+1$ such that for any $(j, i) \in \Delta$ and any $1 \leq q \leq j-1$

$$L_{A^n-qB} \left( a^{[m_0+l]j,i+2} - \sum_{k=2}^{n-2} \sum_{s=1}^{l+1} \sigma_{k,s} a^{[m_0+l]j,i+2}_{k,s} \right) = 0. \quad (12.13)$$

Notice that (12.12) is an invariant way of expressing the fact that

$$a^{[m_0+l]j,i+2} - \sum_{k=2}^{n-2} \sigma_{k,0} a^{[m_0+l]j,i+2}_{k,0} = Q^{[m_0+l-2]}_{j,i},$$

where $Q^{[m_0+l-2]}_{j,i}(\xi_{j+1}, \ldots, \xi_i)$ are homogeneous polynomials of degree $m_0 + l - 2$ depending on the indicated variables only. Similarly, (12.13) is an invariant way of expressing the fact that

$$a^{[m_0+l]j,i+2} - \sum_{k=2}^{n-2} \sum_{s=1}^{l+1} \sigma_{k,s} a^{[m_0+l]j,i+2}_{k,s} = Q^{[m_0+l-2]}_{j,i},$$

where $Q^{[m_0+l-2]}_{j,i}(\xi_{j+1}, \ldots, \xi_i)$ are homogeneous polynomials of degree $m_0 + l - 2$ depending on the indicated variables only.

Observe that checking the conditions (12.7) and (12.12) or the conditions (12.6) and (12.13) involves only differentiation of polynomials and algebraic operations. Therefore Theorem 12.4 (i) followed by a successive application of Theorem 12.8 (resp. Theorem 12.4 (ii) followed by a successive application of Theorem 12.7) yields to a constructive procedure that allows us to check whether a given system can be transformed into feedforward (resp. strict feedforward) form. Moreover, for any system satisfying the conditions (12.6) and (12.13) (resp. (12.7) and (12.12)), we can calculate, step by step, explicit feedback transformations bringing it into feedforward (resp. strict feedforward) form using transformations constructed in [81] and [77] and presented in Section 3.

Finally, observe that the condition (12.13) can be seen as a natural generalization of (12.12). Indeed, in the second sum of (12.13) we could start the summation with $s = 0$. It is so, because the action of $Y_{k,0}^{[l+1]}$ on all components, starting form the second one, can be compensated by that of $Y_{k,s}^{[l+1]}$, for $s \geq 1$, and the action of $Y_{k,0}^{[l+1]}$ on the first component is irrelevant since the first component can be arbitrary in any feedforward system.
12.5 Feedforward and strict feedforward systems on \( \mathbb{R}^4 \)

The aim of this subsection is to illustrate our results by comparing feedforward and strict feedforward systems in \( \mathbb{R}^4 \). Notice that this is the lowest dimension in which both classes are nontrivial since in \( \mathbb{R}^3 \) all systems with controllable linearization can be brought to the feedforward form via feedback.

**Feedforward case.** Consider a system on \( \mathbb{R}^4 \) and assume that it is feedback equivalent to feedforward form up to terms of degree \( m \). Then, according to Theorem 12.3, we can assume, without loss of generality, that the system takes the form

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 + \sum_{k=2}^m \hat{f}_1^k(\xi) + \bar{f}_1^{m+1}(\xi) \\
\dot{\xi}_2 &= \xi_3 + \sum_{k=2}^m \hat{f}_2^k(\xi) + \bar{f}_2^{m+1}(\xi) \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= u,
\end{align*}
\]

(12.14)

where the homogeneous vector fields \( \hat{f}_1^k \) and \( \hat{f}_2^k \), for \( 2 \leq k \leq m \), are in feedforward normal form (see Definition 12.1) and \( \bar{f}_1^{m+1} \) is in Kang normal form \( \Sigma^{[m+1]} \). Let us consider, for any \( 1 \leq s \leq m \), the following homogeneous vector field

\[
Y_{2,s}^{[m]} = \xi_1^s \xi_2^{m-s} \frac{\partial}{\partial \xi_1} + L \xi_3 \xi_2^{m-s} \frac{\partial}{\partial \xi_3} + L^2 \xi_4 \xi_2^{m-s} \frac{\partial}{\partial \xi_4}
\]

and construct the corresponding homogeneous system

\[
\Sigma_{2,s}^{[m+1]} : \dot{\xi} = A\xi + Bu + [\bar{f}^{[2]}_{2,s}, Y_{2,s}^{[m]}](\xi),
\]

where Kang-Krener quadratic normal form \( \bar{f}^{[2]}_{2,s} \) is given by

\[
\bar{f}^{[2]}_{2,s} = (a \xi_3^2 + b \xi_4^2) \frac{\partial}{\partial \xi_1} + c \xi_4^2 \frac{\partial}{\partial \xi_2}.
\]

Let us denote by \( a_{2,s}^{[m+1]j,i+2} \) and \( d_{2,s}^{[m+1]j,i+2} \), for \( (j, i) \in \Delta \), the homogeneous invariants associated, respectively, to the homogeneous systems \( \Sigma_{2,s}^{[m+1]} \) and to the normal form

\[
\Sigma_{2,s}^{[m+1]} : \dot{\xi} = A\xi + Bu + \bar{f}^{[m+1]}(\xi).
\]

Here \( (A, B) \) is the Brunovský canonical form of dimension 4. Denote \( C_2 = (0, 1, 0, 0) \). We have \( \Delta = \{(1, 0); (1, 1); (2, 0)\} \). Since the only term that is not feedforward is present if \( P_{2,4}^{[m-1]} \) depends
on \( \xi_1 \), we will focus our attention only on the invariants \( a_{2,s}^{[m+1]2,2} \) and \( a_{2,s}^{[m+1]2,2} \) given, respectively, by
\[
a_{2,s}^{[m+1]2,2} = C_2 \partial B_2[f, \phi_{2,s}^{[m]}] \quad \text{and} \quad a_{2,s}^{[m+1]2,2} = C_2 \partial B_2[f^{[m+1]}] = \frac{\partial^2 \xi_2^2 P_{2,4}^{[m-1]}}{\partial \xi_4^2}.
\]
A direct computation gives
\[
a_{2,s}^{[m+1]2,2} = 2bs_{[m]} \xi_1^m \xi_2^m - 2c(m - s) \xi_1^{m-s} - 2c(m - s) \xi_1^{m-s}.\]
Then the system (12.14) is feedback equivalent to feedforward form up to order \( m + 1 \) if and only if there exist real constants \( \sigma_{2,s} \), for \( 1 \leq s \leq m - 1 \), such that
\[
a_{2,s}^{[m+1]2,2} = \sum_{s=1}^{m} \sigma_{2,s} a_{2,s}^{[m+1]2,2} = Q_{[m-1]}(\xi_2, \xi_3, \xi_4),
\]
for some homogeneous polynomial \( Q_{[m-1]}(\xi_2, \xi_3, \xi_4) \), which is equivalent to the condition that
\[
P_{2,4}^{[m-1]}(\xi) = Q_{[m-1]}(\xi_{[1]}, \xi_2) + S_{2}^{[m-1]}(\xi_2, \xi_3, \xi_4).
\]
The codimension \( c_{\text{FF}}(m + 1) \) of the space of homogeneous systems of degree \( m + 1 \), which are feedback equivalent to feedforward form, is equal to the dimension of the space of all homogeneous polynomials of degree \( m - 1 \) of the form
\[
Q_{2}^{[m-1]}(\xi) = \xi_1 \xi_3 R_{1}^{[m-3]}(\xi_1, \xi_2, \xi_3) + \xi_1 \xi_4 R_{2}^{[m-3]}(\xi_1, \xi_2, \xi_3, \xi_4).
\]
We thus get
\[
c_{\text{FF}}(m + 1) = \frac{(m - 1)(m - 2)}{2} + \frac{(m - 2)(m - 1)m}{6} = \frac{(m + 3)(m - 1)(m - 2)}{6}.
\]

**Strict feedforward case.** Now, consider a system on \( \mathbb{R}^4 \) and assume that it is feedback equivalent to a strict feedforward form up to terms of degree \( m \). Then, according to Theorem 12.3, we can assume, without loss of generality, that the system takes the form (12.14), where the homogeneous vector fields \( \xi_{2}^{[k]} \partial / \partial \xi_2 \) and \( \xi_{2}^{[k]} \partial / \partial \xi_3 \), for \( 2 \leq k \leq m \), are in strict feedforward normal form (see Definition 12.1) and \( \bar{f}^{[m+1]} \) in Kang normal form. We assume \( m_0 = 2 \) which yields \( l = m - 1 \), and we consider the vector field
\[
Y_{2,0}^{[m]} = \xi_2^m \frac{\partial}{\partial \xi_2} + L_{A_{\xi}}(\xi_2^m) \frac{\partial}{\partial \xi_3} + L_{2A_{\xi}}(\xi_2^m) \frac{\partial}{\partial \xi_4}
= \xi_2^m \frac{\partial}{\partial \xi_2} + m \xi_2^{m-1} \xi_3 + (m(m - 1) \xi_2^{m-2} \xi_3^2 + m \xi_2^{m-1} \xi_4) \frac{\partial}{\partial \xi_4}
\]
whose corresponding homogeneous system is
\[
\Sigma_{2,0}^{[m+1]} : \dot{\xi} = A_\xi + Bu + \bar{f}^{[2]}(\xi_{2,0}^{[m]}).
\]
Denote by \( a_{2,0}^{[m+1]j,i+2} \) and \( a_{2,0}^{[m+1]j,i+2} \), for \( (j,i) \in \Delta = \{(1,0); (1,1); (2,0)\} \), the homogeneous invariants associated, respectively, to the homogeneous systems \( \Sigma_{2,0}^{[m+1]} \) and to the normal form
\[
\Sigma_{N_{F}}^{[m+1]} : \dot{\xi} = A_\xi + Bu + \bar{f}^{[m+1]}(\xi).
\]
Denote $C_1 = (1,0,0,0)$ and $C_2 = (0,1,0,0)$. We will calculate the invariants $a_{2,0}^{[m+1]1,2}$, $a_{2,0}^{[m+1]2,2}$, and $a_{2,0}^{[m+1]1,3}$ of system $\Sigma_{2,0}^{[m+1]}$. We have
\[
a_{2,0}^{[m+1]1,2} = a_{2,0}^{[m+1]2,2} = C_1 a_B^2 [f, Y_{2,0}], \quad a_{2,0}^{[m+1]2,2} = C_2 a_B^2 [f, Y_{2,0}],
\]
and
\[
a_{2,0}^{[m+1]1,3} = C_1 (a_{AB} x_{2}^{[m]} - a_{A} x_{2}^{[m]})(\pi_3(\xi)),
\]
where $\pi_3(\xi) = (\xi_1, \xi_2, \xi_3)^T$, $x_{2}^{[m]} = a_B^{[m]} Y_{2,0}$, and $x_{2}^{[m]} = a_{AB}^{[m]} - a_{A\xi} a_B^{[m]} [f, Y_{2,0}]$. A direct computation gives
\[
a_{2,0}^{[m+1]1,2} = a_{2,0}^{[m+1]2,2} = -2cmx_2^{m-1}
\]
and
\[
a_{2,0}^{[m+1]1,3} = -4amx_2^{m-1} + 8bn(m-1)(m-2)x_2^{m-3}x_3 - 4cm(m-1)x_2^{m-2}x_3.
\]
In the other hand the invariants associated to the normal form are
\[
\tilde{a}_{2,0}^{[m+1]1,2} = a_B^2 [f, Y_{2,0}] = \frac{\partial^2 \xi_2^2 P_{1,4}^{[m-1]}}{\partial \xi_4^2}, \quad \tilde{a}_{2,0}^{[m+1]1,3} = a_B^2 [f, Y_{2,0}] = \frac{\partial^2 \xi_2^2 P_{2,4}^{[m-1]}}{\partial \xi_3^2},
\]
and
\[
\tilde{a}_{2,0}^{[m+1]2,2} = a_B^2 [f, Y_{2,0}] = \frac{\partial^2 \xi_2^2 P_{2,4}^{[m-1]}}{\partial \xi_4^2}.
\]
Then the system (12.14) is feedback equivalent to a strict feedforward form up to order $m+1$ if and only if there exist a real constant $\sigma_2,0$ such that
\[
\tilde{a}_{2,0}^{[m+1]1,2} - \sigma_2,0 \tilde{a}_{2,0}^{[m+1]1,2} = Q_{1,4}^{[m-1]}(\xi_2, \xi_3, \xi_4), \quad \tilde{a}_{2,0}^{[m+1]1,3} - \sigma_2,0 \tilde{a}_{2,0}^{[m+1]1,3} = Q_{1,3}^{[m-1]}(\xi_2, \xi_3),
\]
and
\[
\tilde{a}_{2,0}^{[m+1]2,2} - \sigma_2,0 \tilde{a}_{2,0}^{[m+1]2,2} = Q_{2,4}^{[m-1]}(\xi_3, \xi_4),
\]
for some homogeneous polynomials $Q_{1,4}^{[m-1]}(\xi_2, \xi_3, \xi_4)$, $Q_{1,3}^{[m-1]}(\xi_2, \xi_3)$, and $Q_{2,4}^{[m-1]}(\xi, \xi_4)$. The above conditions are equivalent to the fact that
\[
P_{1,4}^{[m-1]}(\xi) = S_{1,4}^{[m-1]}(\xi_2, \xi_3, \xi_4), \quad P_{1,3}^{[m-1]}(\xi) = S_{1,3}^{[m-1]}(\xi_2, \xi_3), \quad P_{2,4}^{[m-1]}(\xi) = \lambda \xi_2^{m-1} + S_{2,4}^{[m-1]}(\xi_3, \xi_4).
\]
The codimension $c_{SFF}(m+1)$ of the space of homogeneous systems of degree $m+1$, which are feedback equivalent to strict feedforward form, is equal to the dimension of the space of all homogeneous vector fields of degree $m-1$ of the form
\[
\left( \xi_1 \xi_2^2 R_{1,3}^{[m-2]}(\xi_1, \xi_2, \xi_3) + \xi_1 \xi_2 R_{1,4}^{[m-2]}(\xi_1, \xi_2, \xi_3, \xi_4) \right) \frac{\partial}{\partial \xi_1} + \xi_4^2 \left( \xi_1 \xi_2 R_{2,4}^{[m-2]}(\xi_2, \xi_3, \xi_4) + \xi_2 R_{2,4}^{[m-2]}(\xi_2, \xi_3, \xi_4) \right) \frac{\partial}{\partial \xi_2},
\]
with $R_{2,4}^{[m-2]}(\xi_2, 0, 0) = 0$, that is, $\tilde{R}_{2,4}^{[m-2]}(\xi_2, 0, 0) = \xi_3 \tilde{R}_{2,4}^{[m-3]}(\xi_2, \xi_3) + \xi_4 \tilde{R}_{2,4}^{[m-3]}(\xi_2, \xi_3, \xi_4)$. We thus get
\[
c_{SFF}(m+1) = \frac{(m+1)m(m-1)}{6} + \frac{m(m-1)}{2} + \frac{(m+1)m(m-1)}{6} + \frac{m-2}{1} + \frac{(m-1)(m-2)}{2} = \frac{m^3 + 3m^2 - 4m - 3}{3}.
\]
Remark 12.9 The codimensions $c_{FF}(m+1)$ and $c_{SFF}(m+1)$ are computed for generic systems, in particular in the case when $f^{[2]}_2 \neq 0$. If $f^{[2]}_2 = 0$, then those codimensions are modified as follows

$$
\hat{c}_{FF}(m+1) = c_{FF}(m+1) + 1 = \frac{(m+3)(m-1)(m-2)}{6} + 1
$$

$$
\hat{c}_{SFF}(m+1) = c_{SFF}(m+1) + 1 = \frac{m^3 + 3m^2 - 4m}{3}.
$$

In each case the gap between the two codimensions is equal to $\frac{m^3 + 6m^2 - m - 12}{6}$.

Finally, we will compute the codimension the space of linearizable homogenous systems of degree $m+1$ on $\mathbb{R}^4$. The homogeneous part of degree $m+1$ of (12.14) is given by two polynomials $P_{1,4}$ and $P_{2,4}$ of four variables and one polynomial $P_{1,3}$ of three variables. Linearizability is equivalent to vanishing of all of them and thus the codimension of linearizable homogeneous systems is

$$
c_{FL}(m+1) = \frac{(m-1)m}{2} + \frac{2(m-1)m(m+1)}{6} = \frac{(2m+5)(m-1)m}{6}.
$$

12.6 Geometric characterization of feedforward and strict feedforward systems

In the previous section we proposed a step-by-step constructive method to bring a system into a feedforward form and strict feedforward form whenever it is possible (see also [78], [82], [80]). In [61], the problem of transforming a system, affine with respect to controls, into (strict) feedforward form via a diffeomorphism, i.e., via a nonlinear change of coordinates, was studied. A geometric description of systems in feedforward form has been given in [2]. The conditions of [2], although being intrinsic, are not checkable.

In the present section we look at the problem in the spirit of [2] but we focus our attention on vector fields rather than on invariant distributions. It turns out that feedback equivalence (resp. state-space equivalence) to the strict feedforward form can be characterized by the existence of a sequence of infinitesimal symmetries (resp. strong infinitesimal symmetries) of the system.

12.7 Symmetries and strict feedforward form

In this section we will establish results relating symmetries and strict feedforward forms. To start with, recall two basic notions of equivalence of control systems. The word smooth will mean throughout $C^\infty$-smooth and all control systems are assumed to be smooth.

Two control systems

$$
\Sigma : \dot{x} = f(x) + g(x)u, \quad x \in X
$$

and

$$
\tilde{\Sigma} : \dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\tilde{u}, \quad \tilde{x} \in \tilde{X}
$$

are called state space equivalent, shortly $S$-equivalent, if there exists a smooth diffeomorphism $\phi : X \rightarrow \tilde{X}$, such that

$$
\phi_* f = \tilde{f} \quad \text{and} \quad \phi_* g = \tilde{g};
$$
(we take \( u = \hat{u} \)), and they are called feedback equivalent, shortly \( F\)-equivalent, if there exists a smooth diffeomorphism \( \phi : X \to \tilde{X} \) and smooth functions \( \alpha, \beta \), satisfying \( \beta(\cdot) \neq 0 \), such that

\[
\phi_*(f + g\alpha) = \tilde{f} \quad \text{and} \quad \phi_*(g\beta) = \tilde{g}.
\]

Recall also (see Section 11.4) that a vector field \( v \) on an open subset \( X \subset \mathbb{R}^n \) is an infinitesimal symmetry of the system \( \Sigma \) if the (local) flow \( \gamma^v_t \) of \( v \) is a local symmetry of \( \Sigma \), for any \( t \) for which it exists.

We will also be dealing with the following stronger notions. A diffeomorphism \( \phi : X \to X \) is a strong symmetry of \( \Sigma \) if it preserves the vector fields \( f \) and \( g \) (and not only the affine distribution \( \mathcal{A} \) spanned by them), that is, if

\[
\phi_*f = f \quad \text{and} \quad \phi_*g = g.
\]

A local strong symmetry is a local diffeomorphism preserving \( f \) and \( g \). We say that a vector field \( v \) on an open subset \( X \subset \mathbb{R}^n \) is an infinitesimal strong symmetry of \( \Sigma \) if the (local) flow \( \gamma^v_t \) of \( v \) is a local strong symmetry of \( \Sigma \), for any \( t \) for which it exists.

Consider the system \( \Sigma \) and denote by \( \mathcal{G} \) the distribution spanned by the vector field \( g \). We have the following characterization of infinitesimal symmetries and strong symmetries.

**Proposition 12.10**

(i) A vector field \( v \) is an infinitesimal strong symmetry of \( \Sigma \) if and only if

\[ [v, g] = 0, \quad \text{and} \quad [v, f] = 0. \]

(ii) A vector field \( v \), such that \( v(p) \neq 0 \), is an infinitesimal symmetry of \( \Sigma \), locally at \( p \), if and only if

\[ [v, g] = 0 \mod \mathcal{G}, \quad \text{and} \quad [v, f] = 0 \mod \mathcal{G}. \]

in a neighborhood of \( p \).

The second item remains true even if \( g(p) = 0 \). In this case, we have to understand \( \mathcal{G} \) as the module of vector fields generated by \( g \) over the ring of smooth functions.

An infinitesimal symmetry \( v \) is called stationary at \( p \in X \) if \( v(p) = 0 \) and nonstationary if \( v(p) \neq 0 \).

Assume that \( v \) is a strong infinitesimal symmetry of \( \Sigma \), nonstationary at \( p \in X \). Then there exist a neighborhood \( X_p \) of \( p \) and the factor system \( \Sigma/\sim_v \), where the equivalence relation \( \sim_v \) is induced by the local action of the 1-parameter local group defined by \( v \), that is, \( q_1 \sim_v q_2 \) if and only if they belong to the same integral curve of \( v \) (more precisely, to the same connected component of the intersection of an integral curve of \( v \) with \( X_p \)).

**Theorem 12.11** The following condition are equivalent.

(i) \( \Sigma \) is, locally at \( p \in X \), \( S \)-equivalent to the affine strict feedforward form (ASFF);

(ii) Each system \( \Sigma_1, \Sigma_2, \ldots, \Sigma_n \) possesses a strong infinitesimal nonstationary symmetry \( v_i \), where \( \Sigma_1 \) is the restriction of \( \Sigma \) to a neighborhood \( X_p \) and

\[ \Sigma_{i+1} = \Sigma_i/\sim_{v_i}, \]

with \( \sim_{v_i} \) the equivalence relation defined by the local action of the 1-parameter group of \( v_i \);
There exist smooth vector fields \( w_1, \ldots, w_n \), independent at \( p \in X \), such that, locally at \( p \),

\[
[w_i, w_j] \in D_{i-1}, \quad [w_i, g] \in D_{i-1}, \quad [w_i, f] \in D_{i-1},
\]

for any \( 1 \leq i \leq n \) and \( j \leq i \), where \( D_0 = 0 \) and \( D_i = \text{span} \{ w_1, \ldots, w_i \} \);

(iv) There exist smooth vector fields \( \tilde{w}_1, \ldots, \tilde{w}_n \), independent at \( p \in X \), such that, locally at \( p \),

\[
[\tilde{w}_i, \tilde{w}_j] = 0, \quad [\tilde{w}_i, g] \in \tilde{D}_{i-1}, \quad [\tilde{w}_i, f] \in \tilde{D}_{i-1},
\]

for any \( 1 \leq i \leq n \) and \( j \leq i \), where \( \tilde{D}_0 = 0 \) and \( \tilde{D}_i = \text{span} \{ \tilde{w}_1, \ldots, \tilde{w}_i \} \).

In Section 12.8 we will show that the problem of transforming a general system to (SFF) can be reduced to the above theorem by a preintegration. A detailed proof of Theorem 12.11 is given in [70].

The above theorem implies that an invariant characterization of the affine strict feedforward form (ASSF) involves vector fields (forming a sequence of infinitesimal symmetries) rather than invariant distributions. To be more precise, a characterization of the affine feedforward form

\[
\begin{align*}
\dot{z}_1 &= f_1(z_1, \ldots, z_n) + g_1(z_1, \ldots, z_n)u \\
\dot{z}_2 &= f_2(z_2, \ldots, z_n) + g_2(z_2, \ldots, z_n)u \\
& \vdots \\
\dot{z}_{n-1} &= f_{n-1}(z_{n-1}, z_n) + g_{n-1}(z_{n-1}, z_n)u \\
\dot{z}_n &= f_n(z_n) + g_n(z_n)u,
\end{align*}
\]

(AFF)

was obtained by Astolfi and Mazenc [2] in terms of invariant distributions as follows:

**Proposition 12.12** The system \( \Sigma \) is locally equivalent to the affine feedforward form (AFF) if and only if there exists a sequence of distributions

\[
D_1 \subset \cdots \subset D_n,
\]

where \( D_i \) is involutive and of rank \( i \), such that

\[
[D_i, g] \subset D_i, \quad [D_i, f] \subset D_i.
\]

A first guess for a characterization of the affine strict feedforward form (ASFF) could be (compare [2]) the existence of a nested sequence of involutive distributions \( D_i \), of constant rank \( i \), satisfying

\[
[D_i, g] \subset D_{i-1}, \quad [D_i, f] \subset D_{i-1}.
\]

This is not a correct answer for two reasons. Firstly, the latter conditions are not invariant, that is, even if they are satisfied for some vector fields \( w_1, \ldots, w_i \) spanning \( D_i \) then, in general, for other generators of the same distribution \( D_i \), we will have on the right the inclusion in \( D_i \) (and not in \( D_{i-1} \)). Secondly, the above conditions, even reformulated in terms of vector fields, are not sufficient for equivalence to (ASSF). Indeed, the condition that there exist linearly independent vector fields \( w_1, \ldots, w_n \) such that

\[
[w_i, g] \in D_{i-1}, \quad [w_i, f] \in D_{i-1}.
\]
for any \(1 \leq i \leq n\), where \(D_0 = 0\) and \(D_i = \text{span} \{w_1, \ldots, w_i\}\) are involutive, does not imply \(S\)-equivalence to (ASFF) unless we assume an additional property on the \(w_i's\): like the first condition of (iii) (which is the weakest possible) or the first condition of (iv), which is the strongest one.

We have an analogous result for feedback equivalence to strict feedforward form, where the role of strong infinitesimal symmetries is replaced by that of infinitesimal symmetries. To state it, we need the following considerations. We will write \(\Sigma(f, g)\), to denote the system \(\Sigma\) defined by the pair of vector fields \((f, g)\). Assume that \(v\) is an infinitesimal symmetry of \(\Sigma(f, g)\), nonstationary at \(p \in X\), that is, such that \(v(p) \neq 0\). Then the second part of Proposition 2 implies that there exits a feedback pair \((\alpha, \beta)\) such that \(v\) is a strong infinitesimal symmetry of the system \(\Sigma(f, g)\), where \(\tilde{f} = f + g\alpha\) and \(\tilde{g} = g\beta\). Thus there exists a neighborhood \(X_p\) of \(p\) in which the factor system \(\tilde{\Sigma}/\sim_v\) system is well defined, where the equivalence relation \(\sim_v\) is induced by the local action of the 1-parameter local group defined by \(v\). Notice that given a system \(\Sigma\), there are many systems \(\tilde{\Sigma}(\tilde{f}, \tilde{g})\), feedback equivalent to \(\Sigma\), and such that \(v\) is a strong infinitesimal symmetry of \(\tilde{\Sigma}\). We will denote by \(\tilde{\Sigma}\) any of those systems. Actually, any two such systems are equivalent by a feedback pair \((\tilde{\alpha}, \tilde{\beta})\), where the functions \(\tilde{\alpha}\) and \(\tilde{\beta}\) are constant on the trajectories of \(v\).

**Theorem 12.13** The following condition are equivalent.

(i) \(\Sigma\) is, locally at \(p \in X\), \(F\)-equivalent to the affine strict feedforward form (ASFF) satisfying \(g_n \neq 0\);

(ii) Each system \(\Sigma_1, \Sigma_2, \ldots, \Sigma_n\) possesses an infinitesimal symmetry \(v_i\), where \(\Sigma_1\) is the restriction of \(\Sigma\) to a neighborhood \(X_p\) and

\[
\Sigma_{i+1} = \tilde{\Sigma}_i/\sim_{v_i},
\]

where \(\sim_{v_i}\) is the equivalence relation induced by the local action of the 1-parameter group of \(v_i\), and such that \(v_i\) and the control vector field \(g_i\) of \(\Sigma_i\) are independent, for \(1 \leq i \leq n - 1\);

(iii) There exist smooth vector fields \(w_1, \ldots, w_n\), independent at \(p \in X\), such that, locally at \(p\),

\[
[w_i, w_j] \in D_{i-1}, \quad [w_i, g] \in D_{i-1} + G, \quad [w_i, f] \in D_{i-1} + G,
\]

for any \(1 \leq i \leq n\) and \(j \leq i\), where \(D_0 = 0\) and \(D_i = \text{span} \{w_1, \ldots, w_i\}\) and, moreover, \(g(p) \notin D_{n-1}(p)\);

(iv) There exist smooth vector fields \(\tilde{w}_1, \ldots, \tilde{w}_n\), independent at \(p \in X\), such that, locally at \(p\),

\[
[\tilde{w}_i, \tilde{w}_j] = 0, \quad [\tilde{w}_i, g] \in \tilde{D}_{i-1} + G, \quad [\tilde{w}_i, f] \in \tilde{D}_{i-1} + G,
\]

for any \(1 \leq i \leq n\) and \(j \leq i\), where \(\tilde{D}_0 = 0\) and \(\tilde{D}_i = \text{span} \{\tilde{w}_1, \ldots, \tilde{w}_i\}\) and, moreover, \(g(p) \notin \tilde{D}_{n-1}(p)\).

The assumption \(g(p) \notin D_{n-1}(p)\) can be dropped (equivalently, we allow for \(g_n = 0\)) if we understand the conditions (iii) and (iv) as well as those of the second part of Proposition 2 in the sense of module of vector fields and not of distributions.

A proof of the above theorem follows the same line as that of Theorem 12.11, the only difference is to show that in the successive steps, the existence of infinitesimal symmetries does not depend on the choice of \(\tilde{\Sigma}_i\) in \(\Sigma_{i+1} = \tilde{\Sigma}_i/\sim_{v_i}\).
12.8 Strict feedforward form: affine versus general

In this section we will show that the problem of transforming a general control system to the strict feedforward form can be reduced to that for affine systems by taking the preintegration. The same procedure of extension has been already used for the problems of linearization and decoupling [93] and equivalence to the $p$-normal form [69].

Consider a general nonlinear control system

$$\Pi : \dot{x} = f(x, u),$$

where $x \in X$, an open subset of $\mathbb{R}^n$, $u \in \mathbb{R}$. Together with $\Sigma$, we consider its extension (preintegration)

$$\Pi^e : \dot{x}^e = f^e(x^e) + g^e(x^e)w,$$

where $x^e = (x, u) \in X \times \mathbb{R}^1$, $w \in \mathbb{R}$, and the dynamics are given by $f^e(x^e) = f(x, u) + 0 \cdot \frac{\partial}{\partial u}$ and $g^e(x^e) = \frac{\partial}{\partial u}$. Notice that $\Sigma^e$ is a control-affine system controlled by the derivative $\dot{u} = w$ of the original control $u$.

Recall that $L_0$ denotes the Lie ideal generated by $\{f_u - f_{\bar{u}}, u, \bar{u} \in U\}$ in the Lie algebra $\mathcal{L}$ of the system $\Pi$. Assume that $\dim L_0(p) = n$.

**Proposition 12.14** The system $\Pi$ is $S$-equivalent (resp. $F$-equivalent), locally at $(x_0, u_0)$, to the strict feedforward form (SFF) if and only if the extension $\Pi^e$ is, locally at $x_0^e = (x_0, u_0)$, $S$-equivalent (resp. $F$-equivalent) to the affine strict feedforward form (ASFF).

The proof is based on showing that a diffeomorphism bringing $\Pi^e$ into the (ASFF) is of a special form: states depend on states only and the control is preserved. In particular, we show the following statement, which is of independent interest.

**Corollary 12.15** If the system $\Sigma$ is in an affine strict feedforward form (ASFF) satisfying $g_n \neq 0$, then it is $S$-equivalent to another (ASFF), for which $g_1 = \cdots = g_{n-1} = 0$.

12.9 Strict feedforward systems on the plane

In this section we will describe strict feedforward systems on the plane. Consider a system $\Sigma$ on an open subset $X$ of $\mathbb{R}^2$ and suppose that $g(p) \neq 0$. We define the multiplicity of $\Sigma$ at $p$ as the smallest integer $\mu$, such that $g$ and $\text{ad}_p^\mu f$ are linearly independent at $p$. Notice that the notion of multiplicity is feedback invariant (see, e.g., [44]). If the multiplicity is $\mu = 1$, then the system is feedback linearizable and thus feedback equivalent to (ASFF). The case of multiplicity $\mu \geq 2$ is described by the following:

**Proposition 12.16** Consider a system $\Sigma$ on open subset $X$ of $\mathbb{R}^2$ and suppose that $g(p) \neq 0$ and that it has multiplicity $\mu \geq 2$ at $p$.

(i) If $f$ and $g$ are linearly dependent at $p$, then $\Sigma$ is locally $F$-equivalent to the strict feedforward form (ASFF) if and only

$$f = \gamma \text{ad}_p^\mu f \mod \mathcal{G},$$

where $\gamma$ is a smooth function such that the smooth function $\varphi$ defined by

$$f = \varphi \text{ad}_p^\mu f \mod \mathcal{G}$$

is described by the following:
is divisible by \( \gamma^\mu \). Moreover, in this case \( \Sigma \) is locally \( F \)-equivalent to
\[
\dot{z}_1 = z_2^\mu \\
\dot{z}_2 = \nu.
\]

(ii) If \( f \) and \( g \) are linearly independent at \( p \), then \( \Sigma \) is locally \( F \)-equivalent to the strict feedforward form if and only
\[
ad_g f = \gamma ad_g^2 f \mod G,
\]
where \( \gamma \) is a smooth function such that the smooth function \( \psi \) defined by
\[
ad g f = \psi ad_g^\mu f \mod G
\]
is divisible by \( \gamma^\mu - 1 \). Moreover in this case \( \Sigma \) is locally \( F \)-equivalent to
\[
\dot{z}_1 = 1 + \epsilon z_2^\mu \\
\dot{z}_2 = \nu.
\]

In [44] it is proved that any planar system with a finite multiplicity \( \mu \) at \( p \) is locally feedback equivalent to the following system around \( 0 \in \mathbb{R}^2 \):
\[
\dot{z}_1 = z_2^\mu + a_{\mu-2} z_2^{\mu-2} + \cdots + a_1 z_2 + a_0, \\
\dot{z}_2 = \nu,
\]
where the smooth functions \( a_i \), for \( 0 \leq i \leq \mu - 2 \), depend on \( z_1 \) only and satisfy \( a_i(0) = 0 \) (except for \( a_0 \) in the case \( f \) and \( g \) independent at \( p \)). Moreover, we can always normalize one of the functions \( a_i \) (in particular, we can take \( a_0 = 1 \) if \( a_0(0) \neq 0 \)) and then the infinite jets of all remaining functions are feedback invariant. Proposition 12.16 implies that among all planar system only those are \( F \)-equivalent to the affine strict feedforward form for which all the above invariants are identically zero.

13 Analytic normal forms: a class of strict feedforward systems.

In the previous sections we have developed the theory of feedback classification following a formal approach introduced by Kang and Krener. Although the normal forms obtained are formal the theory has proved to be very useful in analyzing structural properties of nonlinear control systems. It has been used to study bifurcations of nonlinear systems [50], [53], has led to a complete description of symmetries around equilibrium, presented in Section 11, (see also [70], [71]), and allowed to characterize systems equivalent to feedforward and strict feedforward forms (see Section 12 and [78], [80], [82]).

A natural question to ask is whether normal and canonical forms presented in earliest sections are convergent or not. It is already known that the problem of convergence is difficult even for dynamical systems whose convergence depend on the location of the eigenvalues of the linear part. Those eigenvalues stand to be invariants for dynamical systems which is a first difference with control systems because the notion of eigenvalues is meaningless. It has been proved (see [1], [10])
that a dynamical system is biholomorphically equivalent to its linear part if the spectrum of its linearization is not resonant and belongs either to the Poincaré domain or to the Siegel domain with type $(C, \nu)$. When the spectrum is resonant and belong to the Poincaré domain, then the Poincaré-Dulac Theorem shows that the dynamical system is biholomorphically equivalent to a polynomial vector field. We will not recall explicitly those results here because of space limitations and we send the reader to the existing literature.

For control systems, Kang [48] derived from [54], and [55] (see also [36] and Proposition 9.4) that if an analytic control system is linearizable by a formal transformation, then it is linearizable by an analytic transformation. Kang [48] also gives a class of non linearizable 3-dimensional analytic control systems which are equivalent to their normal forms by analytic transformations. Those are the only results about convergence of normal forms known to us to this date.

In this section we study a class of nonlinear systems called special strict feedforward forms, and we show that this class could be brought to its normal form (actually canonical form) via analytic transformations.

Consider an analytic single-input control system $\dot{x} = f(x, u)$, in strict feedforward form, that is, such that $f_j(x, u) = f_j(x_{j+1}, \ldots, x_n, u), \quad 1 \leq j \leq n$.

Notice that each component decomposes uniquely as

$$f_j(x, u) = a_j(x_{j+1}) + F_j(x_{j+1}, \ldots, x_n, u), \quad \text{with} \quad F_j(x_{j+1}, 0, \ldots, 0) = 0. \quad (13.1)$$

A special strict feedforward form (SSFF) is an analytic strict feedforward form for which

$$a_j(x_{j+1}) = k_j x_{j+1}, \quad \text{whenever} \quad \frac{\partial a_j}{\partial x_{j+1}}(0) \neq 0. \quad (13.2)$$

The main result of this section is as follows [86].

**Theorem 13.1** Consider an analytic special strict feedforward form (SSFF) given by (13.1)-(13.2). There exists an analytic feedback transformation that brings the system (13.1)-(13.2) into the normal form

$$
\Pi_{NF} : \begin{cases}
\dot{z}_1 &= z_2 + \sum_{i=3}^{n+1} z_i^2 P_{1,i}(z_2, \ldots, z_i) \\
\vdots & \\
\dot{z}_j &= z_{j+1} + \sum_{i=j+2}^{n+1} z_i^2 P_{j,i}(z_{j+1}, \ldots, z_i) \\
\vdots & \\
\dot{z}_{n-1} &= z_n + z_{n+1}^2 P_{n-1,n+1}(z_{n+1}) \\
\dot{z}_n &= v,
\end{cases} \quad (13.3)
$$

where $P_{j,i}(z_{j+1}, \ldots, z_i)$ are analytic functions of the indicated variables, and $z_{n+1} = v$.

The main remark is that the normal form itself is in a strict feedforward form. Moreover, this normal form coincides with the canonical form as defined in [81]. Indeed we have

**Theorem 13.2** Two special strict feedforward systems (SSFF)$_1$ and (SSFF)$_2$ are feedback equivalent if and only if their normal forms are equal (after possible reparametrization $z_i = \lambda x_i$).
The proof of Theorem 13.1 is detailed in [86]. The proof of Theorem 13.2 follows automatically after normalization of the first nonlinearizable homogeneous vector field because no components of the (SSFF) system depend on the first variable $x_1$.

It has been proved in [86] that special strict feedforward forms define the only class of strict feedforward systems that can be brought to a normal form and still being in strict feedforward form.

Indeed, if

$$\dot{z} = \tilde{f}(z, v),$$

is another analytic strict feedforward form, that is, such that

$$\tilde{f}_j(z, u) = \tilde{a}_j(z_{j+1}) + \tilde{F}_j(z_{j+1}, \ldots, z_n, v), \quad \text{with} \quad \tilde{F}_j(z_{j+1}, 0, \ldots, 0) = 0 \quad (13.4)$$

for any $1 \leq j \leq n$, then we have

**Theorem 13.3** The system (13.4) is feedback equivalent to a (SSFF) if and only if

$$\tilde{a}_j(z_{j+1}) = \tilde{k}_j z_{j+1}, \quad \text{whenever} \quad \frac{\partial \tilde{a}_j}{\partial z_{j+1}}(0) \neq 0,$$

that is, the system is in (SSFF) in its coordinates.

Whether it is possible to bring any strict feedforward system into its normal form by analytic transformation is unclear yet, but if true, the normal form will no longer be in strict feedforward form. We send the reader to [86] for more details.

### 14 Conclusion

This survey, as its name indicates, is an attempt to summarize the diverse results about normal forms obtained in the past two decades using a formal approach. Starting from the pioneer work of Poincaré (Section 2), we then described, in Section 3, normal forms for single-input control systems obtained by Kang and Krener using classical Poincaré’s technique. The work of Kang and Krener was completed by the authors to obtain canonical forms, dual normal and dual canonical forms. Those results was exposed in Sections 4, 5 and 6. Results on previously mentioned sections concern single-input control systems with controllable linearization. The uncontrollable linearization case as well as the multi-input case came as a generalization of the precedent results, respectively, in Sections 7, and 8. In each of those sections, the results obtained has been compared to results in earliest sections, and shown to be their generalizations. The feedback linearization of control systems was originally the first problem dealt with feedback classification. However, we have chosen to introduce the results obtained in that matter only in Section 9 in order to make a parallelism between the formal approach that provides a step-by-step procedure of linearization and the classical approach of distributions. Although the main results concern continuous-time control systems, it would be a crime of lèse-majesty not to mention the discrete-time case. Thus we have devoted the Section 10 to normal forms of discrete-time control systems. This formal approach, introduced by Kang and Krener for control systems, has proved to be very useful in analyzing systems. Would we have enough space we might widen our survey, among other topics, to bifurcations, stabilization and observability of control systems.
complete description of symmetries of control systems has been obtained by the authors using this formal approach and those results formed the Section 11. The same approach has led to a step-by-step characterization of systems feedback equivalent to feedforward systems or to strict feedforward systems. For each degree of homogeneity, necessity and sufficient conditions was obtained and presented in Section 12. The amazing point about these notions is that they are all related in a nice way and have the same roots: normal and canonical forms. Indeed, we have shown that symmetries are described by canonical forms either in the analytic or formal category. In the other hand side, feedforward and strict feedforward systems are geometrically characterized in Section 12.6 using symmetries. Finally, in Section 13, it turns out that one of the biggest class (ever found) of control systems that could be brought to a normal and canonical form, using analytic transformations, is a class of strict feedforward systems. The important number of references listed in this survey illustrates, if necessary, the attractiveness of the notions presented here, and we have certainly omit many others. The interested reader will probably complete our work in that matter.

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